Patterns and Permutations

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Fix $w \in S_n$ and $p \in S_k$, with $k \leq n$.

Then w has a p-pattern if there are $i_1 < \cdots < i_k$ such that $w(i_1) \cdots w(i_k)$ and $p(1) \cdots p(k)$ are in the same relative order.

Otherwise, w avoids p.

Example. Let $w = 74\mathbf{1362}5$, $p = 1243$, and $q = 1234$.
1365 is an occurrence of p in w, and w avoids q.

Pattern avoidance: enumeration and characterization

$\boldsymbol{S_n(p)}$ = permutations in S_n that avoid the pattern p.

p and q are Wilf equivalent, if $|S_n(p)| = |S_n(q)|$ for all n.

Theorem. All six permutations in S_3 are Wilf equivalent.

Theorem. $|S_n(123)| = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem. The stack-sortable permutations in S_n are $S_n(231)$.

Theorem. The fully commutative elements in S_n are $S_n(321)$.

These have no long braid moves in their reduced decompositions.

Permutation notation

To study patterns, we write a permutation in one-line notation.

We can also write a permutation as a product of simple reflections, equivalently giving a reduced word.

Example. 4213 equals $s_1 s_3 s_2 s_1 = s_3 s_1 s_2 s_1 = s_3 s_2 s_1 s_2$, so it has three reduced words: $R(4213) = \{1321, 3121, 3212\}$.

These two notations look very different, but we can translate between them!

Vexillary permutations

A permutation is vexillary if it is 2143-avoiding.

Example. 3641572 is vexillary, but **3**64**175**2 is not.

There are many equivalent "classical" definitions of vexillary.

There is also a new vexillary characterization ...

Theorem. [T] p is vexillary iff for every w with a p-pattern, $\exists\ \boldsymbol{j} \in R(w)$ "containing" some $\boldsymbol{i} \in R(p)$.

More pattern-related results

Let $\boldsymbol{C(w)}$ be the set of equivalence classes of $R(w)$, where two reduced words are equivalent if they differ by short braid relations (commuting elements).

Theorem. [T] If w contains a p-pattern, then $|C(w)| \geq |C(p)|$.

Theorem. [T] A regular $2n$-gon of side-length 1 can be tiled by centrally symmetric $2k$-gons of side-length 1 iff $k \in \{2, n\}$.

Patterns and the Bruhat order

The Bruhat order gives a partial ordering to a Coxeter group. The principal order ideal of w is $B(w) = \{v \leq w\}$.

Theorem. [T] For $p \in S_k$ and $n > k \geq 3$, the set $S_n(p)$ is never a nonempty order ideal.

Theorem. [T] For $p \in S_k$ and $q \in S_l$, and $n \geq k, l \geq 3$, the set $S_n(p, q)$ is a nonempty order ideal only for:

$$S_n(321, 3412),$$
$$S_n(321, 231) = B(n12 \cdots (n-1)), \text{ and}$$
$$S_n(321, 312) = B(23 \cdots n1).$$

Boolean elements in the Bruhat order

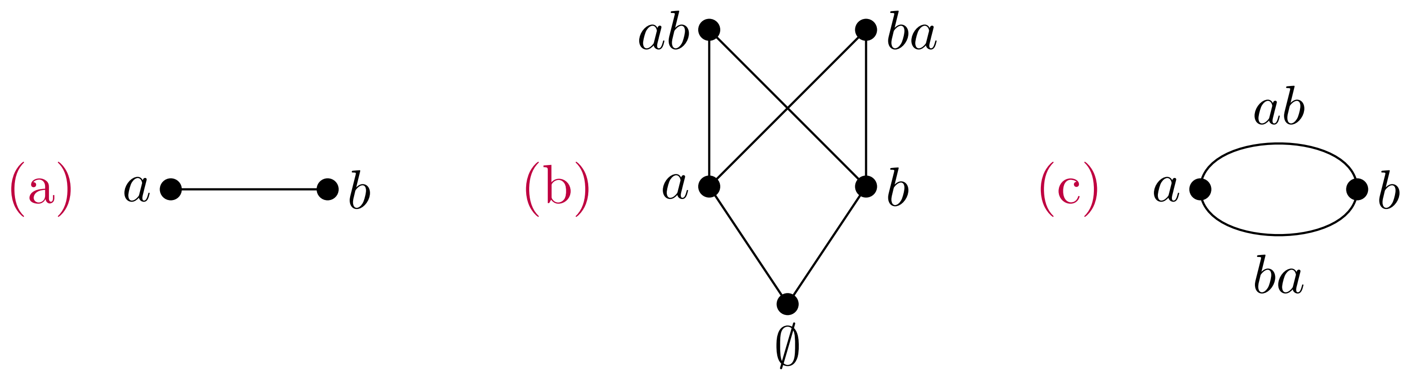
The boolean elements in any Coxeter system (W, S) form an ideal $\mathbb{B}(W, S)$ which is a simplicial subposet. So it is the face poset of a regular cell complex $\Delta(W, S)$.

Theorem. [T] $B(w)$ is boolean iff $w \in S_n$ avoids 321 and 3412.

We study the homotopy type of the geometric realization $|\Delta(W, S)|$.

Theorem. [Ragnarsson-T] For every finitely generated Coxeter system (W, S), $\exists\ \beta(W, S) \in \mathbb{N}$ so that $|\Delta(W, S)| \simeq \beta(W, S) \cdot S^{|S|-1}$. Moreover, $\beta(W, S)$ can be computed recursively.

Example.



(a) The graph K_2. (b) The poset $\mathbb{B}(K_2)$. (c) The boolean complex $\Delta(K_2)$, where $|\Delta(K_2)|$ is homotopy equivalent to S^1.

The unlabeled Coxeter graphs of the Coxeter groups $A_2, B_2/C_2, G_2$ and $I_2(m)$ are all the same as K_2.

What do we do with permutation patterns?

There are two main activities related to permutation patterns: enumeration and characterization.

I am most interested in the latter: determining phenomena characterized by pattern avoidance or containment.

I collect this information in the Database of Permutation Pattern Avoidance.

Database of Permutation Pattern Avoidance

The aim of this database is to provide a resource of phenomena characterized by avoiding a finite number of permutation patterns.

ID:	P0013
Patterns:	2 4 1 3
	3 1 4 2
Title:	Permutations that can be sorted by an unlimited number of pop-stacks in series
	Separable permutations
References:	D. Avis and M. Newborn, On pop-stacks in series
	M. D. Atkinson and J.-R. Sack, Pop-stacks in parallel
	P. Bose, J. Buss, and A. Lubiw, Pattern matching for permutations
Enumeration:	Schroeder numbers
OEIS:	A006318