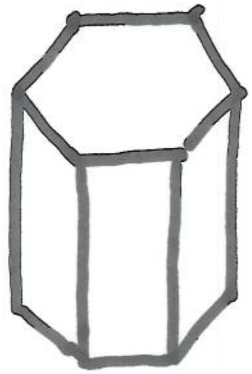
The cd-index of stratified manifolds

Richard Ehrenborg, IAS + UK
Mark Goresky, IAS
Margaret Readdy, IAS + UK.

UNIVERSITY OF KENTUCKY



P, n-dim'l polytope

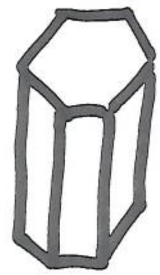
The <u>flag f-vector</u>

$$f_S = \# \; F_{i_1} \subsetneq \cdots \subsetneq F_{i_k},$$

$$S = \{ i_1 < \cdots < i_k \} \subseteq \{0, 1, \ldots, n-1\}$$

$$\dim F_{i_j} = i_j.$$

2.



s	f_s
$\emptyset$	1
0	12
1	18
2	8
01	$12(3)=36$
02	36
12	$18(2)=36$
012	72

The <u>flag h-vector</u>

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T .$$

S	f_S	h_S
$\emptyset$	1	1
0	12	11
1	18	17
2	8	7.
01	36	7
02	36	17
12	36	11
012	72	1

[Stanley].

$h_S = h_{\bar{S}}$

$S \subseteq \{0, \ldots, n-1\}$

$u_S = u_0 \cdots u_{n-1}$

where

$$u_i = \begin{cases} a & \text{if } i \notin S \\ b & \text{if } i \in S \end{cases}$$

S	f_S	h_S	w_S
$\emptyset$	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72	1	bbb

The <u>ab-index</u>

$$\Psi(P) = \sum_{S} h_S \cdot w_S$$

ex. $\Psi(\text{Prism}(⬡)) =$

$$1\,aaa + 11\,baa + 17\,aba$$
$$+ 7\,aab + 7\,bba + 17\,bab$$
$$+ 11\,abb + 1\,bbb$$

Theorem: [Bayer - Klapper]
For face lattices of polytopes, more generally, graded Eulerian posets, the ab-index can be written uniquely in terms of

$$c = a+b$$
$$d = ab+ba$$

(noncommutative). The resulting polynomial is the <u>cd-index</u>

Eulerian: $\mu(x,y) = (-1)^{\rho(x,y)}$
Equivalently, in each non-trivial interval $[x,y]$

$$\#\text{elts of odd rank} = \#\text{elts of even rank}$$

ex. (cont'd)

Ψ(Prism(⬡)) =

$$1aaa + 11baa + 17aba + 7aab$$

$$+ 7bba + 17bab + 11abb + 1bbb$$

$$= (a+b)^3 + 6(a+b)(ab+ba)$$

$$+ 10(ab+ba)(a+b)$$

$$= c^3 + 6cd + 10dc$$

Some cd-history.

[Bayer-Billera]. Gen'd Dehn-Sommerville relations.

[Bayer-Klapper] The cd-index provides a natural basis which removes these linear flag vector relns

[Stanley] $\Psi \geq 0$ for S-shellable posets.

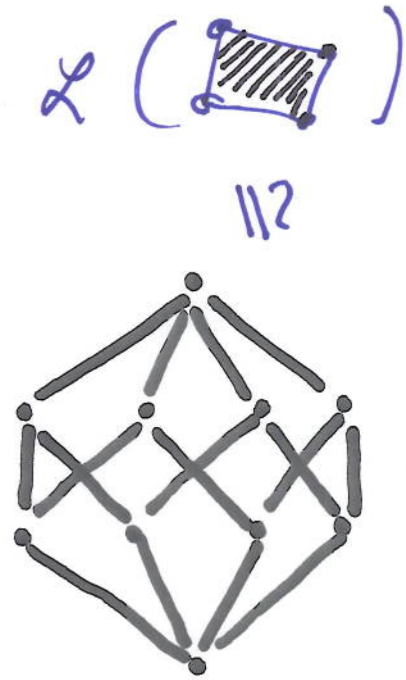
[Ehrenborg-R]. Ψ has a coalgebraic structure (ex. geom. operations reflected as a derivation on Ψ).

[Billera, Ehrenborg, Karu]* Inequalities for Ψ, and hence, for flag-h & flag-f vectors.

* Various permutations & combinations of these authors.

ex. The n-gon, $n \geq 2$

S	f_S	h_S
$\emptyset$	1	1
0	n	$n-1$
1	n	$n-1$
01	$2n$	1



$$\Psi(n\text{-gon}) = aa + (n-1)(ba+ab) + bb$$

$$= c^2 + (n-2) \cdot d.$$

ex. 1-gon

S	f_S	h_S
$\emptyset$	1	1
0	1	0
1	1	0
01	1	0

Not Eulerian

Motivation

$$\mathrm{link}_e(v) = \cdot \quad \cdot$$

$$\chi(\cdot \quad \cdot) = 2$$

ex. Return to the 1-gon

s	$\bar{f}_s$	$\bar{h}_s$
$\emptyset$	1	1
0	1	0
1	1	0
01	$1 \cdot \underset{\underset{2}{\parallel}}{x}(\mathrm{lnk}_e(\cdot))$	1

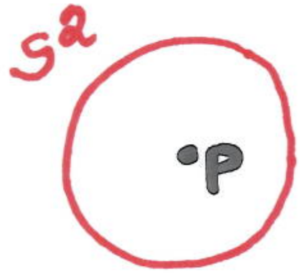
$$\Phi(\text{1-gon}) = a^2 + b^2$$

$$= c^2 - d$$

So,

$$\Phi(\text{n-gon}) = c^2 + (n-2)d \quad (n \geq 1)$$

Another example



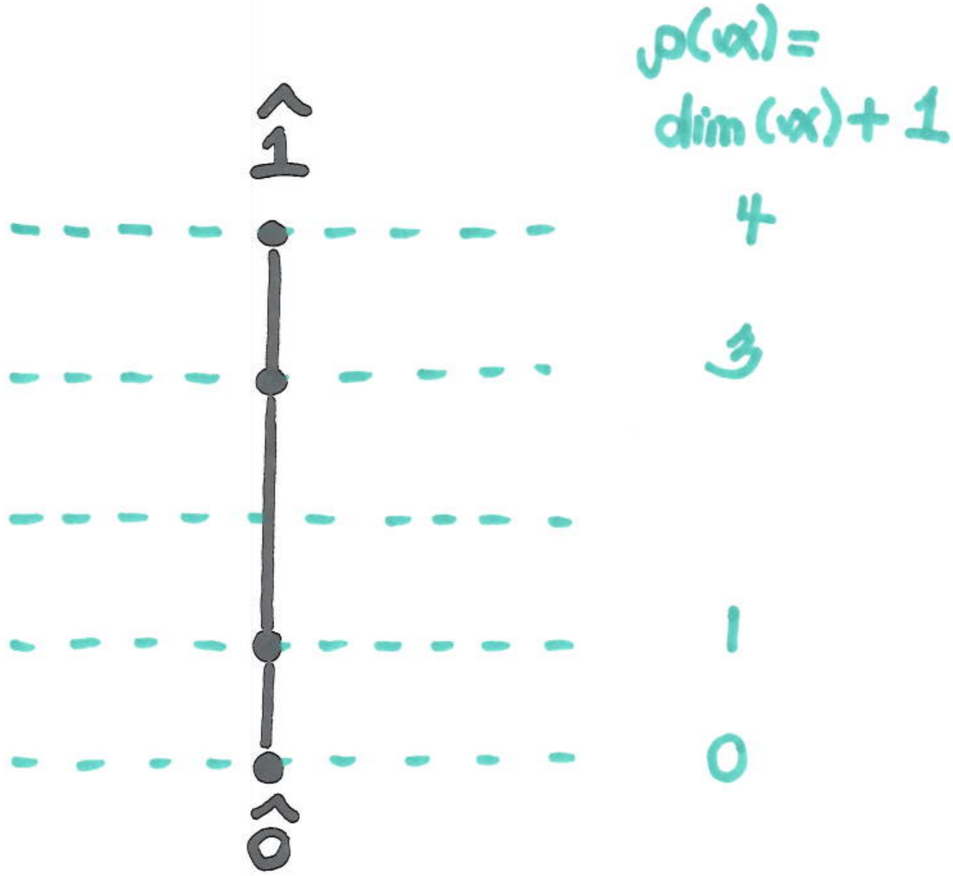
s	$\overline{f_g}$	$\overline{h_g}$
$\emptyset$	1	1
0	1	0
1	0	-1
2	1	0
01	0	0
02	$1 \cdot \chi(S^1) = 0$	-1
12	0	0
012	0	1

$$\Phi\left(\overset{S^2}{\odot}\right) = aaa - aba - bab + bbb$$

$$= c^3 - dc - cd.$$

Note:

$\mathcal{F}\left(\overset{S^2}{\odot}\right) =$



Not graded

def. A <u>quasi-graded poset</u> $(P, \rho, \bar{\zeta})$ consists of

i. P a finite poset (not necessarily graded).

ii. $\rho : P \to \mathbb{N}$, order-preserving ($x < y \Rightarrow \rho(x) < \rho(y)$)

iii. $\bar{\zeta} \in I(P)$, the weighted zeta function, satisfying $\bar{\zeta}(x, x) = 1 \quad \forall x \in P$.

Define

$$\bar{f}_S = \sum_{c} \bar{\xi}(c),$$

where $c = \{\hat{0} = x_0 < \cdots < x_{k+1} = \hat{1}\}$,

$\rho(x_i) = s_i$, $\quad S = \{s_1, \ldots, s_k\}$

and

$$\bar{\xi}(c) = \bar{\xi}(x_0, x_1) \cdot \bar{\xi}(x_1, x_2) \cdots \bar{\xi}(x_{k-1}, x_k)$$

The <u>ab-index of $(P, \rho, \bar{\xi})$</u> is

$$\Phi(P) = \sum_{S} \bar{h}_S \cdot u_S.$$

The <u>Eulerian condition</u>

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x,y) \cdot \bar{\zeta}(y,z) = 0$$

$\bar{\zeta} = \zeta$ gives the <u>classical Eulerian condition</u>.

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} = 0$$

Theorem: For an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$, its ab-index can be written uniquely as a cd-index.

Is this legal?

YES.

W a Whitney stratification

$$W = \bigcup_{\alpha \in P} S_\alpha$$

W satisfies Whitney's conditions A and B.

THE FINE PRINT

Definition Let W be a closed subset of a smooth manifold M, and suppose W can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of W satisfying the *condition of the frontier*:

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_{\mathcal{P}} Y.$$

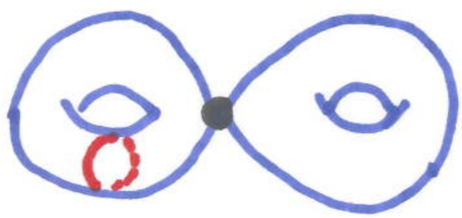
This implies the closure of each stratum is a union of strata. We say W is a *Whitney stratification* if

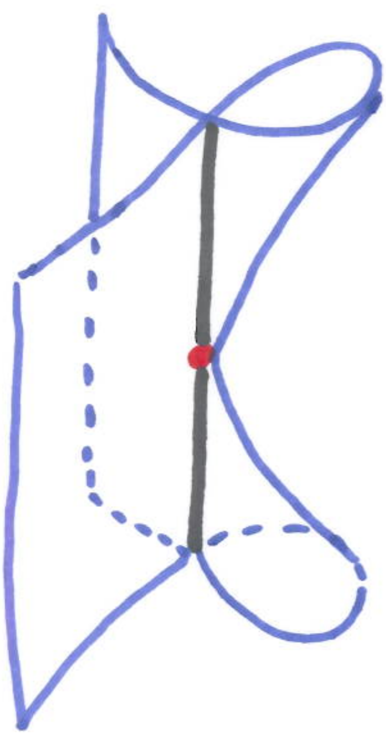
1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of M (not necessarily connected).
2. If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the inclusions

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau$$

hold.

Examples





Theorem: Let W have a Whitney stratification with

$$\rho(x) = \begin{cases} 0 & \text{if } x = \hat{0} \\ \dim(x) + 1 & \text{otherwise} \end{cases}$$

and

$$\bar{\zeta}(x,y) = \begin{cases} \chi(y) & \text{if } x = \hat{0} \\ \chi(\mathrm{link}_y(x)) & \text{if } \hat{0} < x < y. \end{cases}$$

Then the face poset $\mathcal{F}(W)$ is an Eulerian quasi-graded poset.

Cor: M a smooth manifold,
W a Whitney stratification
of the boundary of M.
Then the ab-index of
the face poset $\mathcal{F}(W)$
can be written as a
cd-index.

Thank you!