Parity Based Virtual Link Invariants

Heather A. Dye

McKendree University

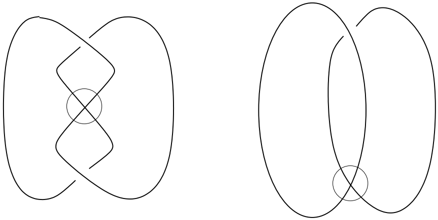
May 2011

Virtual Links

A virtual link diagram is a decorated immersion of n copies of S^1 with two types of crossings: classical and virtual.

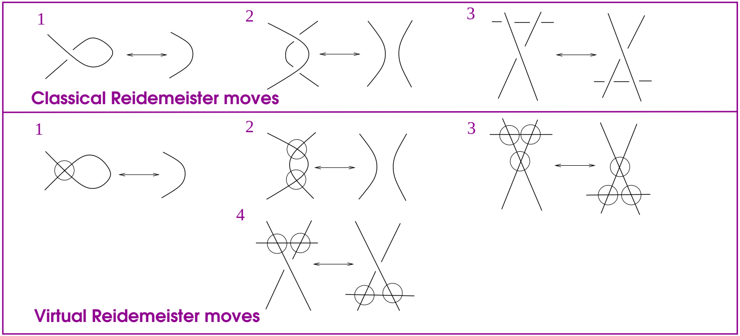
Classical crossings have under/over markings.

Virtual crossings are solid and circled.



Reidemeister Moves

Two link diagrams are said to be equivalent if one can be transformed into the other by a sequence of Reidemeister moves. A virtual link is an equivalence class of virtual link diagrams.

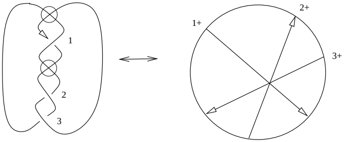


In an oriented diagram, each crossing has an orientation.

Gauss Diagrams

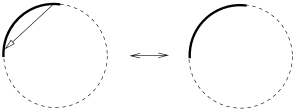
Gauss diagrams are oriented circles with signed, directed chords. Equivalence classes are determined by the local moves shown below, which are analogs of the classical Reidemeister moves.

Virtual knots are in one to one correspondence with equivalence classes of Gauss diagrams.

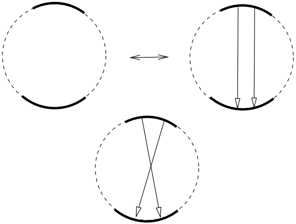


Converting a knot to a Gauss diagram

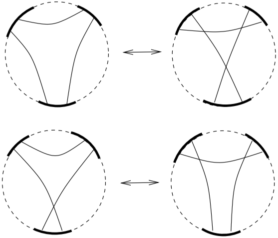
1



2



3



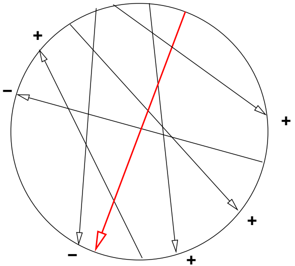
The virtual Reidemeister moves do not appear in the Gauss diagrams.

Parity in Gauss Diagrams

Parity in Gauss diagrams

Gaussian:

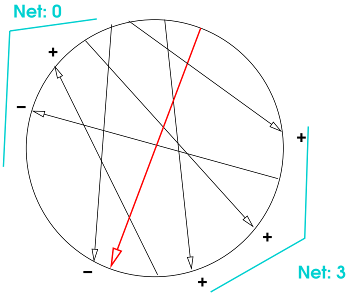
The number of intersecting chords mod two.



Gaussian Parity: 1

Oriented:

Uses the direction and signs of the chords.



Oriented Parity: 3

The absolute value of the difference between the sum of signs on either side of chord.

Linking numbers

To obtain parity from oriented knots, we introduce linking numbers.

Consider a two component link with components A and B.
Let A_B denote the set of crossings, c, where A overpasses B.

We define:

$$l(A,B) = \sum_{A_B} \mathrm{sgn}(c)$$

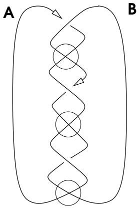
$$L(A,B) = l(A,B) + l(B,A)$$

$$L'(A,B) = | \, l(A,B) - l(B,A) \, |$$

Note that L'(A,B) = L'(B,A).

Hence, for a link K, we write L'(K).

For a classical link, L'(K) =0.

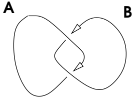


$l(A,B) = -1$

$l(B,A) = 2$

$L(A,B) = 1$

$L'(A,B) = 3$



$l(A,B) = -1$

$l(B,A) = -1$

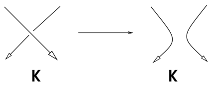
$L(A,B) = -2$

$L'(A,B) = 0$

Parity in virtual knots

Parity in Knots

In a knot, smoothing a crossing vertically produces a 2 component link.



Theorem 1:

We define A_i to be the set of all crossings c with $L'(K_c) = i$.

Let $a_i = \sum_{A_i} \mathrm{sgn}(c)$.

The vector $< a_1, a_2, a_3, a_4 \ldots >$ is an invariant of knots.

Theorem 2:

Designate a component of K_c as A and the other component as B.

Realize each virtual crossing as a classical crossing by designating "A" as the overpassing arc.

Let R denote the set of realized crossings.

We define $L_v(K_c) = \sum_R \mathrm{sgn}(c)$.

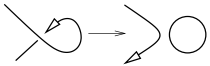
We define B_i to be the set of all crossings c with $L_v(K_c) = i$.

Let $b_i = \sum_{B_i} \mathrm{sgn}(c)$.

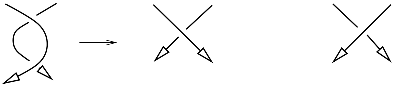
The vector $< b_1, b_2, b_3, b_4 \ldots >$ is an invariant of knots.

Proof of Invariance

Proof of Theorem 1:

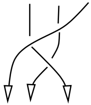


$L'(k_C) = 0$

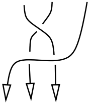


Net contribution: 0

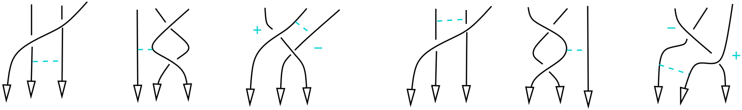
$L'(K\ ) = |\ a - (-1)\ |$ $L'(K\ ) = |\ a + 1\ |$



Net difference: 0



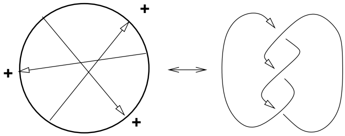
There is a correspondence between the smoothed diagrams.



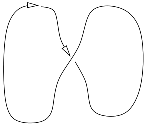
Pairs of crossings

Pairs of intersecting chords

Pairs of intersecting chords correspond to pairs of crossings that have the following property:
when both crossings are smoothed vertically we again obtain a knot.



Smoothed version:



For a knot diagram K, we will let P denote the set of pairs crossings that correspond to intersecting chords with opposite parity.

For a classical knot, the set P is empty.

The η invariant

A formal sum

Let K_p denote the knot diagram obtained from the knot diagram K by smoothing a pair of crossings from the set P.

We define a formal sum of knots with coefficients in Z_2

$$\eta(K) = \sum_{p \varepsilon P} c\, K_p$$

Theorem:

For a knot K, $\eta(K)$ is invariant under the Reidmeister moves and crossing change.

That is $\eta(K)$ is a homtopy invariant.

Invariance

The invariance of $\eta(K)$

Chords corresponding to Reidemeister I moves do not intersect other chords.

Both chords corresponding to a Reidemeister II move have the same parity. So we need only consider 1 crossing from the move and an exterior crossing.

Smoothing these pairs of crossings produces two knots in the same homotopy class.

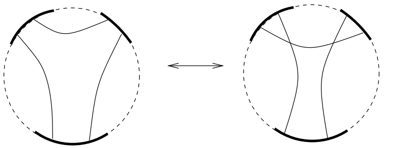
Hence, the net contribution is zero.

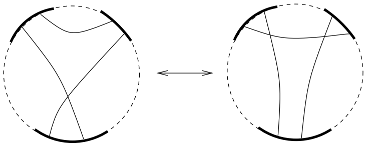
We consider the Reidemeister III move. We have two cases:

1. An exterior and an interior crossing.
2. Two interior crossings.

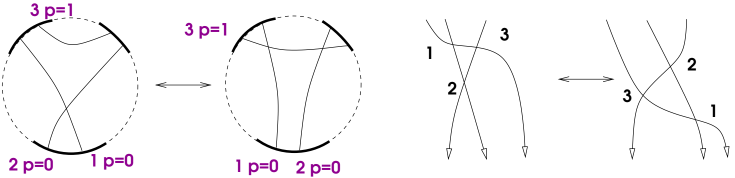
Case 1 is clear.

We expand the crossings in case 2. As Gauss diagrams, we obtain:



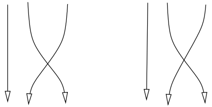


The invariance of η (K) – a sample Reidemeister III move.



On the left hand side, there are no pairs to smooth.

On the right hand side, smooth pairs (1,3) and (2,3).



Pair (1,3) Pair (2,3)

The net contribution of homotopy classes modulus two is zero.

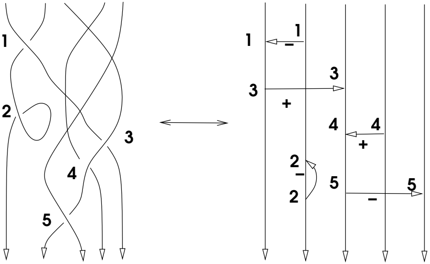
To finish the proof, we check all cases of parity.

Long virtual links

A long virtual link is an oriented n−n virtual tangle with ordered components.

Gauss diagrams of a ordered long virtual links consist of a set of ordered, directed edges.

For each component, we record crossing information (number, orientation, over/under) along the edges. We then connect corresponding points on the edges.



Two crossings, a and b, intersect if a "b" label occurs between the two "a" labels

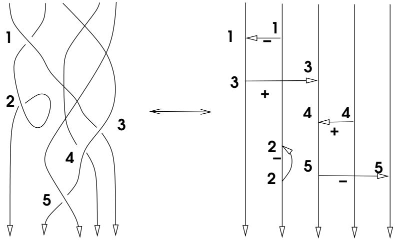
The parity of a crossing is determined by the number of labels between the two labels of a crossing.

Here:

4 and 5 intersect.
4 has odd parity.
2 has even.

Extending the invariant

Use the definitions of intersection and parity to extend the invariant



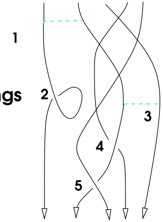
	Parity	Intersects
1	Odd	3
2	Even	
3	Even	1
4	Odd	5
5	Odd	4

We smooth the crossings 1 and 3.

Producing a sum of long virtual links.

Each term in the sum has n−2 crossings

Coefficients are modulo 2.



Questions

1. The vectors and the η invariant are degree one Vassilliev invariants. Is it possible to construct higher degree Vassilliev invariants using the similar methods?
2. Can we extend the invariants to links using methods similar to those in the paper: Homotopy invariants of Gauss phrases?
3. Can we combine the vectors $\langle a_1, a_2, \ldots \rangle$ and $\langle b_1, b_2, \ldots \rangle$ to produce a more powerful invariant?
4. The vectors can be used to determine a lower bound on the virtual and classical crossing number. Can we obtain stronger topological results?
5. How does the formal sum relate to other invariants of knots and links? Is it part of a family of other invariants?

Bibliography

- MR2666478 Gibson, Andrew . Homotopy invariants of Gauss phrases. Indiana Univ. Math. J. 59 (2010), no. 1, 207–229.
- MR2681114 Manturov, V. O. Parity in knot theory. (Russian) Mat. Sb. 201 (2010), no. 5, 65–110; translation in Sb. Math. 201 (2010), no. 5-6, 693–733
- MR2371713 (2009a:57007) Kamada, Naoko. Miyazawa polynomials of virtual knots and virtual crossing numbers. Intelligence of low dimensional topology 2006, 93100, Ser. Knots Everything, 40, World Sci. Publ., Hackensack, NJ, 2007, 57M25 (57M27)
- MR2399131 (2009f:57022) Turaev, Vladimir . Cobordism of knots on surfaces. J. Topol. 1 (2008), no. 2, 285–305.