Lauren Williams

Pos. Bases for cluster algebras from surfaces

Late 1990's: Fomin & Zelevinsky trying to understand Lusztig's theory of total pos. & canon. basis in a "concrete" way. This led them to introduce cluster algebras, which have now been linked to: quiver reps, Poisson geom, Teichmuller theory, tropical geom, etc.

What is a cl. alg? It is a comm. algebra w/ some distinguished generators - cluster variables - which satisfy some very rigid combinatorial conditions.

Presentation: Usually when one encounters an algebra it is given by a set of generators + relations. In contrast, a cluster algebra is typically specified by a "seed" (which includes a cluster = some cluster variables) together w/ a procedure for generating the rest of the cluster algebra - the other cluster variables as well as some 3-term relations.

Def: (F & Z) A clust. alg. $\mathcal{A}$ is a certain subalgebra of $k(x_1, \dots, x_n)$, the field of rational functions over $\{x_1, \dots, x_n\}$. Generators are constructed by a series of exchange relations which in turn induce all relations satisfied by the generators.

Def: A seed for $\mathcal{A}$ is an initial cluster $\underline{x} = \{x_1, \dots, x_n\}$ and an $n \times n$ skew-symmetrizable integral matrix B ($d_i b_{ij} = -d_j b_{ji}$ for $d_i, d_j > 0$) (For simplicity, restricting to the coeff-free case)

Lauren Williams

From this seed, can <u>mutate</u> in each of n directions, obtaining n more seeds.

Columns of B encode the <u>exchange relations</u>:

For $k \in \{1,\dots,n\}$, $$x_k x_k' = \prod_{b_{ik}>0} x_i^{|b_{ik}|} + \prod_{b_{ik}<0} x_i^{|b_{ik}|}$$

This defines a new cluster variable x_k'.

For $k \in \{1,\dots,n\}$, $\exists$ another seed for $\mathcal{A}$ consisting of the clusters $\{x_1,\dots,\hat{x}_k,\dots,x_n\} \cup \{x_k'\}$ and matrix $\mu_k(B)$, where

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k=i \text{ or } k=j \\ b_{ij} & \text{if } b_{ik}b_{kj} \le 0 \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

<u>Rk:</u> $\mu_k(B)$ is again skew-symmetrizable and $\mu_k^2 = \mathrm{id}$.

Start from the initial seed & apply all possible sequences of mutations: this produces the set of all cluster variables (possibly infinite).

<u>Def:</u> The cluster algebra $\mathcal{A}(B)$ is the subalgebra of $k(x_1,\dots,x_n)$ generated by all cluster variables.

<u>Note:</u> Every cluster variable can be expressed as rational expression in the initial cluster variables (or the variables of an arbitrary cluster)

Lauren Williams

Laurent phenomenon (F-Z): This rational expression is actually a Laurent polynomial.

Positivity Conj: All coefficients are positive.

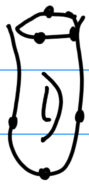
Note: Laurent phenomenon is true for any clust alg (FZ). The positivity conj. is expected to be true " " proved in many special cases.

Open problems about cluster algebras:
1. Most famous one is Pos. Conj.
2. Another important problem is to construct a basis for each clust alg w/ good positivity properties.
(Motivation: analogy w/ Lusztig's dual canonical bases)

Interesting class of cluster algebras is the class coming from surfaces:
- is large class of clust alg's
- related to Teichmuller theory

Gekhtman-Shapiro-Vainshtein
Fock-Goncharov
Fomin-Shapiro-(Dylan) Thurston: can associate a clust. alg $\mathcal{A}(S,M)$ to any bordered surface w/ marked points (S,M) $\rightarrow$ by assoc. an exchange matrix B to a triangulation
(Exchange matrix then determines $\mathcal{A}$ (up to coeffs))

Lauren Williams



Def: Let S be a connected oriented 2-dim. Riemann surface w/ (possibly empty) boundary. Fix finite set M of <u>marked points</u> in S. Marked points in the interior are <u>punctures</u>.

Def: An <u>arc</u> γ in (S, M) is a curve in S (considered up to isotopy) s.t.

- the endpoints of γ are in M.
- γ doesn't intersect itself (except maybe at endpoints)
- γ is not contractible into M or onto bdy of S

Def: Two arcs are <u>compatible</u> if they don't intersect in $\mathrm{int}(S)$.

Def: An <u>ideal triangulation</u> is a max'l collection of distinct pairwise compatible arcs.

Teichmuller: vertices at marked pts, arcs are geodesics of ∞ length

Suppose ideal triangulations of (S, M) have n arcs.

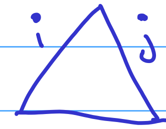
Can associate a matrix $B(T)$ to an ideal triangulation T.

Easiest to define when no self-folded Δ's. Then

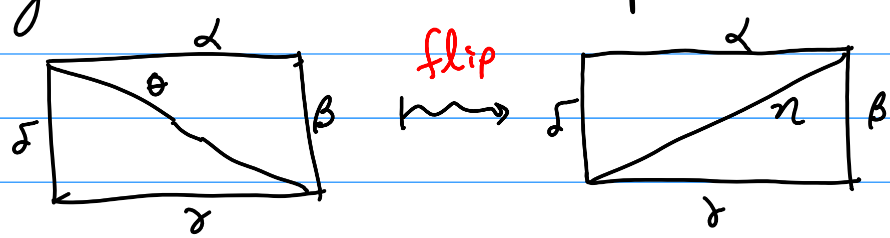
$B(T) = (b_{ij})$ where:

Lauren Williams

$$b_{ij} = \#\left\{\begin{array}{l}\text{triangles w/ sides } i \,\&\, j\text{, w/}\\ j \text{ following } i \text{ in clockwise order}\end{array}\right\} - \#\left\{\begin{array}{l}\text{triangles w/ sides } i \,\&\, j\text{, w/}\\ j \text{ following } i \text{ in counterclockwise order}\end{array}\right\}$$

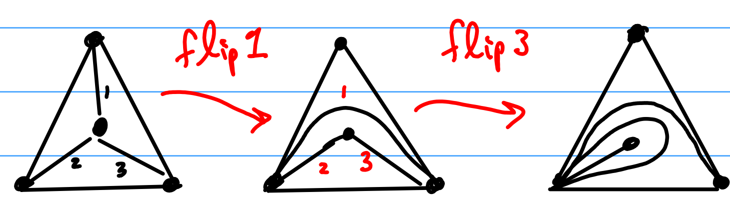


Giving an exchange matrix defines a cluster algebra.
Roughly speaking
cluster variables x_γ $\longleftrightarrow$ arcs γ
clusters $\longleftrightarrow$ triangulations
exchange relation $\longleftrightarrow$ flips



Exchange relation: $x_\theta x_\eta = x_\alpha x_\gamma + x_\beta x_\delta$.

(I'm hiding under the rug the problem that an arc is a self-folded Δ can't be flipped



)

This algebraic structure had appeared before in Penner's work (~1980) on Decorated Teichmuller spaces

Lauren Williams

Theorem (Musiker–Schiffler–W.) '09 The positivity conjecture holds for any cluster algebra coming from a surface. (includes types A, D, $\tilde{A}$, $\tilde{D}$ ...)

We proved the main theorem by providing a combinatorial formula for all clust. variables.

Given any ideal triangulation T of (S, M) and any arc γ in S, need to give an expression for x_γ in terms of the variables x_β $(\beta \in T)$.

Theorem (Musiker–Schiffler–W.) Fix a bordered surface (S, M) & an ideal triangulation T w/ edges $(T_1, \dots, T_n)$.
Let γ be any arc in S. Then there is a graph $G_{\gamma, T}$ s.t.

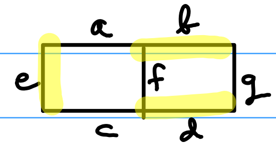
$$x_\gamma = \frac{\sum_P x(P)}{x_{T_1}^{e_1(T,\gamma)} \cdots x_{T_n}^{e_n(T,\gamma)}} \quad \text{where}$$

P ranges over perfect matchings of $G_{\gamma, T}$
$x(P)$ is the weight of P,
$e_i(T, \gamma)$ is the crossing number of T_i and γ.

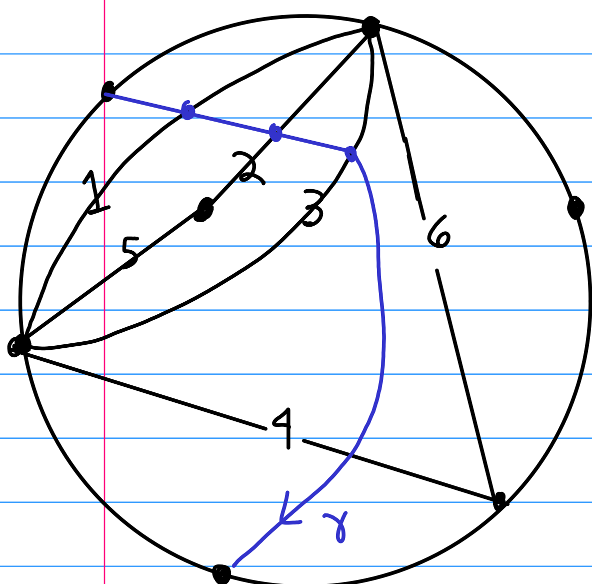
For experts: we have more general formula that includes coeff's.
Also: there are diff formulas for tagged arcs

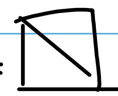
Lauren Williams

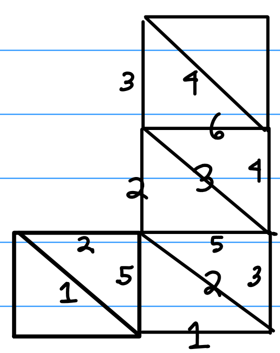
Def: Given a simple undirected graph $G=(V,E)$, a **perfect matching** P is a subset of E s.t. each vertex is incident to exactly one $e \in P$. The weight $x(P)$ = product of all edge variables

$P =$ 

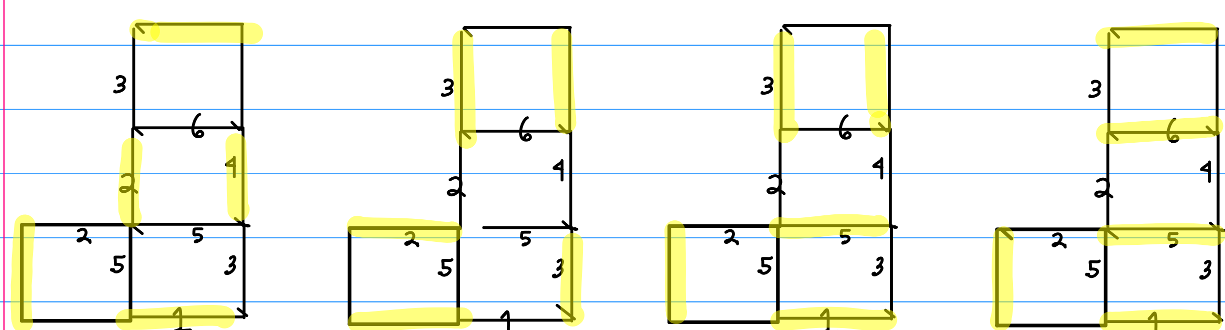
$x(P) = x_b x_d x_e$



Associate a tile $G_j =$  to each intersection point p_j of γ w/ T, whose 2 triangles are labeled according to the labels of the Δ's that p_j sees in T. Glue G_j & G_{j+1} along the side whose label is not crossed by p_j or p_{j+1}, s.t. $\mathrm{rel}(G_j, T) \neq \mathrm{rel}(G_{j+1}, T)$.

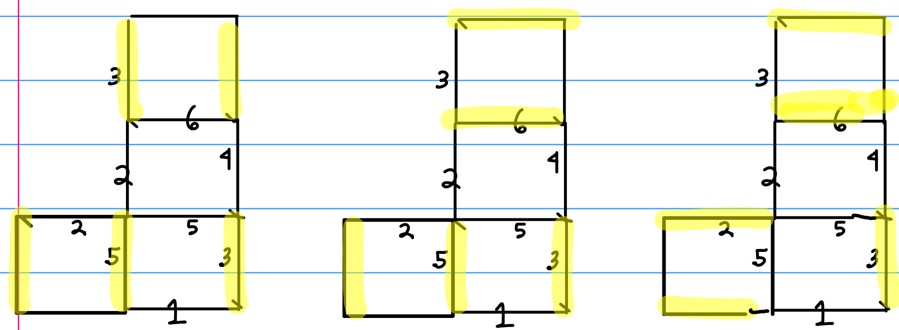


Now remove diagonals to get $G_{\gamma,T}$.



$x_1 x_2 x_4 \quad + \quad x_2 x_3^2 \quad + \quad \cdots$

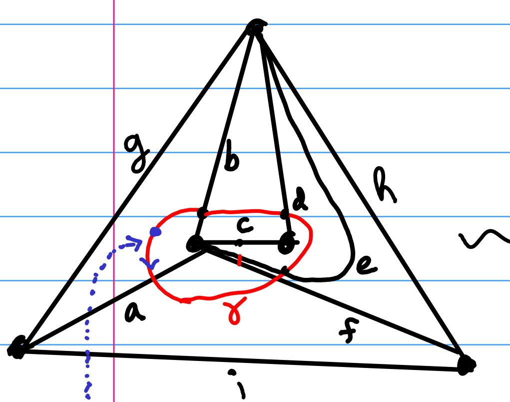
Lauren Williams

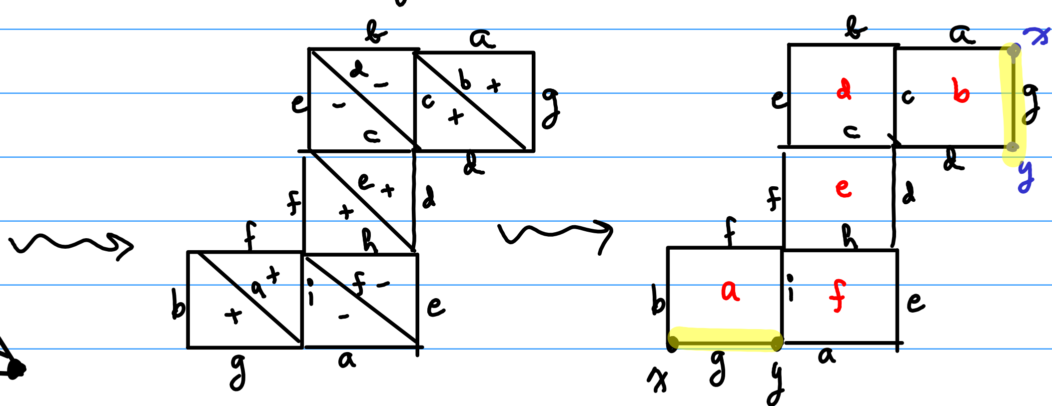


So $X_\gamma = \dfrac{x_1x_2x_4 + x_2x_3^2 + \cdots}{x_1x_2x_3x_4}$

Note: In this way we can associate a Laurent poly to any curve C in surface connecting 2 marked points, even if C has self-intersections.

We can also associate a Laurent poly to a closed loop γ — for this, we take $G_{\gamma,T}$ & make some identifications.





Choose arbitrary start point, & orientation

edge of the triangle where we started, which is not crossed by loop z

Lauren Williams

Let $G_{\gamma,T}$ be the graph (on annulus or Möbius strip) that we get after the identification above. Now we define

Def: (MSW)

$$X_\gamma = \frac{\sum_P x(P)}{x_{T_1}^{e_1(T,\gamma)} \cdots x_{T_n}^{e_n(T,\gamma)}}$$ where

P ranges over ~~good~~ matchings of $G_{\gamma,T}$

$x(P)$ is the weight of P,

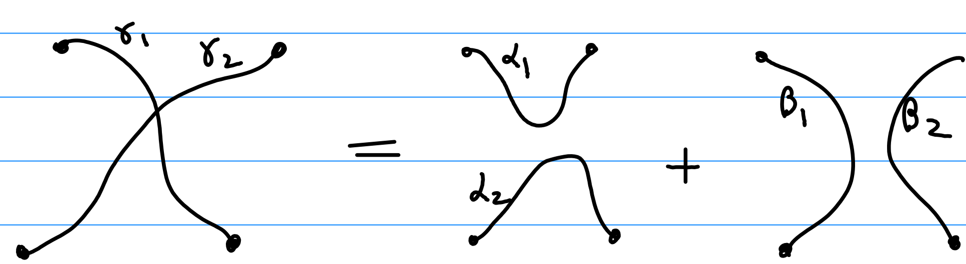
$e_i(T,\gamma)$ is the crossing number of T_i and γ.

We now have way to associate a cluster alg element $x_\gamma \in A(S,M)$ to any curve or closed loop γ in (S,M).

How do we multiply such elements?

Skein Relations:

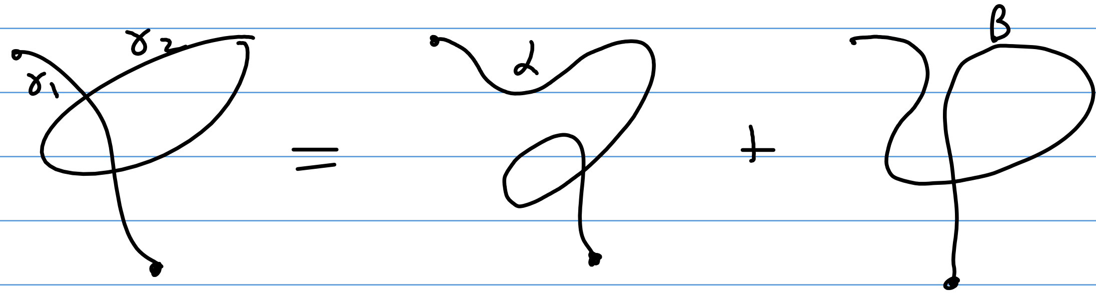
(1)



$$x_{\gamma_1} x_{\gamma_2} = x_{\alpha_1} x_{\alpha_2} + x_{\beta_1} x_{\beta_2}$$

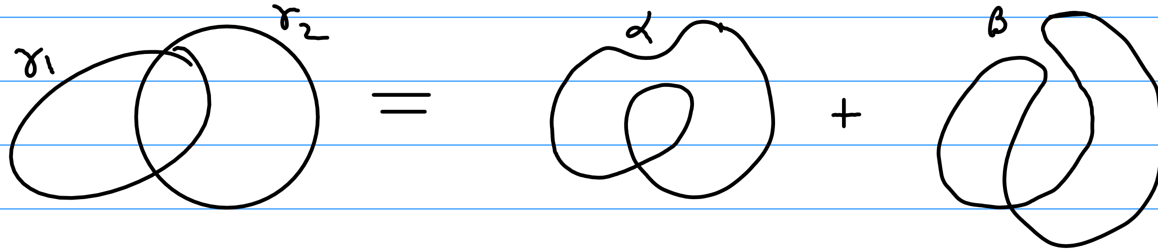
Lauren Williams

(2)



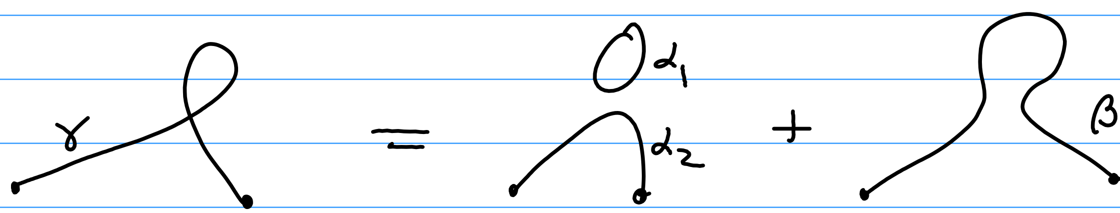
$$x_{\gamma_1} x_{\gamma_2} = x_\alpha + x_\beta$$

(3)



$$x_{\gamma_1}\ x_{\gamma_2} = x_\alpha + \quad + x_\beta$$

(4)



$$x_\gamma = x_{\alpha_1} x_{\alpha_2} + x_\beta$$

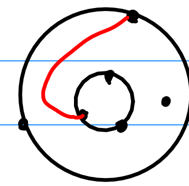
Work in preparation (MSW):
Constructing a positive basis for $A(S,M)$, using monomials obtained from closed loops & arcs in (S,M).

Aside on Teichmuller space

Def: Given (S,M) surface w/ marked points, the Teichmuller space $\mathcal{T}(S,M)$ is the space of metrics on (S,M) which:

- are hyperbolic (constant curvature -1)
- have geodesic boundary at bdy of S
- have cusps at points in M
  (go off to ∞ while area remains bounded)

Considered up to diffeomorphisms homotopic to identity.
$\mathrm{Diff}_0(S,M)$



Want to associate "lengths" to arcs; need to renormalize.

point at ∞

Def: A horocycle (at marked ideal point p) = set of points "equidistant to p."

Def: The decorated Teichmuller space $\widetilde{\mathcal{T}}(S,M)$ is

- a point in $\mathcal{T}(S,M)$
- a choice of horocycle around each cusp from M.

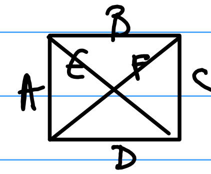
Def (Penner, '88) Given arc α on (S,M) and $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, the length of α w/ respect to Σ is

$\ell_\Sigma(\alpha) =$ length on geodesic representation of α between intersections w/ horocycles around ends

The λ-length of α is $\lambda_\Sigma(\alpha) = \exp\left(\frac{\ell_\Sigma(\alpha)}{2}\right)$.

Lemma: In ideal quadrilateral,

$\lambda_\Sigma(E)\,\lambda_\Sigma(F) = \lambda_\Sigma(A)\,\lambda_\Sigma(C) + \lambda_\Sigma(B)\,\lambda_\Sigma(D).$



Penner '88: Each triangulation Γ of (S,M) gives system of coordinates on (S,M);

Rk: changing the triangulation via a flip is a cluster transformation.

$\therefore$ get (partial) realization of cl. alg. $\mathcal{A}(S,M)$ inside functions on $\tilde{T}(S,M)$:

Base field $\rightsquigarrow$ suitable functions on $\tilde{T}(S,M)$

seeds $\rightsquigarrow$ triangulations

clust variables $\rightsquigarrow$ λ-lengths $\lambda_\Sigma(\alpha)$ $\left(\begin{array}{l}\text{as functions}\\ \text{on } \tilde{T}(S,M)\end{array}\right)$

mutation $\rightsquigarrow$ edge flip

Note: There is a related space of bdd measured laminations. Each triangulation Γ gives syst. of coord's. Coord transformations are tropical cluster transformations.