Partitions and compositions: A tale of two symmetries

Sarah K Mason
Wake forest University

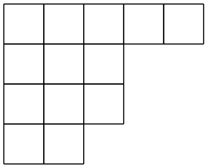
Algebraic Combinatorixx
25 May, 2011

Partition

A **partition** of *n* is a weakly decreasing sequence of positive integers which sum to *n*.

Example: $13 = 5 + 3 + 3 + 2$

$$\lambda = (5, 3, 3, 2)$$

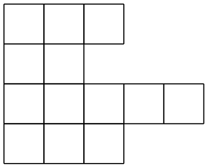


Composition

A **composition** of a positive integer *n* is a sequence of positive integers which sum to *n*.

Example: $13 = 3 + 2 + 5 + 3$

$$\alpha = (3, 2, 5, 3)$$



symmetric functions in n variables (Sym_n)

$\pi f(X) = f(X)$ for any permutation π.
(Indexed by partitions.)

Examples (Sym_3)

- $x_1^2 + x_2^2 + x_3^2$
- $x_1^3 x_2 + + x_1^3 x_3 + x_1 x_2^3 + x_2^3 x_3 + x_1 x_3^3 + x_2 x_3^3$

Non-example

- $x_{1\color{red}{1}}^2 x_2 + x_2^2 x_3^2$ ~~$x_{\color{red}{3}}^2 x_2$~~

quasisymmetric functions in n variables ($QSym_n$)

$\sigma f(X) = f(X)$ for any shift σ of the nonzero exponents.
(Indexed by compositions.)

Examples ($QSym_3$)

- $x_1^2 + x_2^2 + x_3^2$
- $x_1 x_2^3 + x_1 x_3^3 + x_2 x_3^3 =$

  $x_1^{\color{red}{1}} x_2^{\color{red}{3}} x_3^0 + x_1^{\color{red}{1}} x_2^0 x_3^{\color{red}{3}} + x_1^0 x_2^{\color{red}{1}} x_3^{\color{red}{3}}$

Non-example

- $x_1^3 x_2 + x_1^3 x_3$ ~~$x_2^{\color{red}{3}} x_3$~~

Semi-standard Young tableau (SSYT)

rows: weakly decreasing
columns: strictly decreasing

$T =$

9	8	8	6	3
7	6	3		
6	4	1		
3	2			

$$x^T = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

Semi-standard composition tableau (CT)

rows: weakly decreasing
left column: strictly increasing
columns: $a \leq b \Rightarrow b > c$

c	a
	b

$F =$

3	2	1		
6	6	3		
7	4			
9	8	8	6	3

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

Schur functions

$$s_\lambda(x_1, \cdots, x_n) = \sum_{T \in SSYT(\lambda)} x^T$$

Quasisymmetric Schurs

$$\mathcal{QS}_\gamma(x_1, \cdots, x_n) = \sum_{F \in CT(\gamma)} x^F$$

$$s_{2,1}(x_1, x_2, x_3) =$$

2	1
1	

3	1
1	

2	2
1	

3	2
1	

3	1
2	

3	3
1	

3	2
2	

3	3
2	

$$x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

$$\mathcal{QS}_{2,1,3}(x_1, x_2, x_3) =$$

1	1	
2		
3	3	2

1	1	
2		
3	3	3

$$x_1^2x_2^2x_3^2 + x_1^2x_2x_3^3$$

Schur functions

$$s_{2,1}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$
$$+ x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

Quasisymmetric Schurs

$$\mathcal{QS}_{2,1}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

$$\mathcal{QS}_{1,2}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 3 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 3 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 3 & 3 \\ \hline\end{array}$$

$$x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

$$\boxed{s_{2,1}(x_1, x_2, x_3) = QS_{2,1}(x_1, x_2, x_3) + QS_{1,2}(x_1, x_2, x_3)}$$

Proposition (Haglund, Luoto, M., van Willigenburg)

$$s_\lambda = \sum_{\alpha^+ = \lambda} \mathcal{QS}_\alpha$$

Corollary

If a function is **symmetric** and **quasisymmetric Schur positive**, then it is **Schur positive**!

Proposition (Haglund, Luoto, M., van Willigenburg)

$$s_\lambda = \sum_{\alpha^+ = \lambda} \mathcal{QS}_\alpha$$

Proof (example):

9	8	6	1	1
6	6	5		
5	4	2		
3	2			
2				

$$\overset{\rho\overline{\rho}^{1}}{\longleftrightarrow}$$

2	2	2	1	1
3				
5	4			
6	6	6		
9	8	5		

Schur functions...

- form a basis for all symmetric functions.
- are closely related to other symmetric function bases.
- correspond to characters of irr reps of GL_n.
- describe the cohomology of the Grassmannian.
- have many nice combinatorial properties.
- generalize to Macdonald polynomials ($\tilde{J}_\lambda(X; q, t)$).

Quasisymmetric Schur functions...

- form a basis for all quasisymmetric functions.
- are closely related to other quasisymmetric function bases.
- correspond to sums of Demazure characters.
- have many nice combinatorial properties.
- generalize to non-symmetric Macdonald polynomials ($\hat{E}_\gamma(X; q, t)$).

Let $D(T)$ be the set of all i such that $i+1$ appears weakly to the right of i in T, let $\beta(S)$ be the composition obtained from the differences between consecutive elements of a set S, and let F_γ be the fundamental quasisymmetric function corresponding to the composition γ. Then:

Theorem (Gessel 84)

$$s_\lambda = \sum_T F_{\beta(D(T))}$$

where the sum is over all standard contretableaux, T, of shape λ.

Theorem (HLMvW)

$$\mathcal{QS}_\alpha = \sum_T F_{\beta(D(T))}$$

where the sum is over all standard contretableaux, T, of shape $\lambda(\alpha)$ that map under ρ^{-1} to a CT of shape α.

Example

$$
\begin{array}{|c|c|c|}\hline 2 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & 3 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 3 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & 2 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
$$

$$
\downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho
$$

$$
\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
$$

$$
D = \{2\} \qquad D = \{1,3\} \qquad D = \{3\} \qquad D = \{1,4\} \qquad D = \{2,4\}
$$

$$
s_{3,2} = F_{2,3} + F_{1,2,2} + F_{3,2} + F_{1,3,1} + F_{2,2,1}
$$

$$
QS_{2,3} = F_{2,3} + F_{1,2,2}
$$
$$
QS_{3,2} = F_{3,2} + F_{1,3,1} + F_{2,2,1}
$$

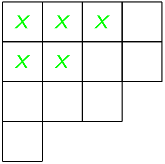
skew shape

diagram for partition λ/μ

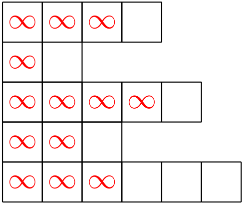
skew diagram

composition diagram with extended basement

$(4, 4, 3, 1)/(3, 2)$



$(4, 2, 5, 3, 6)/(3, 1, 4, 2, 3)$



Yamanouchi

A **Yamanouchi** word is a word in which any prefix contains at least as many occurences of j as of $j+1$, for each $j \geq 1$.

Yamanouchi: 11212321

not Yamanouchi: 1123131

contre-Yamanouchi

A **contre-Yamanouchi** word is a word in which any prefix contains at least as many occurences of j as $j-1$.

contre-Yamanouchi: 33232123

not contre-Yamanouchi: 3321313

Littlewood-Richardson SSYT

A **Littlewood-Richardson SSYT** is a skew tableau whose reading word (L-R, bottom-top) is a Yamanouchi word.

x	x	x	x	1
x	x	1	1	2
1	2	3		

1 2 3 1 1 2 1

Littlewood-Richardson CT

A **Littlewood-Richardson CT** is a skew composition whose reading word (L-R, bottom-top) is a Yamanouchi word.

∞	∞	∞	1		
∞	1				
∞	∞	∞	∞	2	
∞	∞	2			
∞	∞	∞	3	3	3

3 2 3 1 3 2 1

Littlewood-Richardson Rule

In the expansion

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu,$$

the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is the number of LRST of shape ν/λ, weight μ.

Theorem (HLMvW 08)

$$\mathcal{QS}_\alpha \cdot s_\lambda = \sum_\beta c^\beta_{\alpha,\lambda} \mathcal{QS}_\beta,$$

where $c^\beta_{\alpha,\lambda}$ is the number of LRCT of shape β/α and with content λ^*.

Example

$$s_{2,1}s_{2,1} = s_{4,1,1} + s_{4,2} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1} + s_{2,2,2} + s_{2,2,1,1}$$

x	x	1	1
x			
2			

x	x	1	1
x	2		

x	x	1
x	1	2

x	x	1
x	1	
2		

x	x	1
x	2	
1		

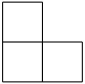
x	x	1
x		
1		
2		

x	x
x	1
1	2

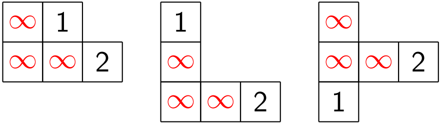
x	x
x	1
1	
2	

Example

$$\mathcal{QS}_{1,2} \cdot s_{1,1} = \mathcal{QS}_{2,3} + \mathcal{QS}_{1,1,3} + \mathcal{QS}_{1,3,1}$$



$(2,1) \rightarrow$



$Sym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[x_1, \ldots, x_n] = \mathbb{Q}[\mathbf{x}]$

$Sym_n(\mathbb{Q}) \hookrightarrow QSym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[\mathbf{x}]$

A classical result

The following are equivalent:

1. Sym_n is a polynomial ring, generated by the elementary symmetric polynomials $\mathcal{E}_n = \{e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\}$;
2. the ring $\mathbb{Q}[\mathbf{x}]$ is a free Sym_n-module;
3. the coinvariant space $\mathbb{Q}[\mathbf{x}]_{\mathfrak{S}_n} = \mathbb{Q}[\mathbf{x}]/(\mathcal{E}_n)$ has dimension $n!$.

Bergeron-Reutenauer conjectures

- $QSym_n(\mathbb{Q})$ is a free module over $Sym_n(\mathbb{Q})$; Garsia-Wallach (2003)
- dim of coinvariant space $QSym_n(\mathbb{Q})/(\mathcal{E}_n)$ is $n!$; Garsia-Wallach (2003)
- Pure, inverting comps index a stable basis for $QSym_n(\mathbb{Q})/(\mathcal{E}_n)$. Lauve-M (2009)

$Sym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[x_1, \ldots, x_n] = \mathbb{Q}[\mathbf{x}]$ $\quad$ $Sym_n(\mathbb{Z}) \hookrightarrow QSym_n(\mathbb{Z}) \hookrightarrow \mathbb{Z}[\mathbf{x}]$

A classical result

The following are equivalent:

1. Sym_n is a polynomial ring, generated by the elementary symmetric polynomials $\mathcal{E}_n = \{e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\}$;
2. the ring $\mathbb{Q}[\mathbf{x}]$ is a free Sym_n-module;
3. the coinvariant space $\mathbb{Q}[\mathbf{x}]_{\mathfrak{S}_n} = \mathbb{Q}[\mathbf{x}]/(\mathcal{E}_n)$ has dimension $n!$.

Bergeron-Reutenauer conjectures

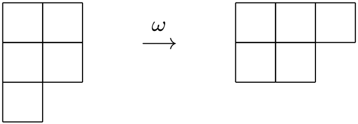
- $QSym_n(\mathbb{Z})$ is a free module over $Sym_n(\mathbb{Z})$; Lauve-M (2009)
- dim of coinvariant space $QSym_n(\mathbb{Z})/(\mathcal{E}_n)$ is $n!$; Lauve-M (2009)
- Pure, inverting comps index a stable basis for $QSym_n(\mathbb{Z})/(\mathcal{E}_n)$. Lauve-M (2009)

$\omega : Sym_n \to Sym_n$

- endomorphism
- $\omega(e_\lambda) = h_\lambda$
- $\omega(s_\lambda) = s_\lambda^T$

Example:

$\omega(s_{2,2,1} s^{col}_{2,2,1}) = s_{3,2} s^{col}_{3,2} = s^{row}_{2,2,1}$



$\omega : QSym_n \to QSym_n$

- endomorphism
- $\omega(F_\beta(x_1, x_2, \ldots, x_n)) = F_{\hat{\beta}}(x_n, \ldots, x_2, x_1)$

Theorem (M.-Remmel)

$\omega(QS^{col}_\alpha(x_1, \ldots, x_n)) = QS^{row}_\alpha(x_n, \ldots, x_1)$

Moreover:

$$s^{row}_\lambda = \sum_{\tilde{\alpha}=\lambda} QS^{row}_\alpha$$

column-strict SSYT

rows: weakly increasing
columns: strictly increasing

$T =$

9	8	8	6	3
7	6	3		
6	4	1		
3	2			

$$x^T = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

column-strict CT

rows: weakly decreasing
left column: strictly increasing
columns: $a \leq b \Rightarrow b > c$

c	a
	⋮
	b

$F =$

3	2	1		
6	6	3		
7	4			
9	8	8	6	3

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

row-strict SSYT

rows: strictly increasing
columns: weakly increasing

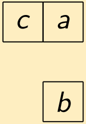
$T =$

9	8	7	3	1
7	6	3		
7	4	1		
5	4			

$$x^T = x_1^2 x_3^2 x_4^2 x_5 x_6 x_7^3 x_8 x_9$$

row-strict CT

rows: strictly decreasing
left column: weakly increasing
columns: $a < b \Rightarrow b \geq c$

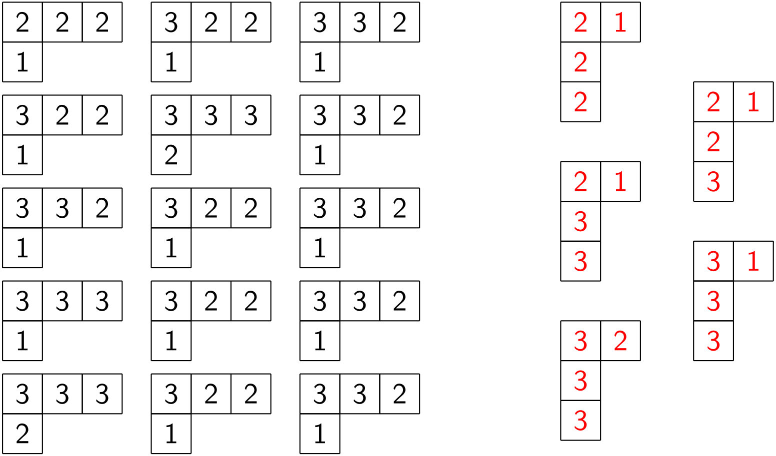


$F =$

5	4	3		
7	6	3		
7	4			
9	8	7	3	1

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

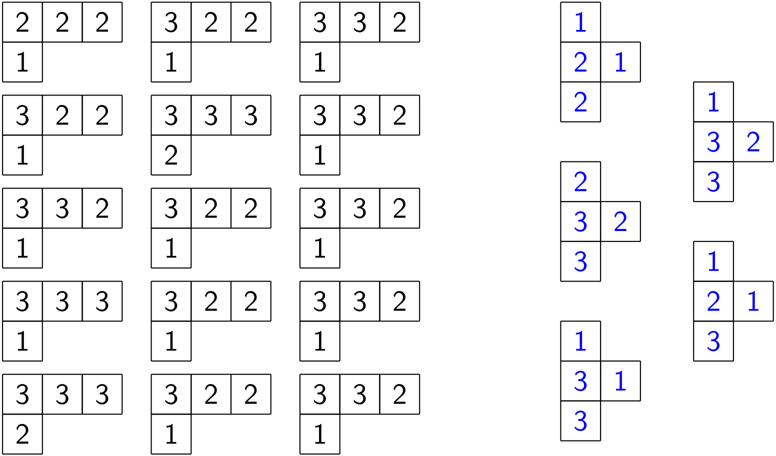
$$
\begin{array}{ccccccc}
s_{2,1,1}(x_1,x_2,x_3) & = & QS_{2,1,1} & + & QS_{1,2,1} & + & QS_{1,1,2} \\
\downarrow \omega & & \downarrow \omega & & \downarrow \omega & & \downarrow \omega \\
s_{3,1}(x_1,x_2,x_3) = s^{row}_{2,1,1} & = & QS^{row}_{2,1,1} & + & QS^{row}_{1,2,1} & + & QS^{row}_{1,1,2}
\end{array}
$$



$$s_{2,1,1}(x_1, x_2, x_3) = QS_{2,1,1} + QS_{1,2,1} + QS_{1,1,2}$$

$$\downarrow \omega \qquad \downarrow \omega \qquad \downarrow \omega \qquad \downarrow \omega$$

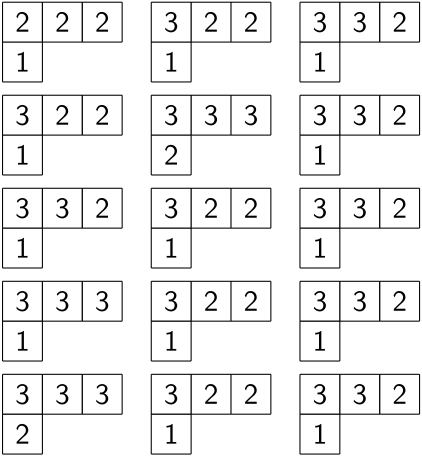
$$s_{3,1}(x_1, x_2, x_3) = s^{row}_{2,1,1} = {\color{red} QS^{row}_{2,1,1}} {\color{blue} + QS^{row}_{1,2,1}} {\color{green} + QS^{row}_{1,1,2}}$$

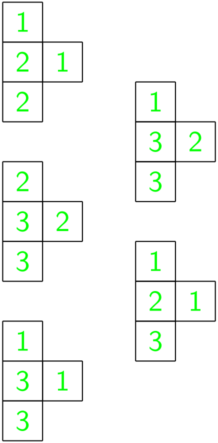


$$s_{2,1,1}(x_1, x_2, x_3) = QS_{2,1,1} + QS_{1,2,1} + QS_{1,1,2}$$

$$\downarrow \omega \qquad \downarrow \omega \qquad \downarrow \omega \qquad \downarrow \omega$$

$$s_{3,1}(x_1, x_2, x_3) = s^{row}_{2,1,1} = QS^{row}_{2,1,1} + QS^{row}_{1,2,1} + QS^{row}_{1,1,2}$$





Further directions

- Quasisymmetric Hall-Littlewood polynomials
- Quasisymmetric Macdonald polynomials
- Quasisymmetric Schur P-functions (in progress here!)
- Representation theoretic interpretation (Steph van Willigenburg and Christine Bessenrodt)
- Multiplication rules (Jeff Ferreira)
- Basis for invariant space $QSym_n^r / Sym_n$ (in progress here!)

THANK YOU!!

- Haglund, Luoto, Mason & van Willigenburg, *Refinements of the Littlewood-Richardson rule*, Trans. Amer. Math. Soc. (2011).
- Lauve & Mason, *QSym over Sym has a stable basis*, J. Combin. Theory Ser. A (to appear).
- Mason & Remmel, *Row-strict quasisymmetric Schur functions* (in preparation).