COMBINATORIAL INTERPRETATIONS OF SYMMETRIC FUNCTION OPERATORS

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Abstract

We consider two symmetric function operators defined by Haglund, Morse and Zabrocki [12] which play an important role in the study of the combinatorics of the space of diagonal harmonics.

In this work, we give combinatorial interpretations of those symmetric function operators applied to basis symmetric functions, especially Schur functions.

We also relate the operators iteratively applied to 1 and the Hall-Littlewood symmetric functions and derive a new way of calculating the charge statistics.

Introduction : Diagonal Harmonics

Consider the ring R_n of coinvariants for the diagonal action of the symmetric group S_n on $\mathbb{C}^n \oplus \mathbb{C}^n$;

$$R_n = \mathbb{C}[x, y]/I,$$

where $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ is the ring of polynomial functions on $\mathbb{C}^n \oplus \mathbb{C}^n$, the symmetric group acts "diagonally" (*i.e.*, permuting the x and y variables simultaneously), and the ideal $I = ((\mathbf{x}, \mathbf{y}) \cap \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n})$ is generated by all S_n-invariant polynomials without constant term. The S_n action respects the double grading

$$R_n = \bigoplus_{r,s} (R_n)_{r,s} \tag{0.1}$$

given by the x and y degrees.

A formula for the character of R_n as a doubly graded S_n module was conjectured in [6] and proved in [17]. The formula expresses the character in terms of Macdonald polynomials, as follows. Let F denote the Frobenius characteristic: the linear map from S_n characters to symmetric functions that sends the irreducible character χ^λ to the Schur function $s_\lambda(X)$. Encoding the graded character of R_n by means of its *Frobenius series*

$$\mathcal{F}_{R_n}(X; q, t) = \sum_{r,s} q^r t^s F\mathrm{char}(R_n)_{r,s}, \tag{0.2}$$

its value is given by the following theorem.

Macdonald Polynomials

- Macdonald [22] : $P_\lambda[X;q,t]$, and the integral form

$$J_\mu[X;q,t] = h_\mu(q,t)P_\mu[X;q,t] = \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q,t)s_\lambda[X(1-t)],$$

  where $h_\mu(q,t) = \prod_{c\in\mu}(1-q^{a(c)}t^{l(c)+1})$.

- $$H_\mu[X;q,t] := J_\mu\left[\frac{X}{1-t};q,t\right] = \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q,t)s_\lambda[X].$$

- Modified Macdonald polynomials

$$\tilde{H}_\mu[X;q,t] := t^{n(\mu)}H_\mu[X;q,1/t] = \sum_{\lambda\vdash|\mu|} \tilde{K}_{\lambda\mu}(q,t)s_\lambda[X],$$

  where $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})$.

Theorem ([17])

Let ∇ be the linear operator defined in terms of the modified Macdonald symmetric functions $\tilde{H}_\mu(X;q,t)$ by

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu, \tag{0.3}$$

where μ is a partition of n, μ' is its conjugate and $n(\mu) = \sum_i (i-1)\mu_i$. Then we have

$$\mathcal{F}_{R_n}(X;q,t) = \nabla e_n[X], \tag{0.4}$$

where $e_n[X]$ is the nth elementary symmetric function.

Combinatorial Consequences

- Garsia, Haiman [6] : $\dim_{\mathbb{C}} R_n = (n+1)^{(n-1)}$,

$$\dim_{\mathbb{C}} R_n^{\epsilon} = C_n = \frac{1}{n+1}\binom{2n}{n},$$

  where R_n^{ϵ} is the subspace of S_n-antisymmetric elements, C_n is the n^{th} *Catalan number*.

- Garsia, Haglund [4, 5] :

$$C_n(q,t) = \sum_{\pi \in L_{n,n}^{+}} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)} = < \nabla e_n, e_n >,$$

  where $C_n(q,t)$ is a q,t-analog of the Catalan number satisfying $C_n(1,1) = C_n$.

Nabla Operator

- ∇ was introduced by F. Bergeron, Garsia, Haiman and Tesler [2].

Table 1 Summary of research on the combinatorics of the nabla operator

Algebraic object	Combinatorial model	Conjectured by	Proved by
$\langle\nabla(e_n), s_{1^n}\rangle$	q,t-Dyck paths	Haglund [11] (cf. [10])	Garsia and Haglund [8, 9]
$\langle\nabla(e_n), h_{1^n}\rangle$	q,t-parking functions	Haglund, Loehr [14, 22]	Open
$\langle\nabla(e_n), e_d h_{n-d}\rangle$	q,t-Schröder paths	Egge, et al [7]	Haglund [13]
$\langle\nabla^m(e_n), s_{1^n}\rangle (m > 1)$	m-Dyck paths	Loehr [21]	Open
$\langle\nabla^m(e_n), h_{1^n}\rangle (m > 1)$	Labeled m-Dyck paths	Loehr and Remmel [24]	Open
$\langle\nabla^m(e_n), h_\mu\rangle (m \geq 1)$	Labeled m-Dyck paths	Haglund et al. [17]	Open
$\langle\nabla(p_n), s_{1^n}\rangle$	q,t-square paths	Loehr and Warrington [25]	Can and Loehr [5]
$\langle\nabla(p_n), h_\mu\rangle$	Labeled square paths	Loehr and Warrington [25]	Open
$\langle\nabla_{q=1}(s_{\lambda/\nu}), s_\mu\rangle$	Digraphs	Lenart [19]	Lenart [19]
$\langle\tilde{H}_\mu, h_\mu\rangle$	Fillings of $\mathcal{F}(\mu)$	Haglund [12]	Haglund et al. [15, 16]

(table from [20]).

Conjectures

- Haglund, Loehr [11] : the Hilbert series of R_n is

$$\mathcal{H}_n(q,t) =< \nabla e_n, e_1^n >= \sum_{r,s} q^r t^s \dim(R_n)_{r,s}$$
$$= \sum_{p \in \mathcal{PF}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}.$$

- Haiman, Haglund, Loehr, Remmel, Uylanov [9] :

$$\langle \nabla e_n[X], h_\lambda[X] e_\mu[X] \rangle = \sum_{f \in \mathcal{PF}_\lambda} t^{area(f)} q^{dinv(f)}.$$

- Loehr, Warrington [20] :

$$< \nabla(s_\lambda, s_{1^n} >= \mathrm{sgn}(\lambda) \sum_{G \in \mathrm{NDP}_\lambda} t^{\mathrm{area}(G)} q^{\mathrm{dinv}(G)}.$$

Plethysm

; simplifies the notation for compositions of power sum functions and symmetric functions.

Definition

Let $E = E(t_1, t_2, \dots)$ be a formal Laurent series with rational coefficients in $t_1, t_2, \dots$. We define the **plethystic substitution** $p_k[E]$ by replacing each t_i in E by t_i^k, i.e.,

$$p_k[E] := E(t_1^k, t_2^k, \dots).$$

For any arbitrary symmetric function f, the plethystic substitution of E into f, denoted by $f[E]$, is obtained by extending the specialization $p_k \mapsto p_k[E]$ to f.

There is a special symbol ϵ which will represent a negative one ;

$$f[\epsilon X] = (-1)^d f[X], \qquad f[-\epsilon X] = \omega(f[X]).$$

(definition ct'd)

Define

$$\Omega[X] = e^{\sum_{k\geq 1} \frac{p_k[X]}{k}}.$$

Then for $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$, we have the following identities

$$\Omega[X+Y] = \Omega[X]\Omega[Y],$$
$$\Omega[X-Y] = \Omega[X]\Omega[Y],$$
$$\Omega[X] = \prod_i \frac{1}{1-x_i} = \sum_{n\geq 0} h_n[X],$$
$$\Omega[-X] = \prod_i (1-x_i) = \sum_{n\geq 0} (-1)^n e_n[X].$$

Jing's Symmetric Function Operators

Jing [18] defined the following symmetric function operators

$$\mathbb{H}_m P[X] = P\left[X - \frac{1-q}{z}\right]\Omega[zX]\Big|_{z^m} \tag{0.5}$$

which have the property that

$$\mathbb{H}_{\mu_1}\mathbb{H}_{\mu_2}\cdots\mathbb{H}_{\mu_{l(\mu)}}(1) = q^{n(\mu)}\tilde{H}_\mu[X;0,1/q], \tag{0.6}$$

for μ a partition.

Symmetric Function Operators

In [12], Haglund, Morse and Zabrocki introduced two new operators on symmetric functions. In plethysistic notation, for any symmetric function $P[X]$,

$$\mathbb{B}_m P[X] = P\left[X + \epsilon\frac{(1-q)}{z}\right]\Omega[-\epsilon z X]\Big|_{z^m}, \tag{0.7}$$

$$\mathbb{C}_m P[X] = \left(-\frac{1}{q}\right)^m P\left[X - \frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^m}, \tag{0.8}$$

where $X = x_1 + x_2 + \cdots$ and $\Omega[zX] = \prod_i \frac{1}{1-zx_i}$. and $\Omega[-\epsilon z X] = \prod_i (1+zx_i)$.
These operators are related to the Jing's operators $\mathbb{H}_m$ by

$$\mathbb{B}_m = \omega \mathbb{H}_m \omega, \qquad \mathbb{C}_m = (-1/q)^{m-1}\mathbb{H}_m^{q\to 1/q}$$

and so

$$\mathbb{C}_m = (-1/q)^{m-1}\omega\mathbb{B}_m^{q\to 1/q}\omega.$$

(Symmetric Function Operators:ctn'd)

Define symmetric functions by

$$B_\alpha[X;q] = \mathbb{B}_{\alpha_1}\mathbb{B}_{\alpha_2}\cdots\mathbb{B}\alpha_{l(\alpha)}(1)$$

and

$$C_\alpha[X;q] = \mathbb{C}_{\alpha_1}\mathbb{C}_{\alpha_2}\cdots\mathbb{C}\alpha_{l(\alpha)}(1),$$

for any composition $\alpha = (\alpha_1, \ldots, \alpha_{l(\alpha)})$. Then for a partition μ,

$$B_\mu[X;q] = q^{n(\mu)}\omega\tilde{H}_\mu[X;0,1/q], \quad (0.9)$$

$$C_\mu[X;q] = q^{-n(\mu)}(-1/q)^{|\mu|-l(\mu)}\tilde{H}_\mu[X;0,q]. \quad (0.10)$$

Hence,

$$B_\mu[X;q] = \omega\sum_\lambda K_{\lambda\mu}(q)s_\lambda[X] = \sum_\lambda K_{\lambda\mu}(q)s_{\lambda'}[X], \quad (0.11)$$

$$C_\mu[X;q] = (-1)^{|\mu|-l(\mu)}(1/q)^{|\mu|-l(\mu)+n(\mu)}\sum_\lambda \tilde{K}_{\lambda\mu}(q)s_\lambda[X]. \quad (0.12)$$

$\mathbb{B}_m$ applied to Schur functions

Theorem

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n and $m > 0$. Then

$$\mathbb{B}_m s_\lambda[X] = \sum_{\gamma \vdash (n+m)} b_{m,\lambda,\gamma} s_\gamma[X],$$

where

(a) $b_{m,\lambda,\gamma} = 0$ *unless* γ *satisfies either*

 (1) γ *arises by adding a vertical strips of size m on the outside of λ, or*

 (2) γ *arises from λ by first removing a nonempty broken rim hook $H = \lambda/\mu$ of λ from λ which starts in the first column to get a partition μ and then adding a vertical strip V of size $m + |\lambda/\mu|$ on the outside of μ to obtain γ so that the all the cells of H lie strictly above all the cells of V.*

(b) $b_{m,\lambda,\gamma} = q^{l(\gamma)-m}$ if γ satisfies the above condition (1), and

(c) if γ satisfies above condition (2) and u is the height of the lowest red cell that was removed from λ, v is the height of the highest row which contains a cell in γ which is to right of the lowest red cell removed from λ, R is the set of red cells in that were removed from λ that are not in any connecting rim hooks, and p is the number of rim hooks that make of the broken rim hook λ/γ, then

$$b_{m,\lambda,\gamma} = (-1)^{p-1}(q^{u-v} - 1)q^{v-m-|\lambda/\gamma|} \prod_{c \in R} w_B(c),$$

where

$$w_B(s) = \begin{cases} -1 & \text{if } s \text{ has a red cell to its right} \\ q & \text{if } s \text{ has a red cell below it, and} \\ q-1 & \text{if } s \text{ is the lowest cell of a rim hook.} \end{cases}$$

$\mathbb{C}_m$ applied to Schur functions

Theorem

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_k)$ *is a partition of* n *and* $m > 0$. *Then*

$$\mathbb{C}_m s_\lambda[X] = \sum_{\gamma \vdash (n+m)} c_{m,\lambda,\gamma} s_\gamma[X],$$

where

(a) $c_{m,\lambda,\gamma} = 0$ *unless* γ *satisfies either*

  (1) γ *arises by adding a horizontal strip of size* m *on the outside of* λ, *or*

  (2) γ *arises from* λ *by first removing a nonempty broken rim hook* $H = \lambda/\mu$ *of* λ *from* λ *which starts in the first row to get a partition* μ *and then adding a horizontal strip* S *of size* $m + |\lambda/\mu|$ *on the outside of* μ *to obtain* γ *so that all the cells of* H *lie strictly below all the cells of* S.

(b) $c_{m,\lambda,\gamma}(q) = (1/q)^{\gamma_1 - m}$ *if* γ *satisfies the above condition* (1), *and*

(c) if γ satisfies condition (2) above, and u is the column of the highest red cell that was removed from λ, v is the column of the first corner cell to the left of the highest red cell of λ, R is the set of red cells that were removed from λ which are not in any connecting rim hooks, and p is the number of rim hooks that make the broken rim hook λ/γ, then

$$c_{m,\lambda,\gamma}(q) = \left(-\frac{1}{q}\right)^{m-1} (-1)^{p-1} \frac{q^{u-v}-1}{q^{u-m-|\lambda/\gamma|(q-1)}} \prod_{s \in R} w_C(s)$$

$$= (-1)^{m+p} \frac{q^{u-v}-1}{q^{u-|\lambda/\gamma|-2}(q-1)} \prod_{s \in R} w_C(s),$$

where

$$w_C(s) = \begin{cases} 1/q & \text{if } s \text{ has a red cell to its right,} \\ -1 & \text{if } s \text{ has a red cell below it, and} \\ \frac{1-q}{q} & \text{if } s \text{ is the lowest cell of a rim hook.} \end{cases}$$

Charge Construction

Note that

$$B_\mu[X;q] = q^{n(\mu)}\omega\tilde{H}_\mu[X;0,1/q] = \omega\sum_\lambda K_{\lambda\mu}(q)s_\lambda[X]$$
$$= \sum_\lambda K_{\lambda\mu}(q)s_{\lambda'}[X]$$

where $K_{\lambda\mu}(q) = \sum_{T\in SSYT(\lambda,\mu)} q^{ch(T)}$. Considering the definition of $B_\mu[X;q]$

$$B_\mu[X;q] = \mathbb{B}_{\mu_1}\mathbb{B}_{\mu_2}\cdots\mathbb{B}_{\mu_{l(\mu)}}(1),$$

and the Schur expansion of $\mathbb{B}_m(s_\lambda)$, we could give a different combinatorial construction of the charge statistics.

Example

For $\mu = (2,1,1)$,

$$B_{(2,1,1)}[X;q] = \sum_{\lambda \vdash 4} \left(\sum_{T \in SSYT(\lambda', \mu)} q^{ch(T)} \right) s_\lambda[X]$$
$$= q^3 s_{1^4}[X] + (q+q^2) s_{211}[X] + q s_{22}[X] + s_{31}[X].$$

On the other hand, we can construct $B_{(2,1,1)}[X;q]$ by adding a vertical strip of size $\mu_3 = 1$ to 1, and add vertical strips of size $\mu_2 = 1$ and $\mu_1 = 2$ to the results, iteratively. We can see the procedure in the diagram given in the following figure.

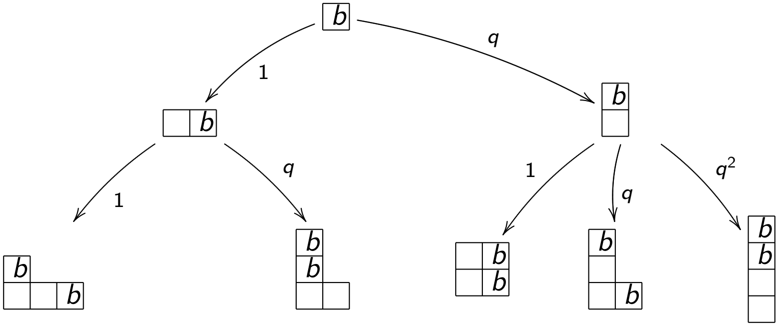


Figure: $B_{(2,1,1)}[X;q]$ diagram.

Recursion of Kostka-Foulkes polynomials

Let $\nu = (\mu_2, \mu_3, \ldots, \mu_{l(\mu)})$. Then

$$\begin{aligned} B_\mu[X;q] &= \mathbb{B}_{\mu_1}\mathbb{B}_{\mu_2}\cdots\mathbb{B}_{\mu_{l(\mu)}}(1) \\ &= \mathbb{B}_{\mu_1}(B_\nu[X;q]). \end{aligned}$$

Thus we get

$$\begin{aligned} B_\mu[X;q] &= \sum_{\lambda \vdash n}\left(\sum_{T\in SSYT(\lambda',\mu)} q^{ch(T)}\right) s_\lambda[X] \\ &= \mathbb{B}_{\mu_1}(B_\nu[X;q]) = \mathbb{B}_{\mu_1}\left(\sum_{\eta\vdash(n-\mu_1)}\left(\sum_{T\in SSYT(\eta',\nu)} q^{ch(T)}\right) s_\eta[X]\right) \\ &= \sum_{\eta\vdash(n-\mu_1)}\left(\sum_{T\in SSYT(\eta',\nu)} q^{ch(T)}\right)\mathbb{B}_{\mu_1}(s_\eta[X]). \end{aligned}$$

By applying the Schur expansion of $\mathbb{B}_{\mu_1}(s_\eta[X])$, we can get the following recursion of Kostka-Foulkes polynomials.

$$K_{\lambda'\mu}(q) = \sum_{T\in SSYT(\lambda',\mu)} q^{ch(T)}$$

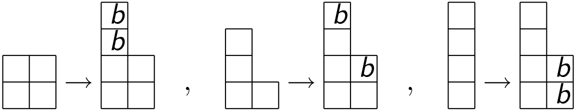
$$= \sum_{\substack{\eta\vdash(n-\mu_1) \text{ such that } \exists\alpha\subseteq\eta,\lambda/\alpha \text{ is a} \\ \text{vertical strip of size } \mu_1+|\eta/\alpha|}} \left(\sum_{T\in SSYT(\eta',\nu)} q^{ch(T)} \right) q^{l(\lambda)-\mu_1-|\eta/\alpha|} w_B(\eta/\alpha)$$

$$= \sum_{\substack{\eta\vdash(n-\mu_1) \text{ such that } \exists\alpha\subseteq\eta,\lambda/\alpha \text{ is a} \\ \text{vertical strip of size } \mu_1+|\eta/\alpha|}} K_{\eta'\nu}(q) q^{l(\lambda)-\mu_1-|\eta/\alpha|} w_B(\eta/\alpha) \quad (0.13)$$

where $w_B(\eta/\alpha) = \prod_{s\in\eta/\alpha} w_{B,\eta/\alpha}(s)$ is the weight function of the removed rim hooks.

Example

Let $\mu = (2,1,1,1,1)$ and consider the coefficient of $s_{(2,2,1,1)}$, for $\lambda = (2,2,1,1)$. Note that there are three possible cases that we can get λ by adding a vertical strip :



The weight of the connecting rim hooks are uniformly q^2. Hence, for $\nu = (1,1,1,1)$, (0.13) gives

$$\sum_{T \in SSYT(\lambda',\mu)} q^{ch(T)} = q^2 \left(\sum_{T \in SSYT((2,2)',\nu)} q^{ch(T)} + \sum_{T \in SSYT((2,1,1)',\nu)} q^{ch(T)} + \sum_{T \in SSYT((1,1,1,1)',\nu)} q^{ch(T)} \right).$$

A tree in the following figure shows the process of adding vertical strips of size 1, 1, 1, 1 and 2 starting from the bottom leaves. The weights on the branches are the weights of the connecting rim hooks. Sum of all the charges of SSYT of shape λ' with weight μ would be the sum of products of weights on the branches connecting each bottom leaf to the root. From this tree, we get

$$\sum_{T \in \mathrm{SSYT}(\lambda',\mu)} q^{ch(T)} = q^5(q-1) + q^4 + q^5 + q^5 + q^6 + q^7 + q^8$$

$$= q^4 + q^5 + 2q^6 + q^7 + q^8.$$

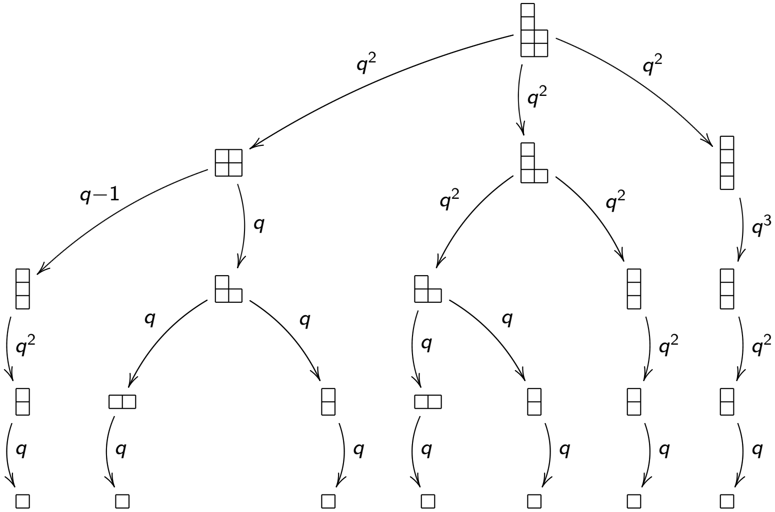


Figure: Charge tree to calculate $K_{\lambda'\mu}(q) = \sum_{T \in \mathrm{SSYT}(\lambda',\mu)} q^{ch(T)}$ for $\mu = (2,1,1,1,1)$ and $\lambda = (2,2,1,1)$

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