AN INTRODUCTION TO QUASISYMMETRIC FUNCTIONS

BASICS

DEF.

The Hopf algebra

$$QSym := \bigoplus_{n \geq 0} QSym^n \subset \mathbb{Q}[x_1, x_2, \ldots].$$

where

$$QSym^n = \mathrm{span}_{\mathbb{Q}} \{ M_\alpha \mid \alpha = \alpha_1 \ldots \alpha_k, \ \sum_{i=1}^k \alpha_i = n \}$$

and $M_0 = 1$

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$ monomial basis

Ex. $M_{21} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \ldots$

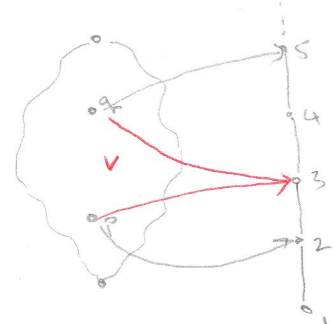
DISCRETE GEOMETRY (GESSEL 84).

(P, γ) a (totally ordered) labelled poset then a (P,γ)-partition is a map f:

i) $f(p) \leq f(q)$

ii) $f(p) = f(q)$ implies label $\gamma(p) < \gamma(q)$

$\mathcal{O}(P,\gamma)$ = set of all (P,γ)-partitions



DEF.

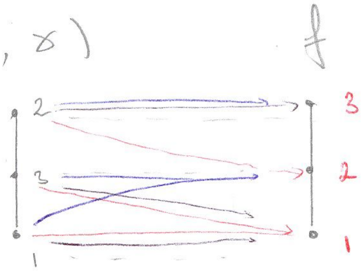
If f a (P,γ)-partition then

$$x^f = \prod_{p \in P} x_{f(p)}.$$

and weight enumerator of (P,γ) is

$$F(P,\gamma) = \sum x^f = \sum F_\delta$$

Ex// chain (ω, γ) $\quad f$



x^f

$x_1^2 x_2$

$x_1^2 x_3$

$x_2^2 x_3$

$x_1 x_2 x_3$

$F(\omega, \gamma) = M_{21} + M_{111}$

FACT: $F(\omega, \gamma)$ is quasisymmetric

DEF//

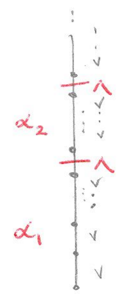
$$F_\alpha = \sum_{\alpha \succcurlyeq \beta} M_\beta \qquad \text{fundamental basis}$$

where $\alpha \succcurlyeq \beta \Rightarrow \underbrace{\beta_1 + \cdots + \beta_{i_1}}_{\alpha_1} \; \underbrace{\beta_{i_1+1} + \cdots + \beta_{i_1+i_2}}_{\alpha_2} \; \cdots \; \underbrace{\beta_m + \cdots + \beta_\ell}_{\alpha_k}$

Ex// $F_{21} = M_{21} + M_{111}$ NOTE:

cut at $\gamma(p) > \gamma(q)$
then component sizes $= 21$

FACT: If (P, γ) a chain and



then $F(P, \gamma) = F_\alpha$

FACT: $\mathcal{O}(P, \gamma) = \dot{\bigcup} \, \mathcal{O}(\omega, \gamma)$ ω linear extension of P

1, take lin' ext of P
2, note desc tunneled
3, $\rightsquigarrow$ $= \sum F_\delta$.

HOPF ALGEBRAS (Ehrenborg 96, Gelfand et al 95, Malvenuto-Reutenauer 95)

$$\begin{tikzcd} \mathrm{NSym} \arrow[rr, leftrightarrow, bend left, "*\ \text{dual}"] \arrow[dr, "\substack{\text{surjection} \\ x\text{-forgetful}}"'] & & \mathrm{QSym} \\ & \mathrm{Sym} \arrow[ur, "\substack{\text{subalgebra:} \\ m_\lambda = \sum_{\tilde{\alpha} = \lambda} M_\alpha}"'] \arrow[loop below, "*"] & \end{tikzcd}$$

$$\mathrm{NSym} = \bigoplus_{n \geq 0} \mathrm{NSym}^n \subset \mathbb{Q}\langle y_1, y_2, \ldots \rangle$$

BASES: NSym$_n$ spanned by

$$\eta_\alpha = \sum_{\alpha \succcurlyeq \beta} (-1)^{k - \Sigma \alpha_i} y_{\beta_1} y_{\beta_2} \cdots y_{\beta_k}$$

complete basis

$$R_\alpha = \sum_{\beta \succcurlyeq \alpha} (-1)^{\#\text{parts } \alpha - \#\text{parts } \beta} \eta_\beta$$

ribbon basis

FACT $\phi : \mathrm{NSym}^* \xrightarrow{\sim} \mathrm{QSym}$

$$\eta^*_\alpha \longmapsto M_\alpha$$

$$R^*_\alpha \longmapsto F_\alpha$$

QUESTION: What about Schur functions? (Bessenrodt, Haglund, Luoto, Mason, vW)

1. As $m_\lambda = \sum_{\tilde\alpha = \lambda} M_\alpha$ does $s_\lambda = \sum_{\tilde\alpha = \lambda} \mathcal{S}_\alpha$?

2. As $\chi(\eta_\alpha) = h_{\tilde\alpha}$ does $\chi(\mathcal{S}_\alpha) = s_{\tilde\alpha}$?

3. $\phi(\mathcal{S}^*_\alpha) = \mathcal{S}_\alpha$? YES!

DEF. Poset $\mathcal{L}_c$ on compositions

1. Add 1 to start: $121 < 1121$

2. 1 to left most part of size: $121 < 221$, $121 < 131$

saturated chains in $\mathcal{L}_c$ $\leftrightarrow$ standard skew composition tableaux (SCT) shape $\beta /\!/ \alpha$

Ex. $23 < 123 < 133 < 143$ $\leftrightarrow$

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3
•  •  2  1
•  •  •
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QUASISYMMETRIC (SKEW) SCHUR FUNCTIONS

$$\mathcal{S}_{\gamma/\!/\beta} = \sum C_{\gamma/\!/\beta\,\delta}\, F_\delta$$

$C_{\gamma/\!/\beta\,\delta}$ = # SCT shape $\gamma/\!/\beta$ where δ is composition $\leftrightarrow$ set of all i st $i+1$ is weakly right.

Ex//

$$\mathcal{S}_{23} = \mathcal{S}_{23/\!/0} = F_{122} + F_{23}$$

from

3	1			2	1	
5	4	2		5	4	3

FACT: like Schur functions - just switch partition to composition

THE KING