Statistical Mechanics Of The Quasi-Geostrophic Equations On A Rotating Sphere

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Abstract

The standard procedure of statistical mechanics is applied to a barotropic model of the atmosphere. We obtain a probability distribution for the coarse-grained potential vorticity maximizing statistical entropy subject to the constraints of conservation of the invariants of the dynamical equations (Energy, Casimirs). We solve the equation of state for the ensemble-mean flow on the rotating sphere and classify the resulting solutions.

Physical Motivations

Our aim is to be able to study directly the equilibrium state (or steady-state in a forced-dissipative context) of the large-scale general circulation without needing to compute the full temporal evolution of the microscopic fields. The tools of statistical mechanics were specifically designed for this purpose.

Statistical Mechanics of a simple atmospheric model

Quasi geostrophic equations: $\partial_t q + (\mathbf{v}.\nabla)q = 0 \quad q = -\Delta\psi + \frac{\psi}{R^2} + h$

(h topography, R Rossby deformation radius)

Conserved quantities: $E = \frac{1}{2}\int_{S^2}(q-h)\psi \quad \Gamma_n = \int_{S^2} q^n$

Potential vorticity is mixed by the flow: initial patches evolve into intricate filamented structures, but their area is conserved (conservation of the moments of potential vorticity). The additional constraint of energy conservation leads to the formation of coherent structures in the final state. Thus, to describe these coherent structures, we introduce a probability distribution for the coarse-grained potential vorticity field:

Ensemble-mean coarse-grained potential vorticity field: $\overline{q}(\mathbf{r}) = \int q\rho(\mathbf{r},q)dq$ with normalization: $\int \rho(\mathbf{r},q)dq = 1$

The probability distribution maximizing the statistical entropy $S[\rho] = -\int \rho(\mathbf{r},q)\ln\rho(\mathbf{r},q)dqd\mathbf{r}$ is $\rho(\mathbf{r},q) = \frac{1}{Z}e^{-\beta\psi q + \sum_n \alpha_n q^n}$

Equation of state: $\overline{q} = F(\psi) = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \psi}$ (Z is the partition function, α_n and β are Lagrange multipliers)

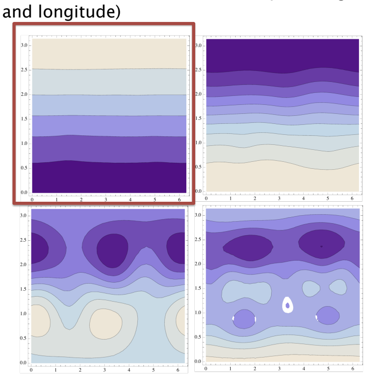
For conservation of energy, circulation and enstrophy, F is linear and the equation of state $\Delta\psi - \lambda(\psi - \langle\psi\rangle) = h - \langle h\rangle$ can be solved analytically in terms of spherical harmonics:

$$\psi = \langle\psi\rangle + \sum_{n=1}^{+\infty}\sum_{m=-n}^{n}\frac{\langle hY_{nm}\rangle}{\beta_n - \lambda}Y_{nm}(\theta,\phi) \qquad \Delta Y_{nm} = \beta_n Y_{nm}, \quad \lambda = \beta + \frac{1}{R^2}$$

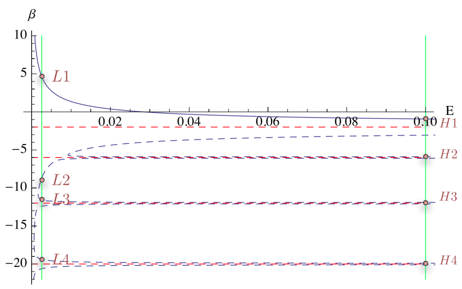
Equilibrium mean flows on the sphere

We take $h = f + f\frac{h_b}{H}$ with f the Coriolis parameter, H the average fluid height and h_b the bottom topography (in a T16 spectral truncation)

Low-Energy Equilibrium Mean Flows
(streamfunction as a function of polar angle and longitude)



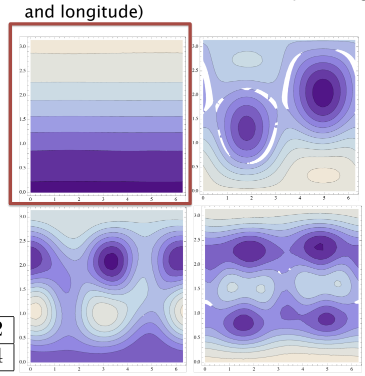
$\boxed{L1}$	$L2$
$L3$	$L4$



Caloric curve β(E): stable solutions (plain blue), saddle-points (dashed blue) and degenerate solutions (dashed red)

Stable Equilibrium

High-Energy Equilibrium Mean Flows
(streamfunction as a function of polar angle and longitude)



$\boxed{H1}$	$H2$
$H3$	$H4$

Conclusion and Prospects

The only stable equilibrium state for quasi-geostrophic flow on a rotating sphere consists of a nearly zonal flow; however, some metastable states, possibly long-lived, do exhibit coherent vortex structures. Addition of an angular momentum conservation constraints yields other possibilities for the structure of the stable equilibrium. Further developments include accounting for some forcing and dissipation in an approximate way.

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