Strengths and Weaknesses of Common Numerical Methods for Simulating Atmospheric Flows

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Broad Categories

- Spatially discrete
  - Finite difference
  - Finite volume

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  - Finite volume
- Series expansions
  - Orthogonal expansion functions
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- Fluid dynamical viewpoint
  - Eulerian
  - Lagrangian
  - Semi-Lagrangian

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Transport in 2D nondivergent flow.

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- Flux form facilitates the construction of schemes with local (cell-wise) and global conservation.
- Advective form helps preserve uniform ψ in non-trivial velocity fields

Finite difference notation

$$\delta_{nx} f(x) = \frac{f(x + n\Delta x/2) - f(x - n\Delta x/2)}{n\Delta x}$$

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Local conservation:

$$\frac{d\phi_{i,j}}{dt} + \delta_x(u_{i,j}\phi_{i,j}) + \delta_y(v_{i,j}\phi_{i,j}) = 0$$

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$$\left(\frac{d\psi}{dx}\right)_j = \delta_{2x}\phi_j + O[(\Delta x)^2], \quad \left(\frac{d\psi}{dx}\right)_j = \frac{4}{3}\delta_{2x}\phi_j - \frac{1}{3}\delta_{4x}\phi_j + O[(\Delta x)^4]$$

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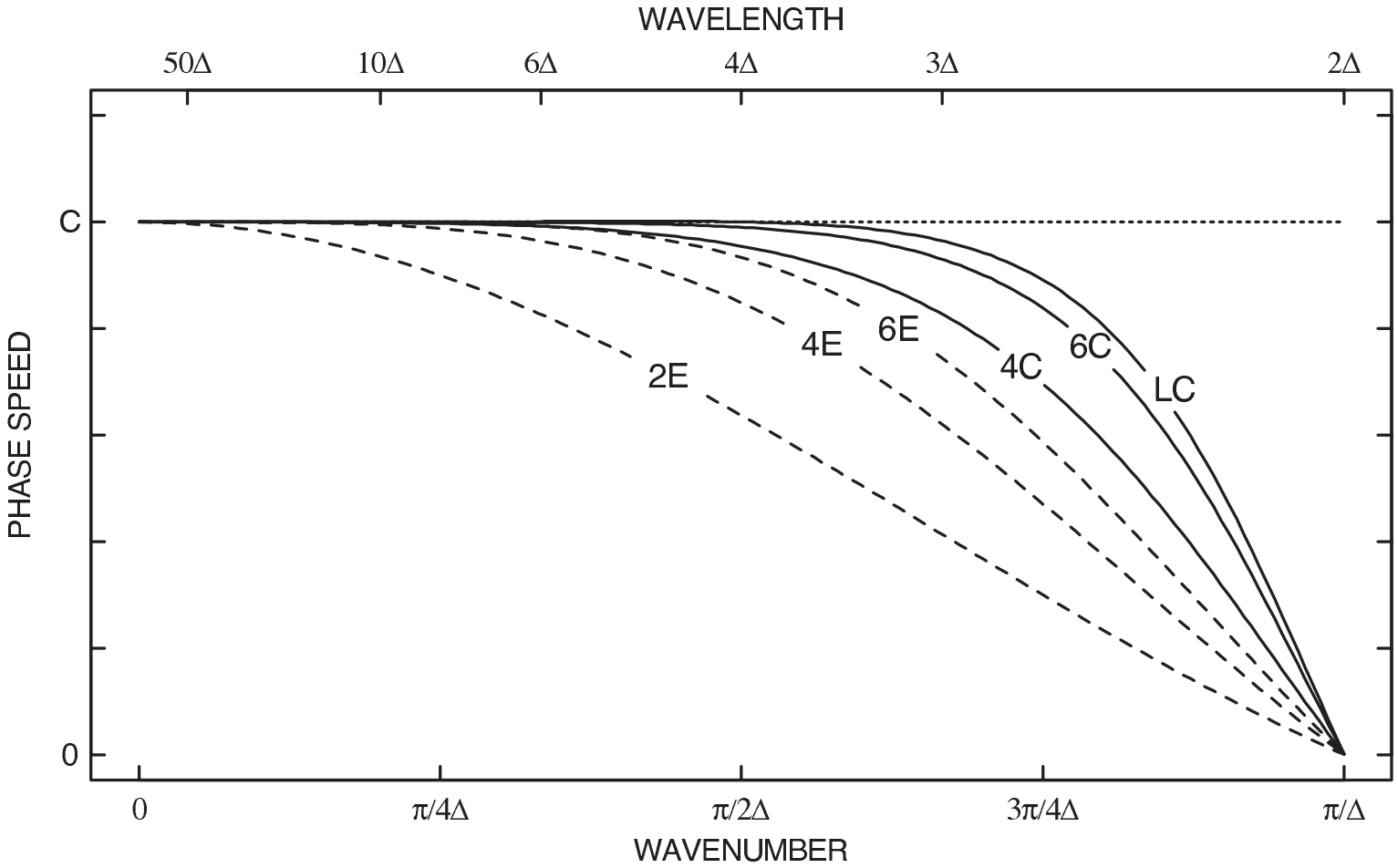
- More advanced approach: a 4th-order compact scheme

$$\frac{1}{24}\left[5\left(\frac{d\psi}{dx}\right)_{j+1} + 14\left(\frac{d\psi}{dx}\right)_j + 5\left(\frac{d\psi}{dx}\right)_{j-1}\right] = \frac{1}{12}\left(11\delta_{2x}\phi_j + \delta_{4x}\phi_j\right)$$

Phase Speed Error in 1D Advection

Semi-discrete approximation to $\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0.$



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- Convergence is never achieved in high Reynolds number atmospheric flow. (Why not?)
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- In atmospheric applications, errors are generally dominated by the most poorly resolved scales.
- High-order schemes *may* treat the marginally resolved scales better.

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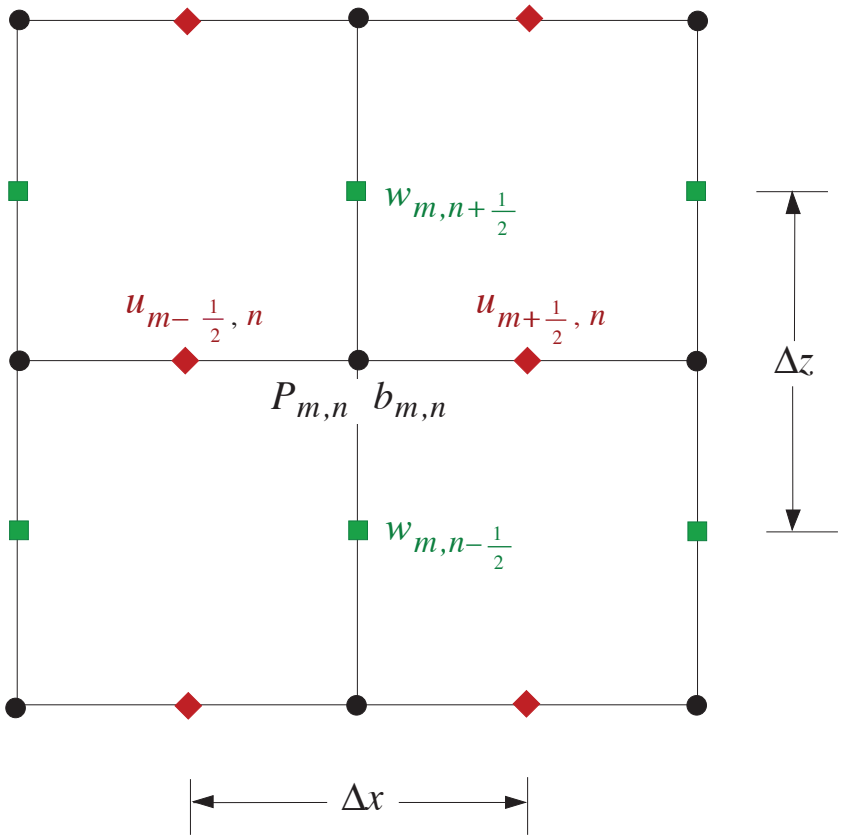
- Yes – if you are trying to achieve convergence.
- Not particularly if you are trying to improve the representation of poorly resolved scales.

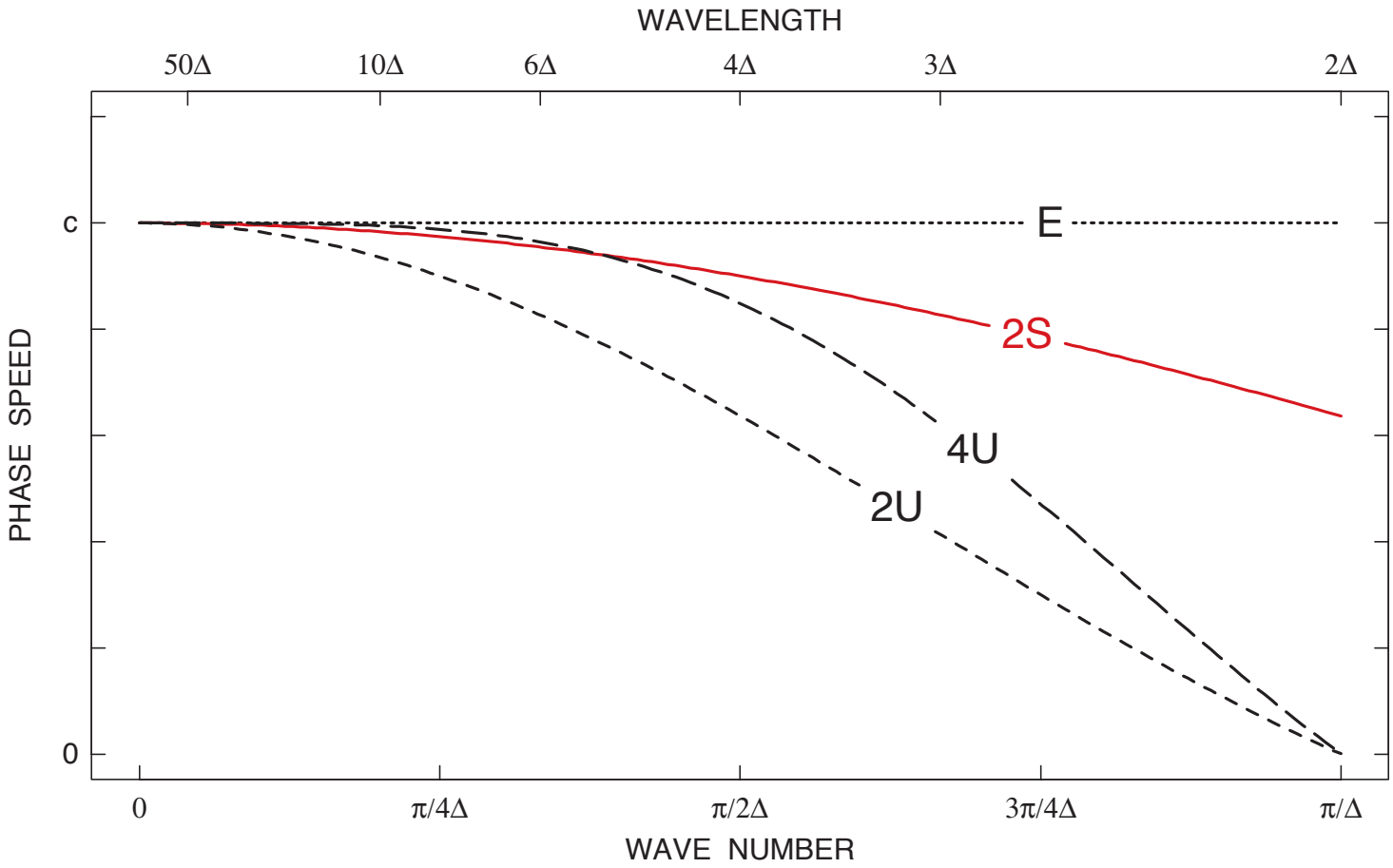
Example: Staggered Meshes

Arakawa C-grid



Example: Staggered Meshes II

Phase speeds using staggered (S) or unstaggered (U) 2nd- or 4th-order differences



Trouble with $2\Delta x$-Waves

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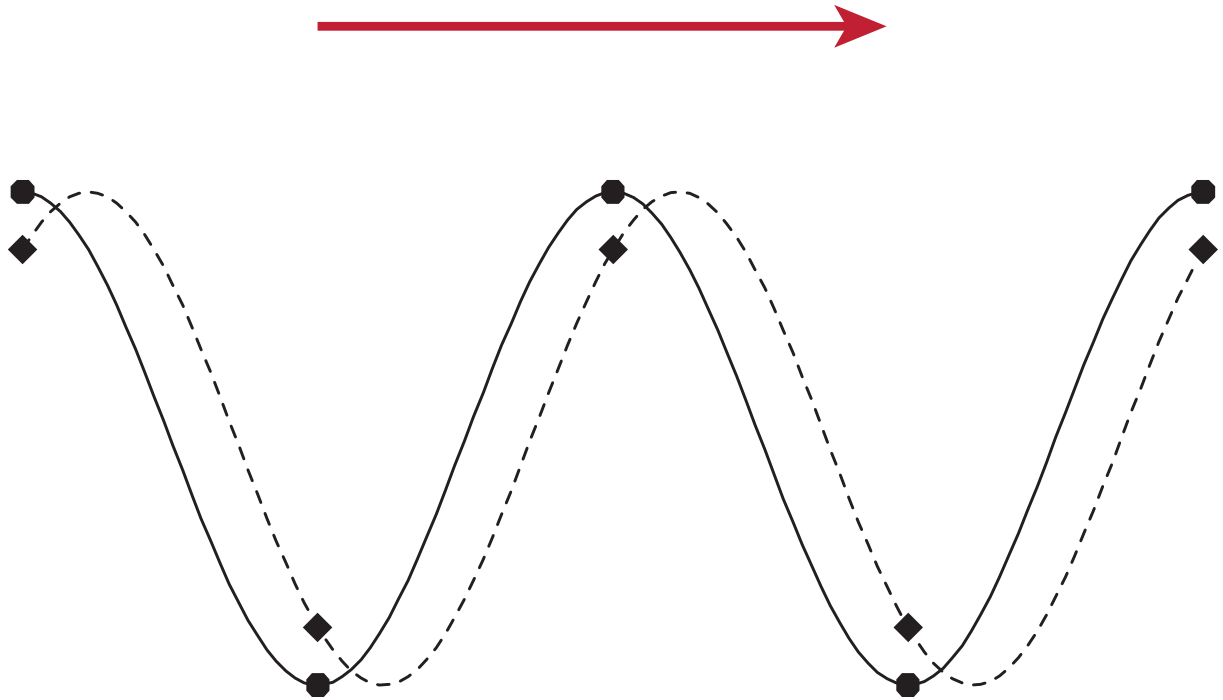
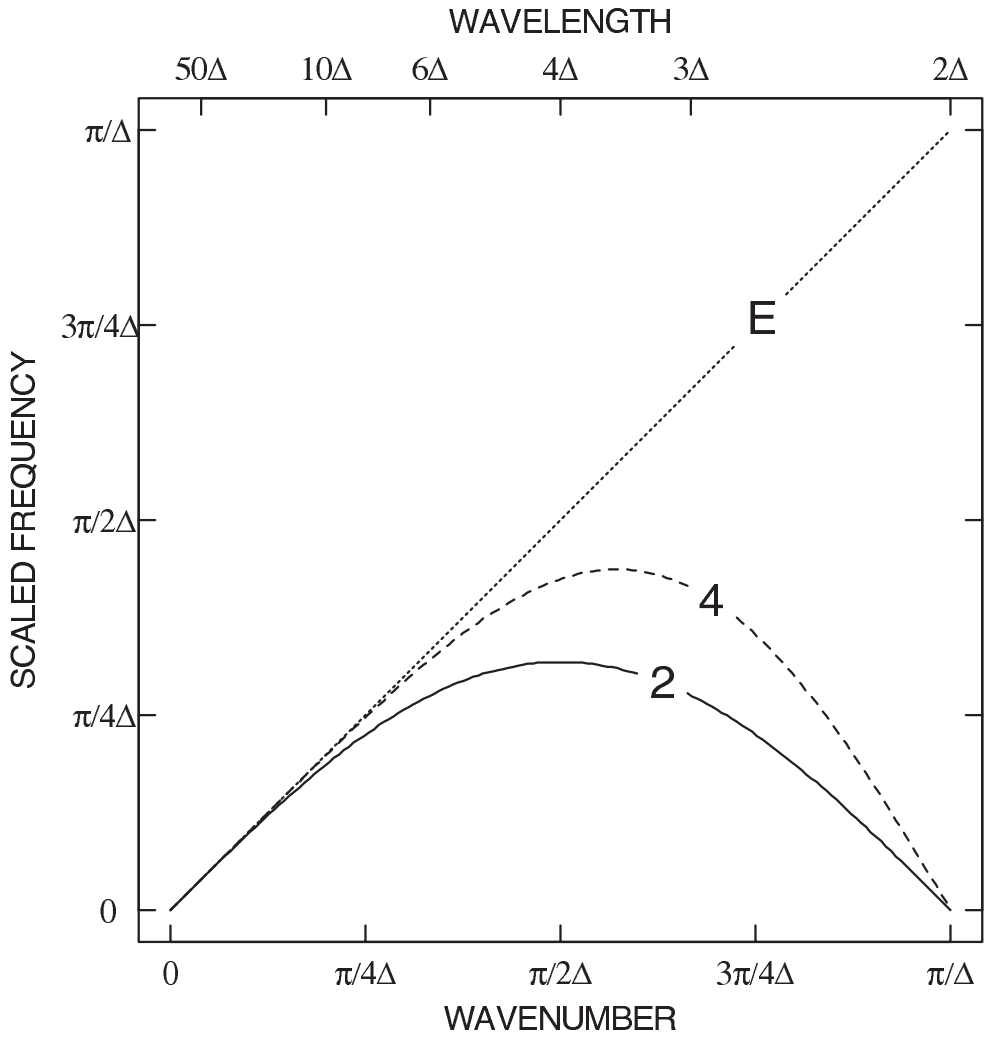


$2\Delta x$-Waves and Negative Group Velocities

Animation of $2\Delta x$-wide spike simulated by upstream and by explicit centered 2nd and 4th-order finite differences.

2Δx-Waves and Negative Group Velocities

Group velocity is $\frac{\partial(\text{frequency})}{\partial(\text{wavenumber})}$



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  - Major focus of finite-volume methods

Finite Volume Methods

- Unknowns are *cell averages*:

$$\phi_j^n \approx \frac{1}{\Delta x} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} \psi(x, n\Delta t)\, dx$$

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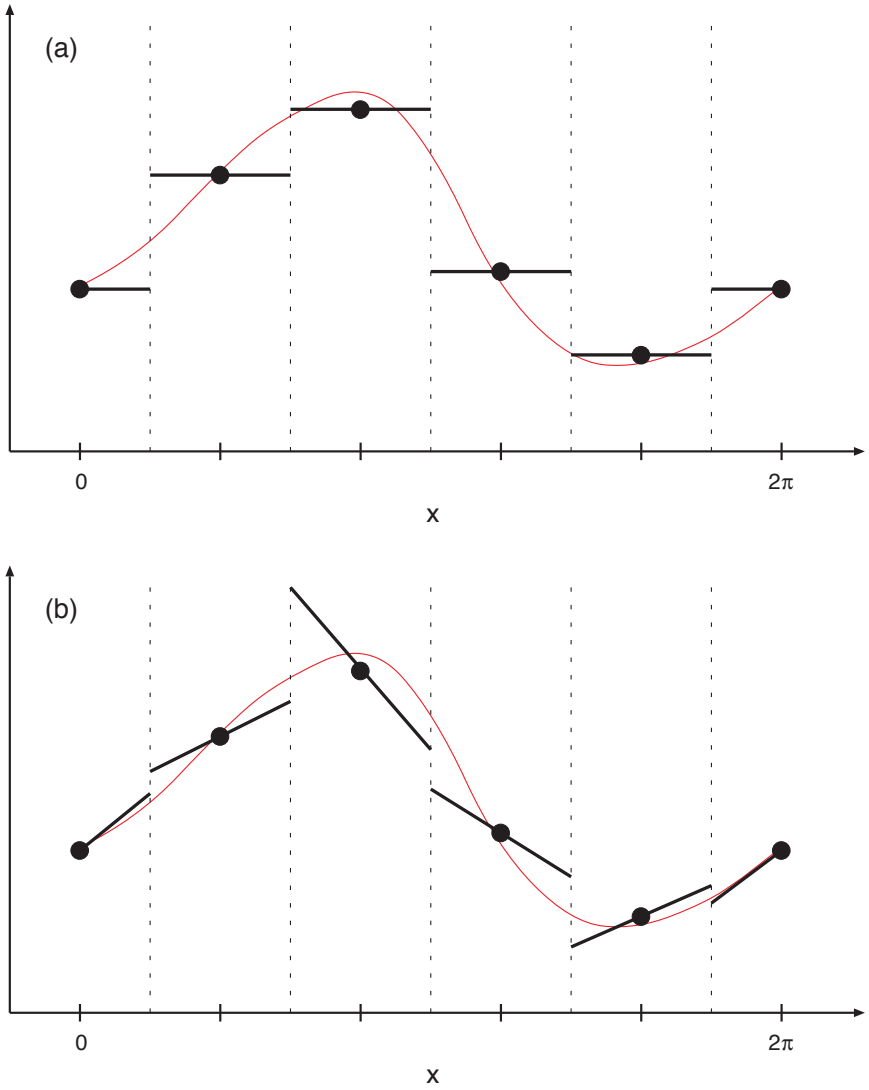
- Fluxes at cell boundaries computed from sub-cell reconstructions
- *Leads directly to two-level forward-in-time schemes*

Sub-Cell Reconstructions



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$$\phi_j^{n+1} = H(\phi_{j-p}^n, \ldots, \phi_{j+q+1}^n),$$

is *monotone* if

$$\frac{\partial\, H(\phi_{j-p}, \ldots, \phi_{j+q+1})}{\partial \phi_i} \geq 0$$

for each integer i in the interval $[j-p,\ j+q+1]$.

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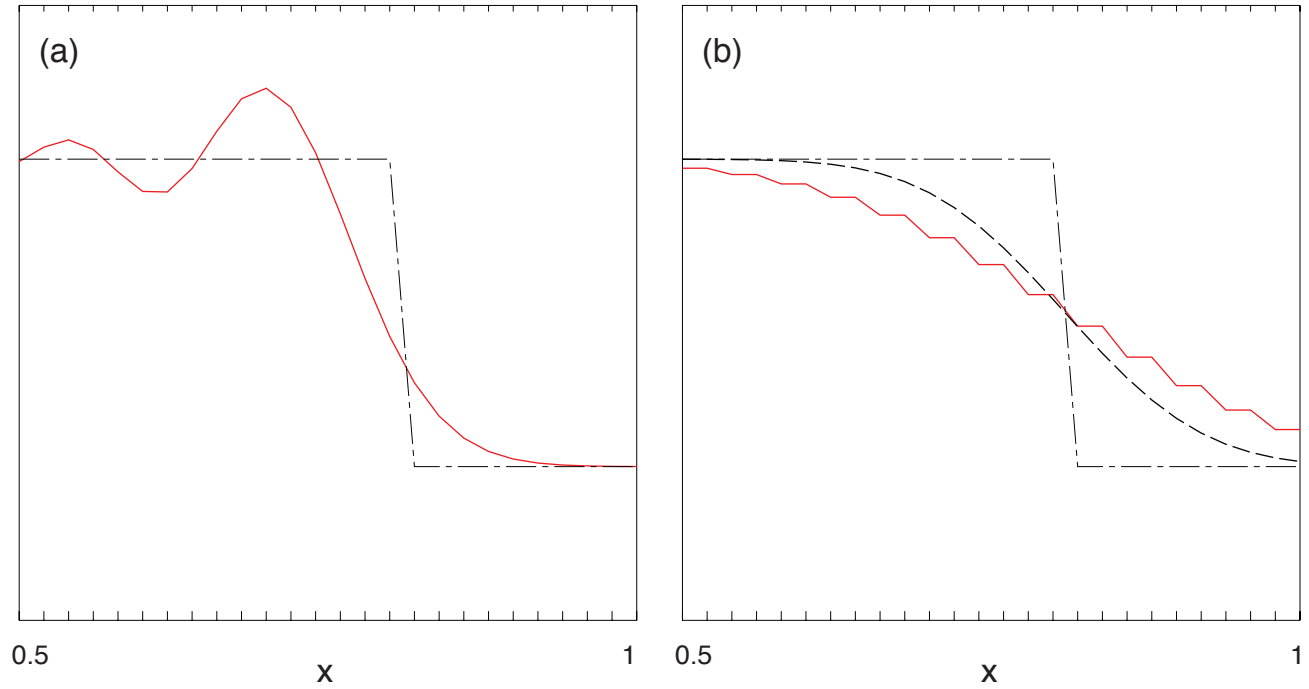
Monotone schemes

- Do not produce spurious ripples.
- Are first-order accurate.

Advection of a Step Function



- Left: 2nd-order centered with global 4th-derivative smoother
- Right: Upstream and Lax-Friedrichs (red) monotone methods

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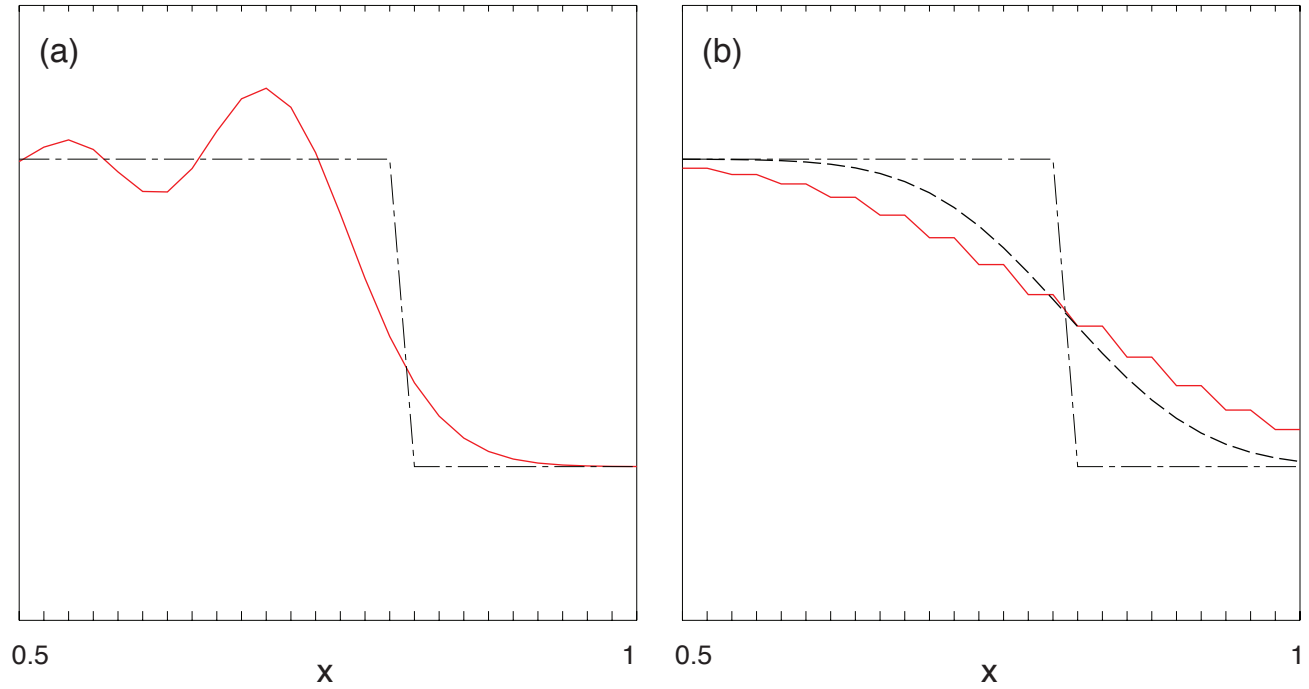

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Strategy: Scheme becomes monotone near discontinuities and is high-order elsewhere.

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When is it important in problems without shocks? *Chemically reacting flow*

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} + c\frac{\partial \psi_1}{\partial x} &= -r\psi_1\psi_2, \\ \frac{\partial \psi_2}{\partial t} + c\frac{\partial \psi_2}{\partial x} &= r\psi_1\psi_2. \end{aligned}$$

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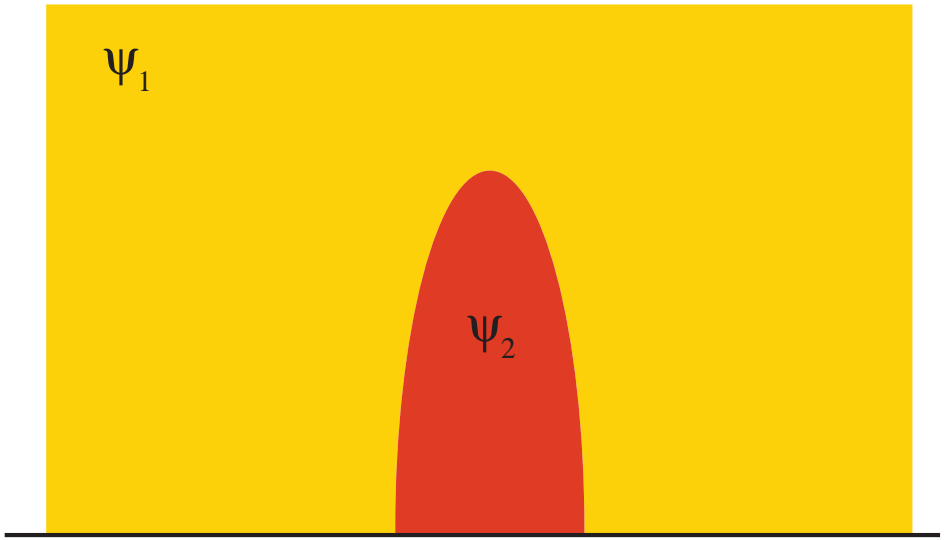

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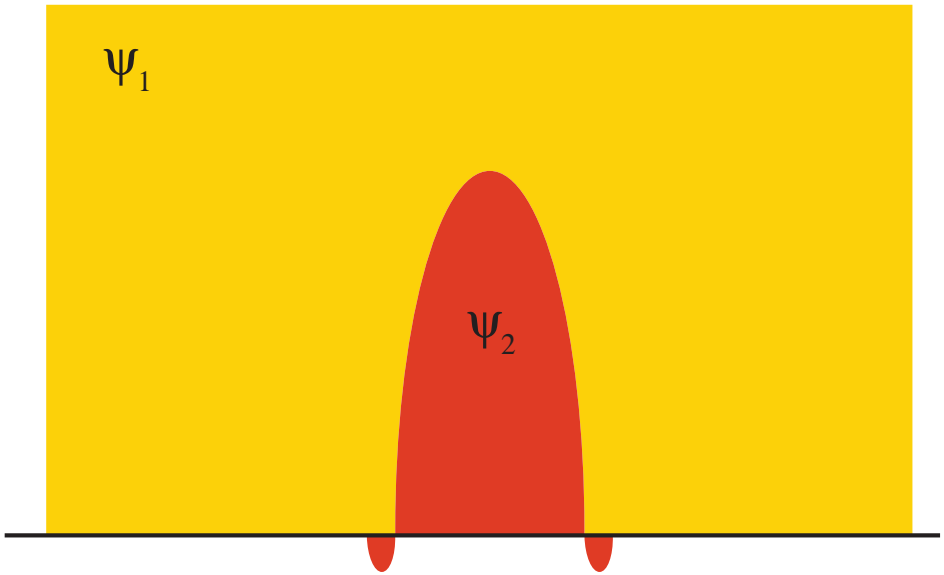
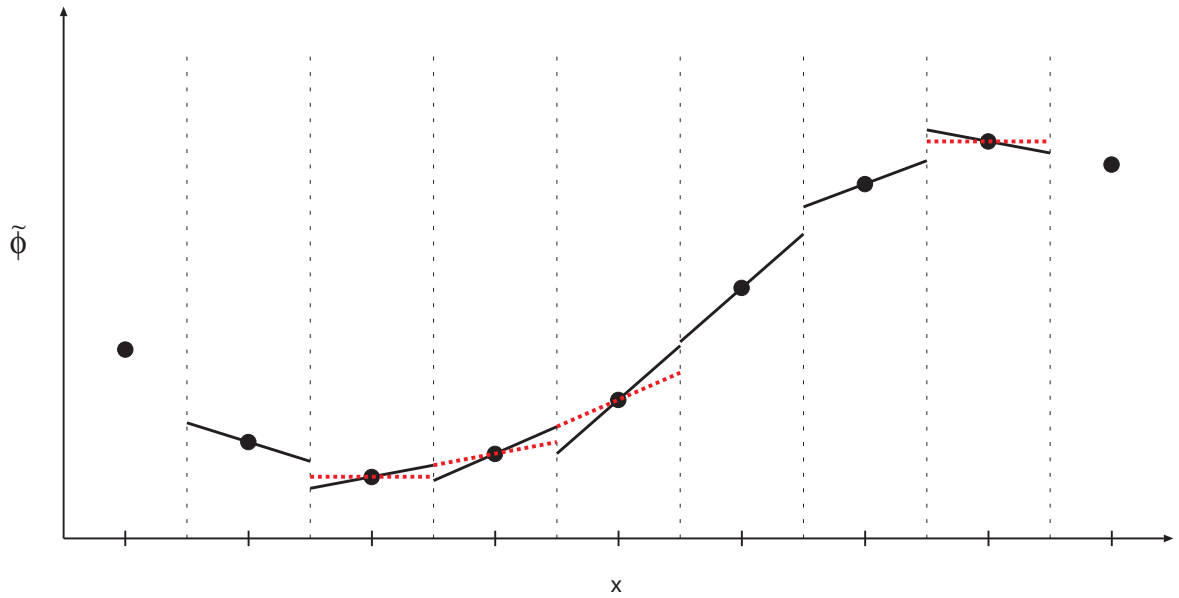


Minmod limiter

One local-smoothing strategy to prevent growth of ripples.



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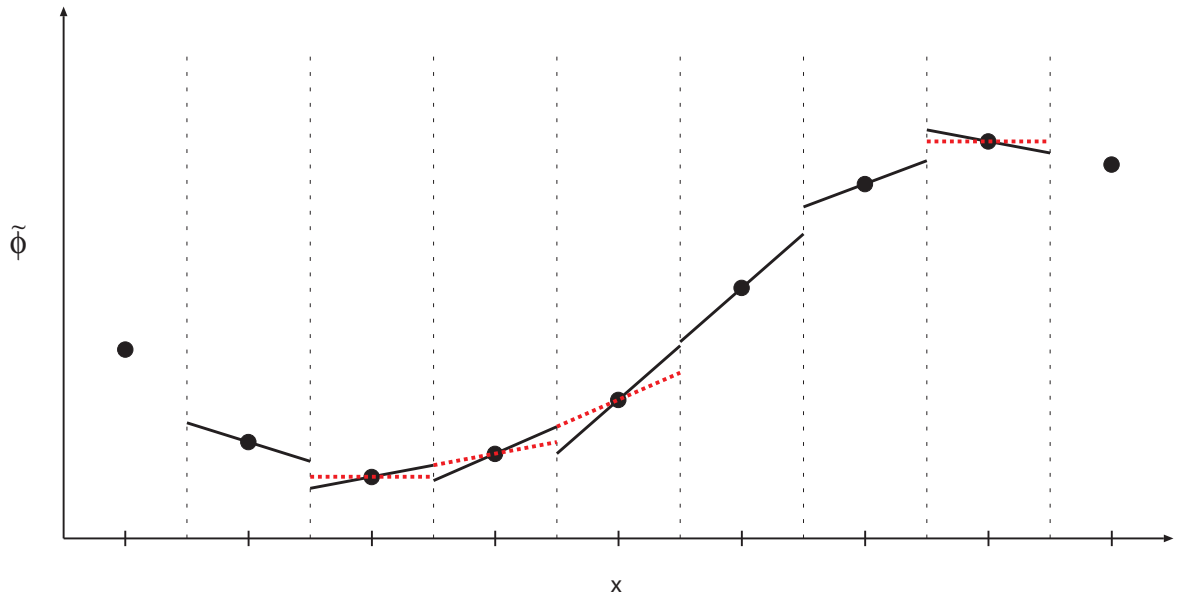

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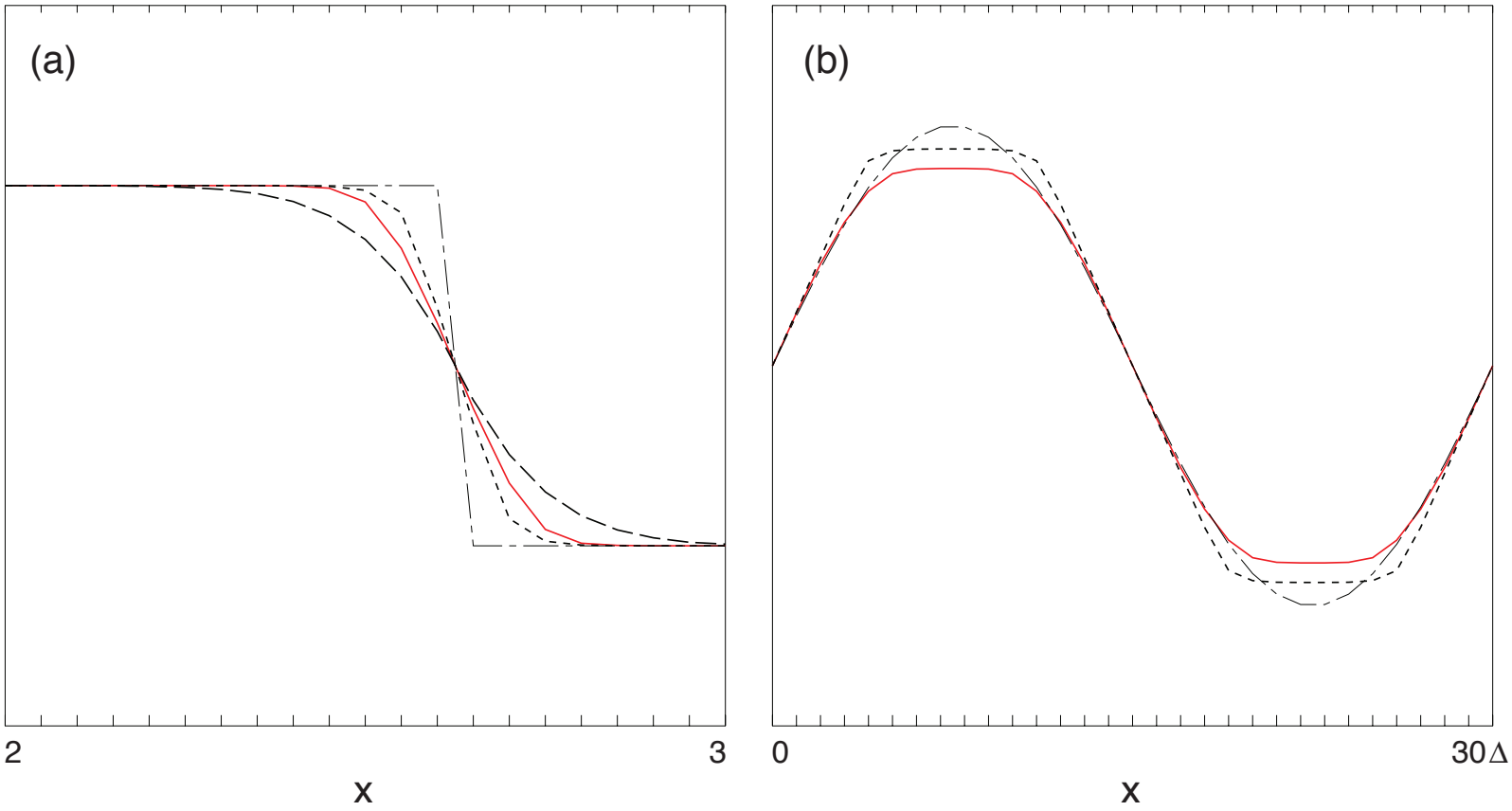
Scheme is TVD, but not monotone.

Limiters in Action

Minmod (long-dashed), MC (red), Superbee (short-dashed)



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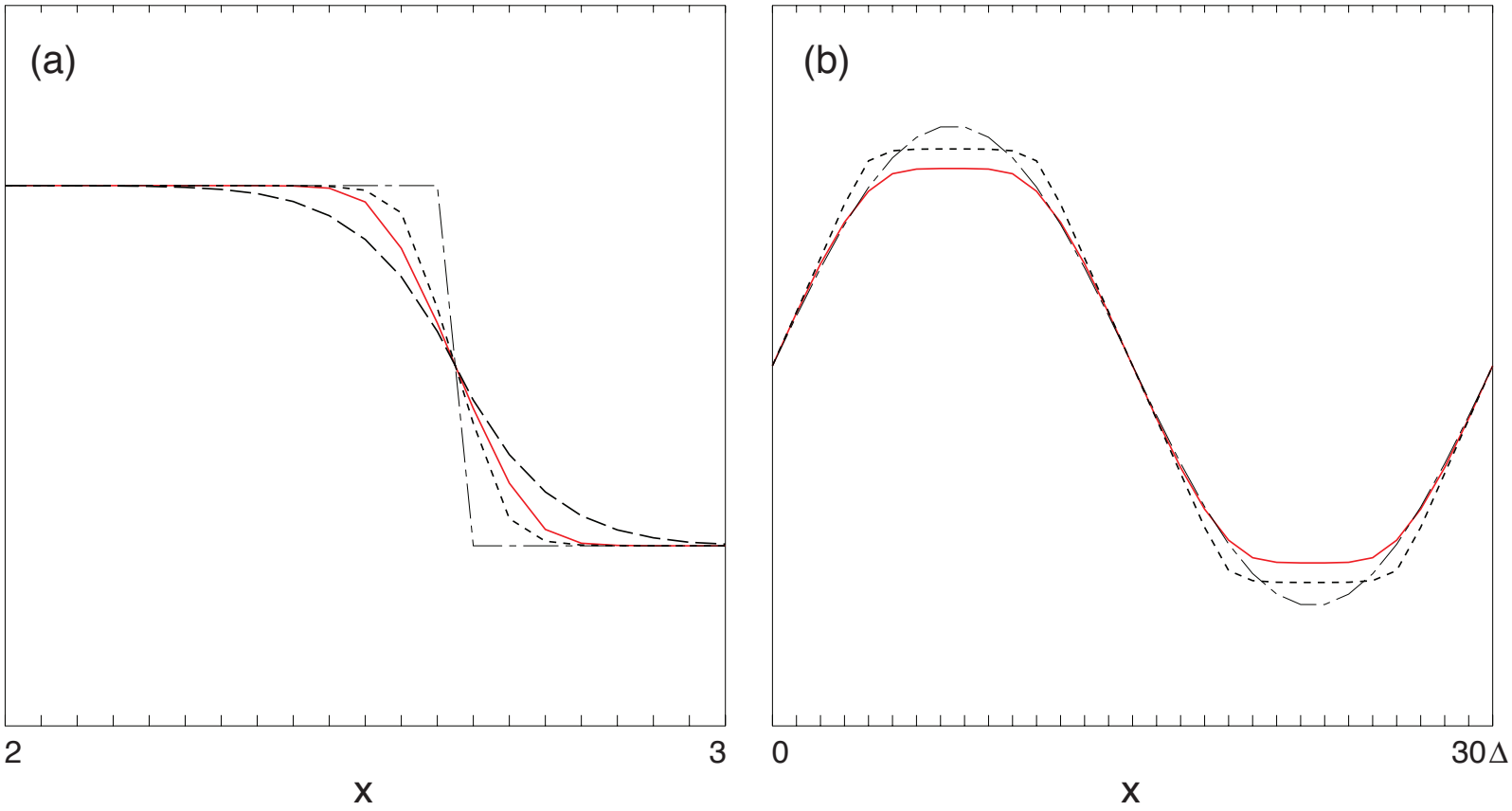

Nice job at jump, but messes up smooth extremum.

Order of Accuracy

Scheme	*Estimated Order of Accuracy*
Upstream	0.9
Minmod Limiter	1.6
Superbee Limiter	1.6
Zalesak FCT	1.7
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Avoid limiting smooth extrema!

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    - Global conservation of ϕ and ϕ^2
  - Collocation requirement that sets the residual to zero at a set of grid points
    - “Pseudo-spectral"
    - 50% faster than spectral
    - Lose conservation

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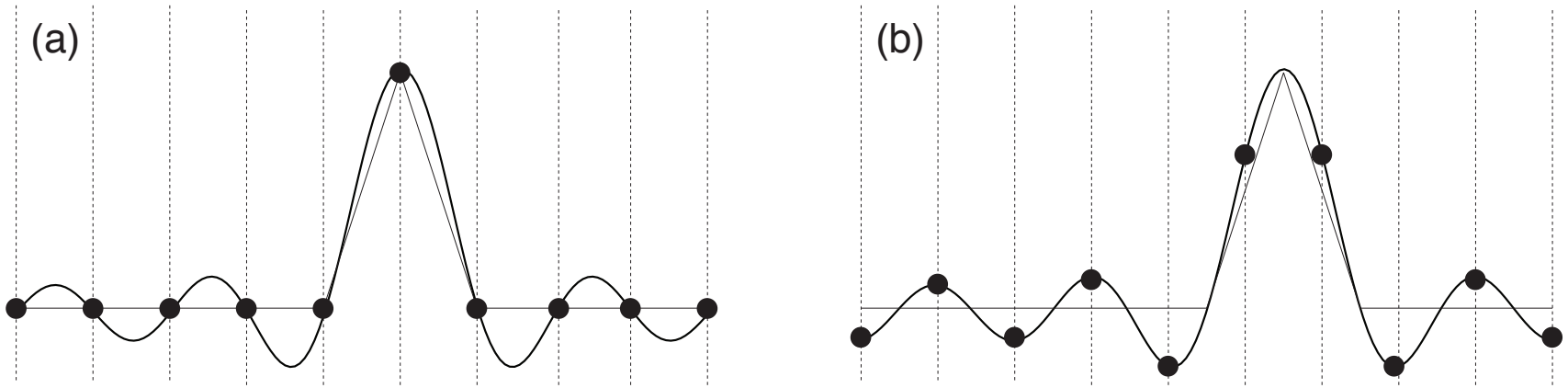
- Not conducive to preserving positivity or treating steep gradients

Global Overshoots and Undershoots with Poor Resolution



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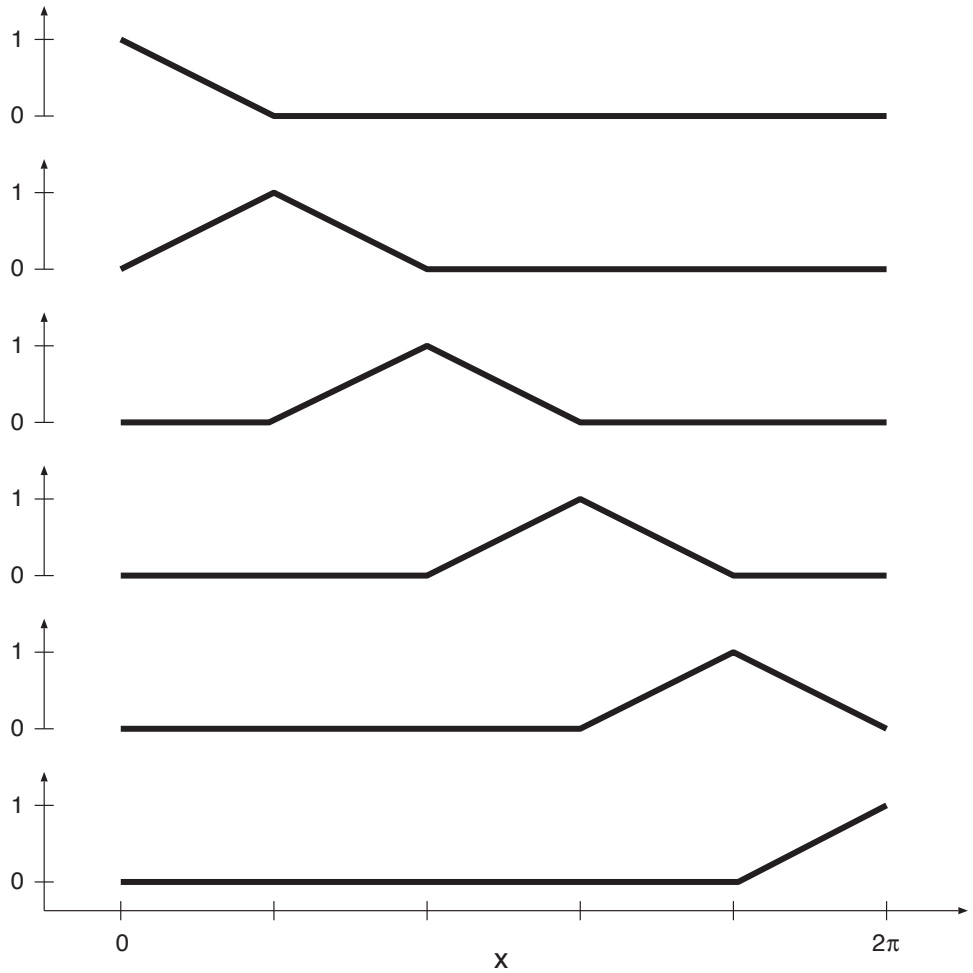
- Common choice for the dynamical variables in global hydrostatic models.
- Moisture variables often finite volume or finite difference.

Finite Element Methods

Piecewise linear elements (chapeau functions)



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- Implicit coupling in quadratic or cubic finite element methods makes them completely impractical for the simulation of waves.
- Higher-order finite element methods are the nevertheless the standard approach for solving may elliptic problems.

Finite Element versus Compact Differencing in 1D

1D advection equation

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Piecewise linear finite-elements:

$$\frac{d}{dt}\left(\frac{a_{j+1} + 4a_j + a_{j-1}}{6}\right) + c\left(\frac{a_{j+1} - a_{j-1}}{2\Delta x}\right) = 0$$

Finite Element versus Compact Differencing in 1D

1D advection equation

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$$

Piecewise linear finite-elements:

$$\frac{d}{dt}\left(\frac{a_{j+1} + 4a_j + a_{j-1}}{6}\right) + c\left(\frac{a_{j+1} - a_{j-1}}{2\Delta x}\right) = 0$$

4th-order compact scheme:

$$\frac{d\phi_j}{dt} + c\left(\frac{d\psi}{dx}\right)_j = 0$$

$$\frac{1}{6}\left[\left(\frac{d\psi}{dx}\right)_{j+1} + 4\left(\frac{d\psi}{dx}\right)_j + \left(\frac{d\psi}{dx}\right)_{j-1}\right] = \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}$$

Finite Element versus Compact Differencing in 1D

Schemes are identical!

Finite Element versus Compact Differencing in 2D

4th-order compact scheme:

$$\frac{d\phi_{i,j}}{dt} + u\left(\frac{d\psi}{dx}\right)_{i,j} + v\left(\frac{d\psi}{dy}\right)_{i,j} = 0$$

$$\frac{1}{6}\left[\left(\frac{d\psi}{dx}\right)_{i+1,j} + 4\left(\frac{d\psi}{dx}\right)_{i,j} + \left(\frac{d\psi}{dx}\right)_{i-1,j}\right] = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}$$

$$\frac{1}{6}\left[\left(\frac{d\psi}{dy}\right)_{i,j+1} + 4\left(\frac{d\psi}{dy}\right)_{i,j} + \left(\frac{d\psi}{dy}\right)_{i,j-1}\right] = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y}$$

Finite Element versus Compact Differencing in 2D

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Finite-elements: huge implicit mess (element couples with itself and 8 neighbors).

The Challenge

How do we

- Extend finite-volume methods to high order (beyond piecewise parabolic or cubic)?

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- Extend finite-element methods to high order while avoiding implicit coupling (and other bad behaviors)?
- Avoid the global coupling and $O(N^2)$ operation counts to update a set of expansion coefficients in spectral or pseudo-spectral methods?

One Solution

Use *h*-*p* methods – break domain into *h* elements and represent the solution within each element by orthogonal polynomials of maximum order *p*.

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- Spectral element methods – solution is continuous at cell boundaries (good for approximating 2nd-order derivatives).
- Discontinuous Galerkin methods – solution is discontinuous across cell boundaries (localizes communication between cells).

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Enforce Galerkin criterion locally, within each element.

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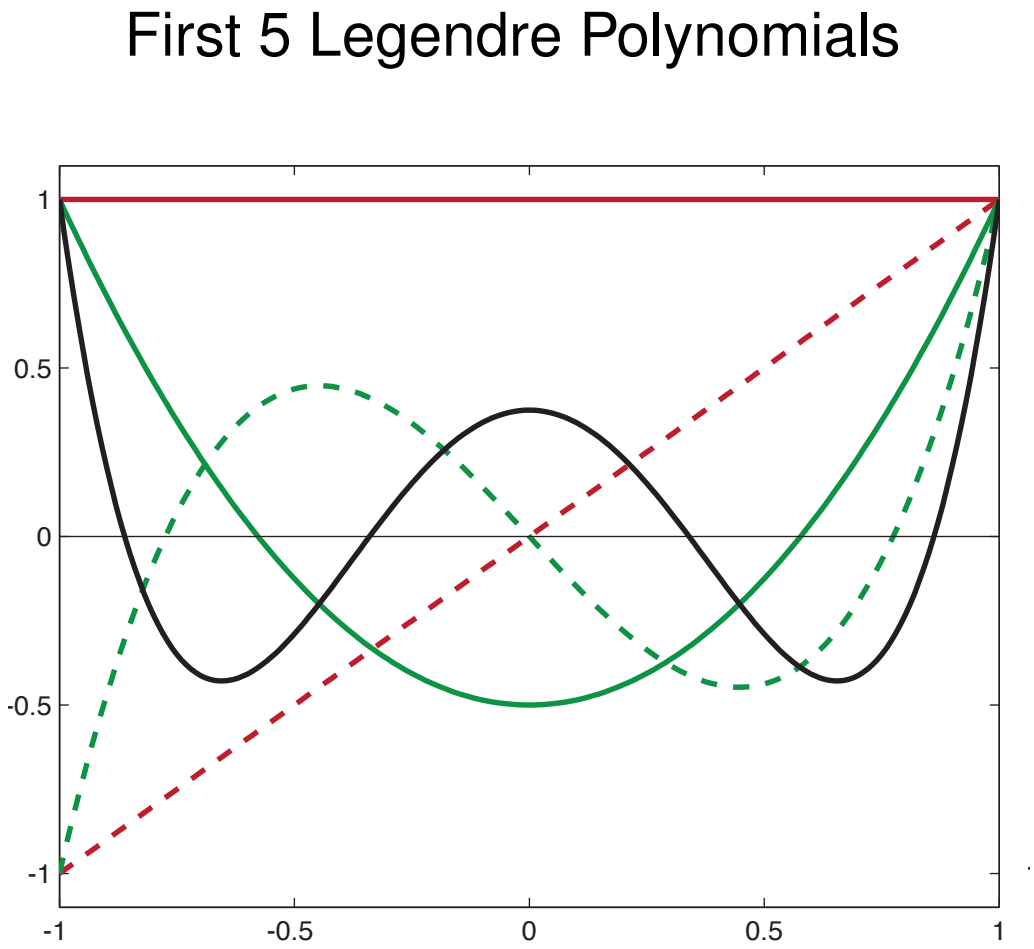
Polynomial structure within each element is either

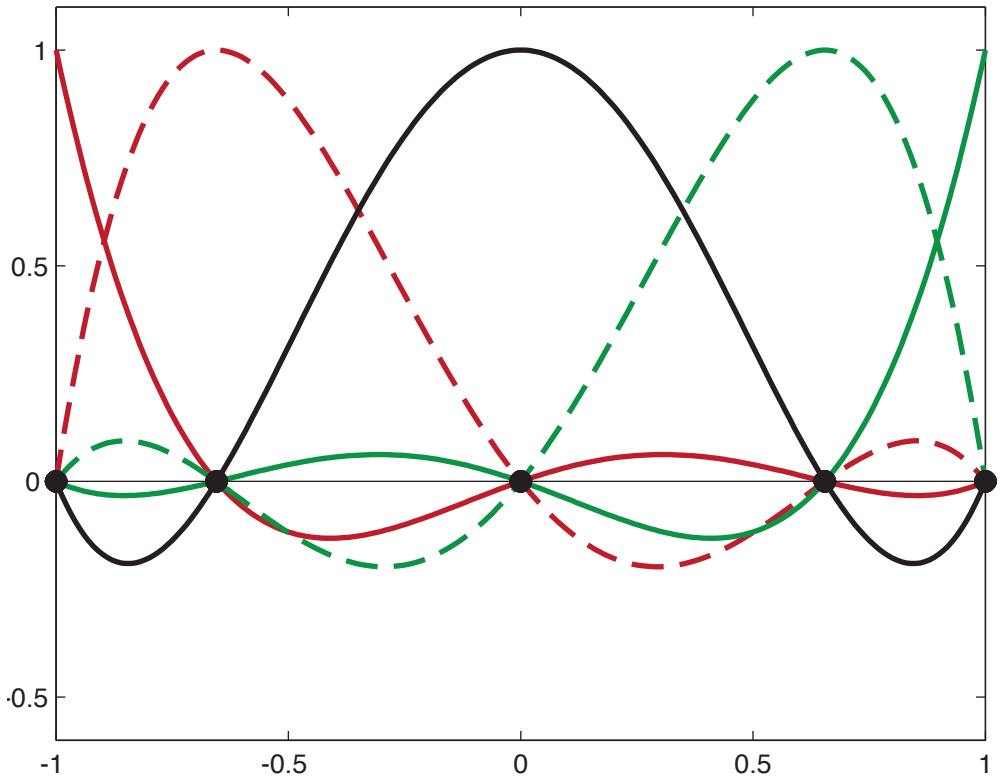
- *Modal* variant: Legendre polynomials
  - Orthogonal on $[-1, 1]$ with weight function unity
- *Nodal* variant: Lagrange polynomials interpolating the Gauss-Legendre-Lobatto (GLL) points
  - Most accurate node placement for quadrature when there is a node at each end of the interval
  - Polynomials are not truly orthogonal
  - Discrete integrals are orthogonal due to the approximate quadrature on the GLL nodes

Modal DG



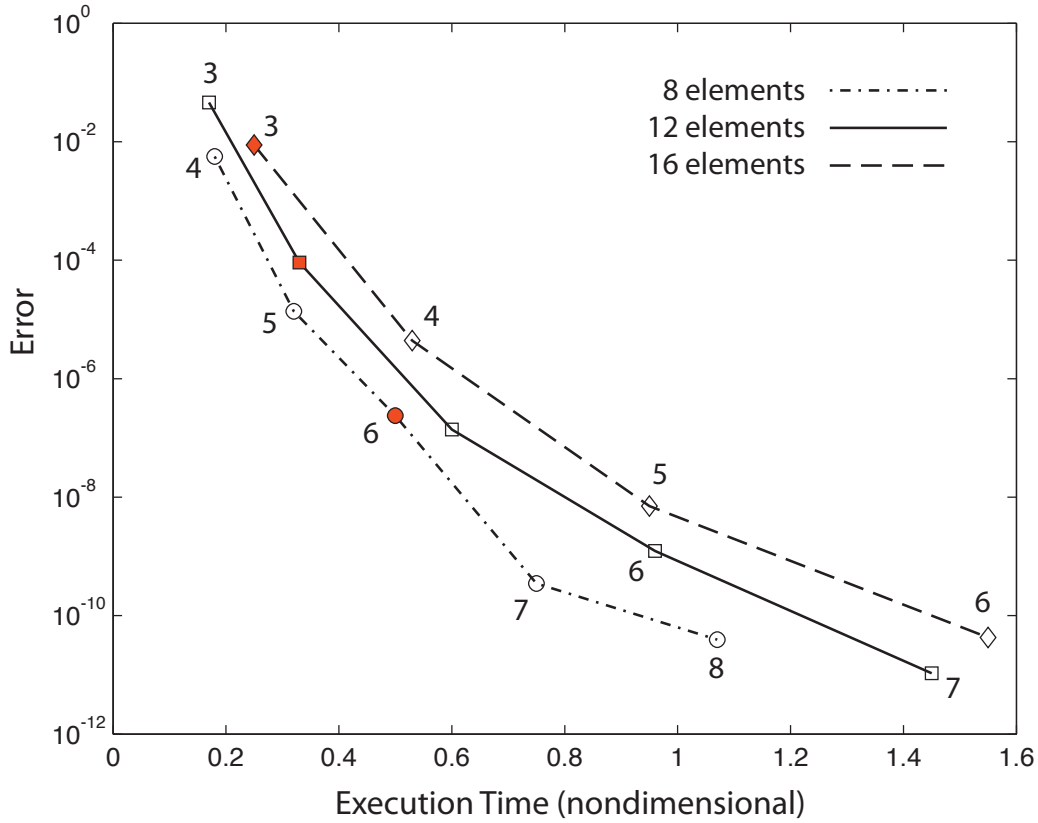
Nodal DG

Lagrange Polynomials Interpolating 5 GLL Nodes



Convergence as a Function of Resolution

Nodal DG errors as a function of execution time



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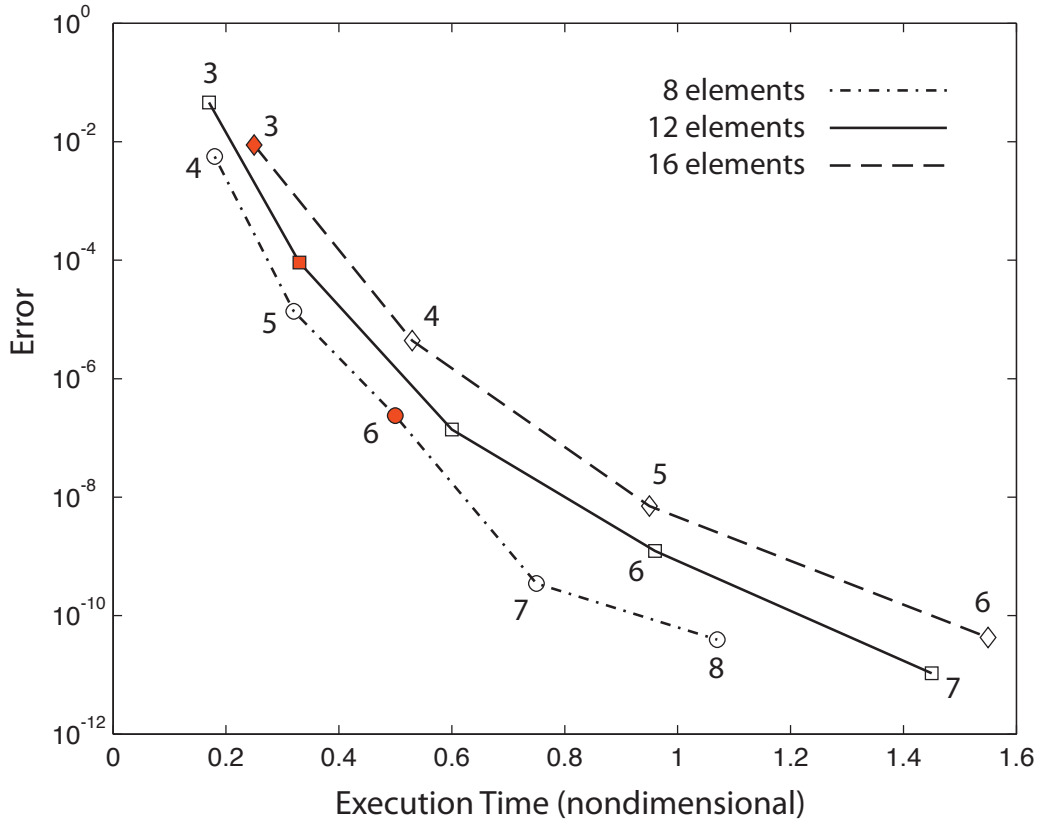
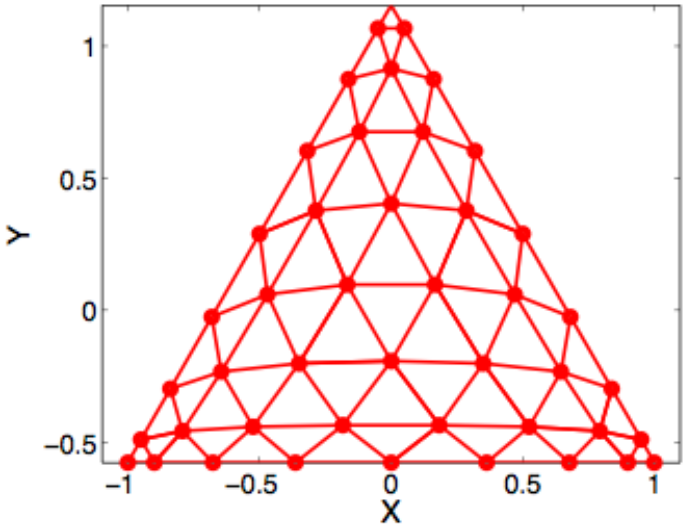


Most efficient way to reduce error is to increase the polynomial order.

DG Considerations



Advantages:

- Suitable for massively parallel computing

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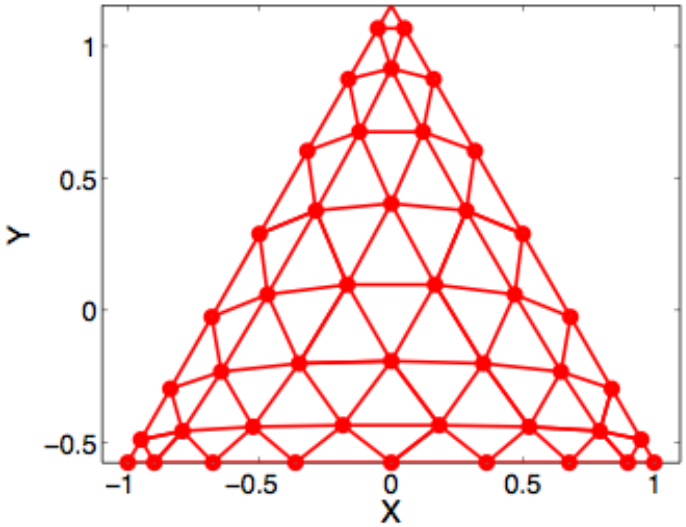

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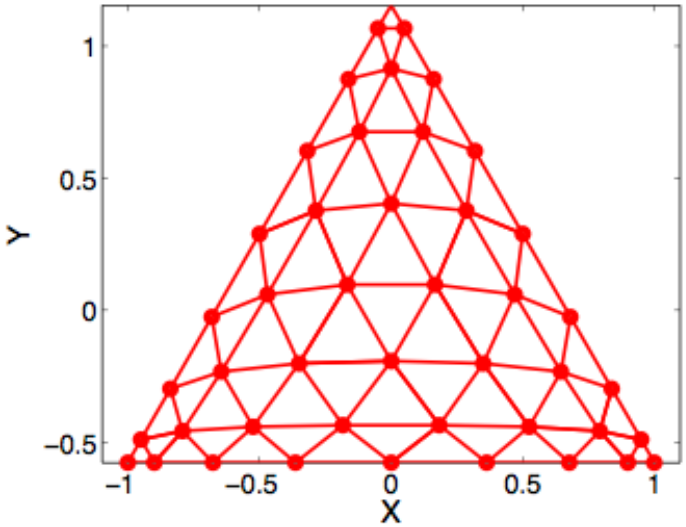

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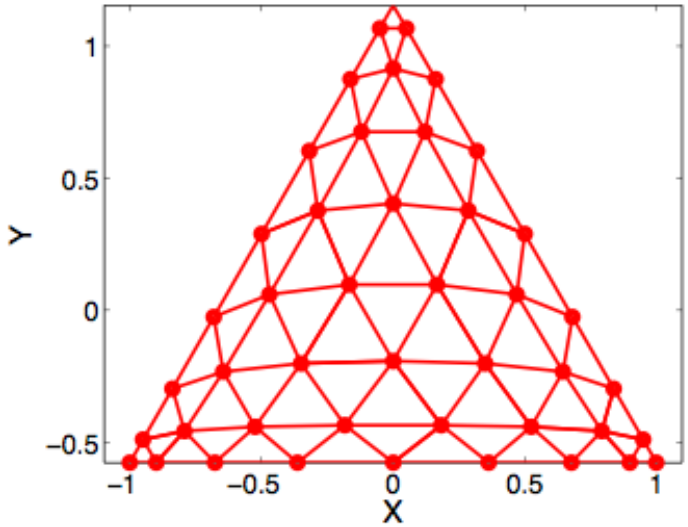

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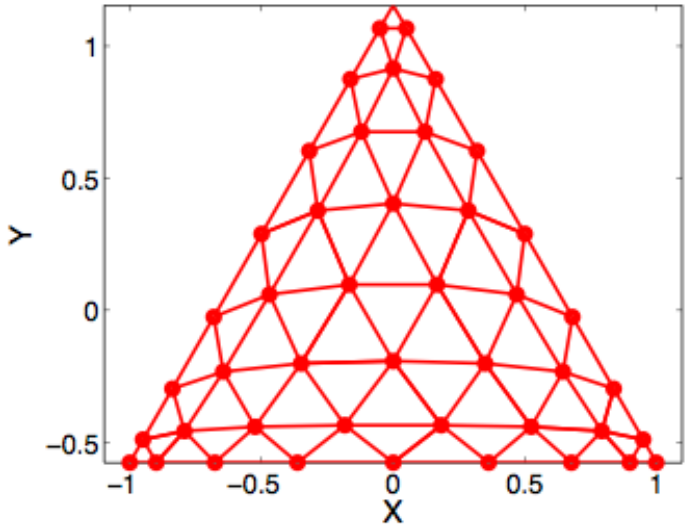
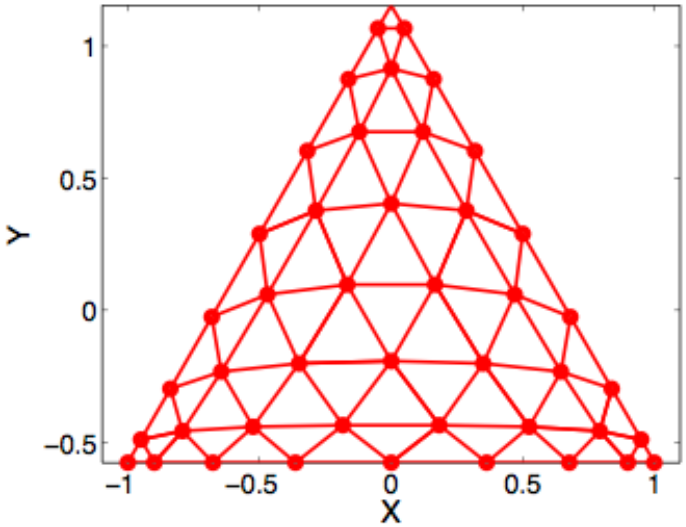


Advantages:

- Suitable for massively parallel computing
  - Communicates via fluxes with nearest neighbors only
  - For high order polynomials, solution relatively insensitive to flux formulation
  - Lots of work to do within each element
- Rapid convergence for smooth solutions

DG Considerations II



Disadvantages:

- Requires very short time step
  - Grid spacing reduced toward element boundaries

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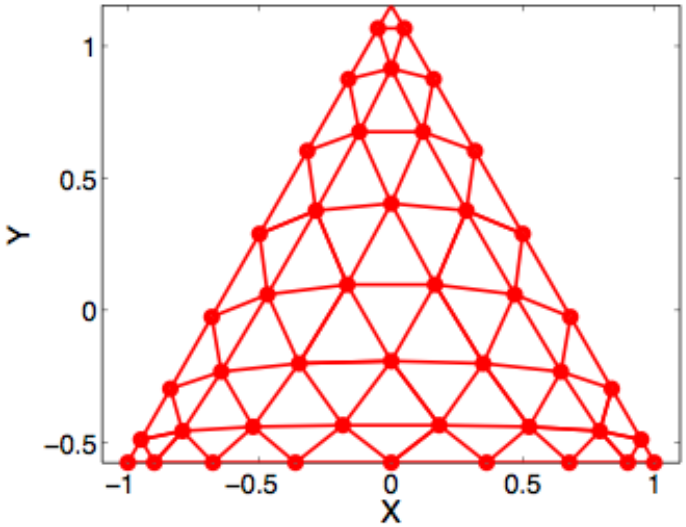

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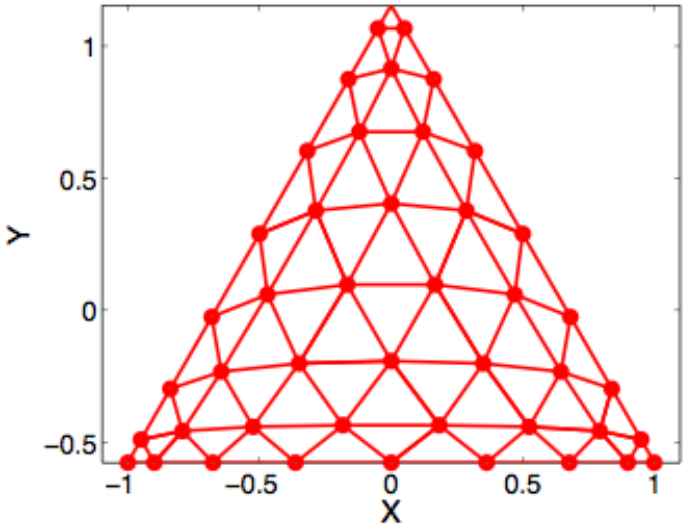

Disadvantages:

- Requires very short time step
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- Discontinuities
  - Can be accommodated across element boundaries by limiting the fluxes
  - Cannot naturally be accommodated within each element

Eulerian

Accelerations evaluated at fixed points.

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Accelerations evaluated along fluid parcel trajectories

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Semi-Lagrangian

Fluid parcels arrive at every node on the specified mesh at the *end* of each time step.

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Semi-Lagrangian approximation to the advection equation is

$$\frac{\phi(x_j, t^{n+1}) - \phi(\tilde{x}_j^n, t^n)}{\Delta t} = 0,$$

where $\tilde{x}_j^n$ denotes the departure point of a trajectory originating at time t^n and arriving at (x_j, t^{n+1}).

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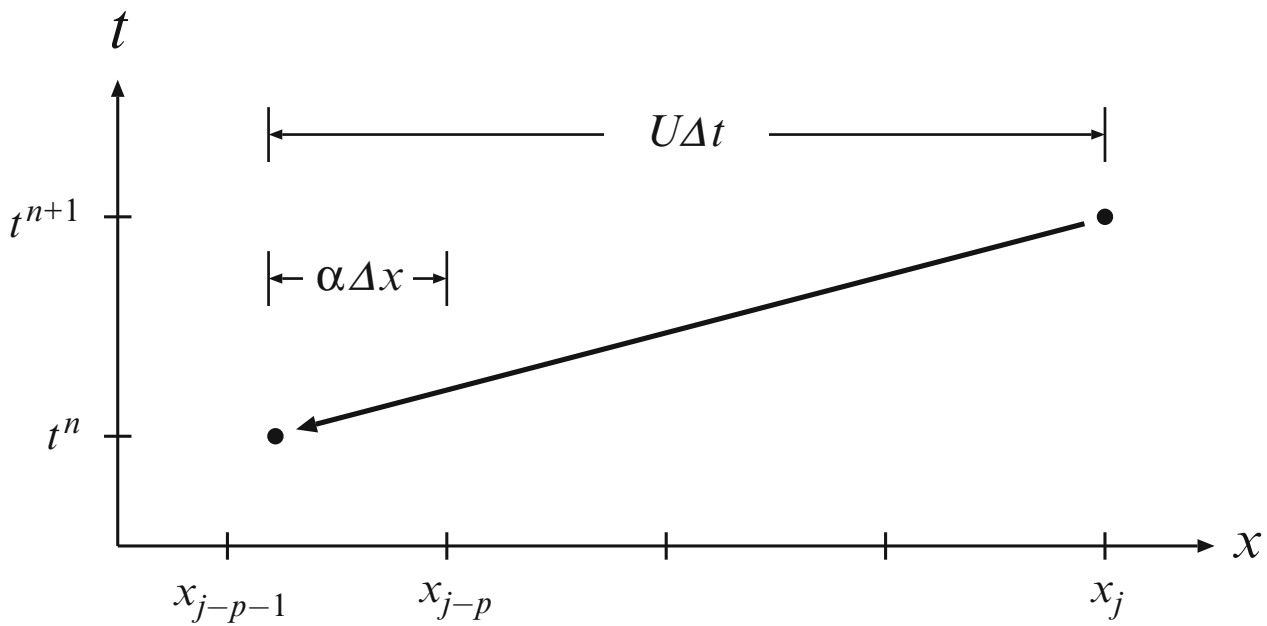
Let p be the integer part of $U\Delta t/\Delta x$, then

$$x_{j-p-1} \leq \tilde{x}_j^n < x_{j-p}$$



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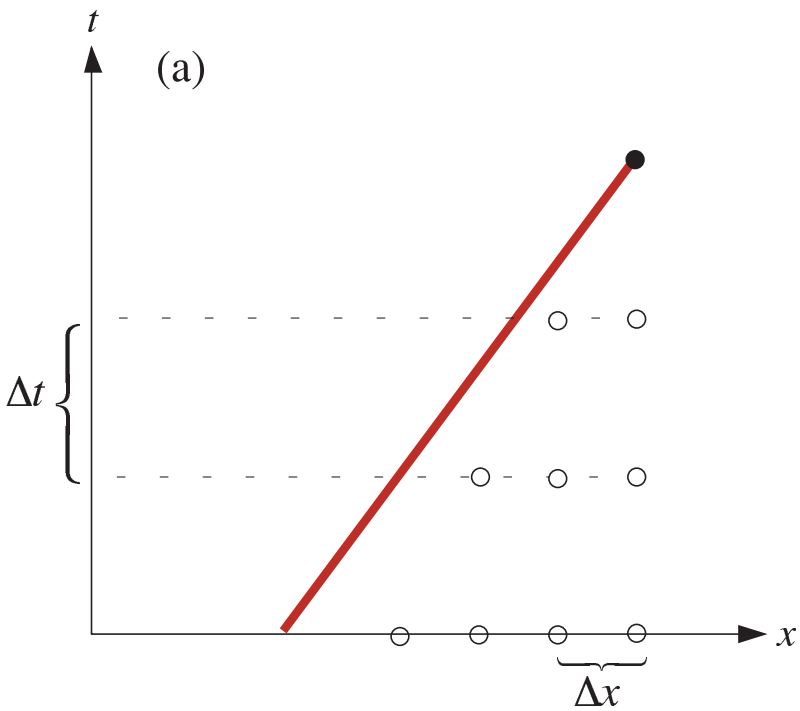
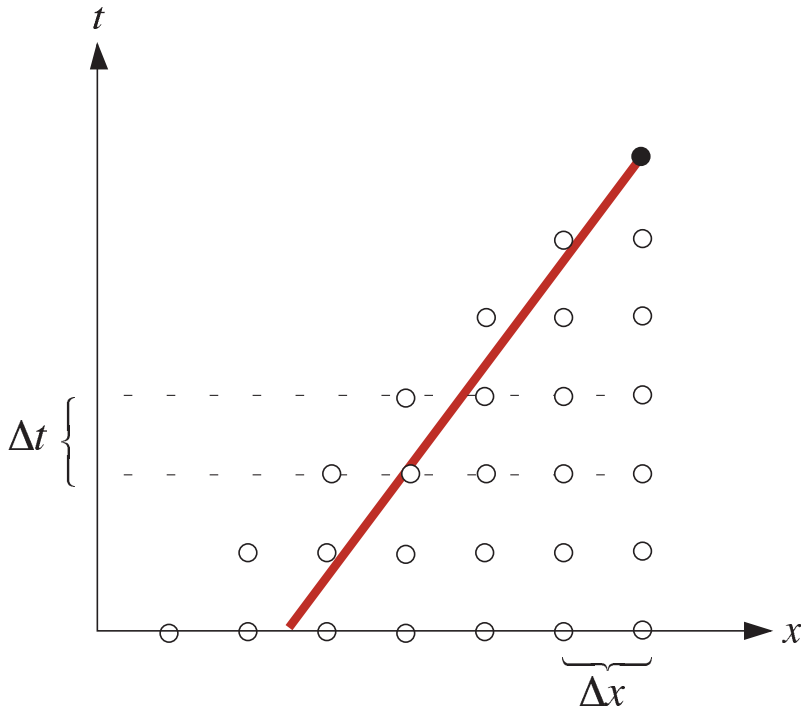
Courant-Freidrichs-Levy Condition: the numerical domain of dependence must include the domain of dependence of the true solution. (Necessary but not sufficient for stability.)

Eulerian CFL

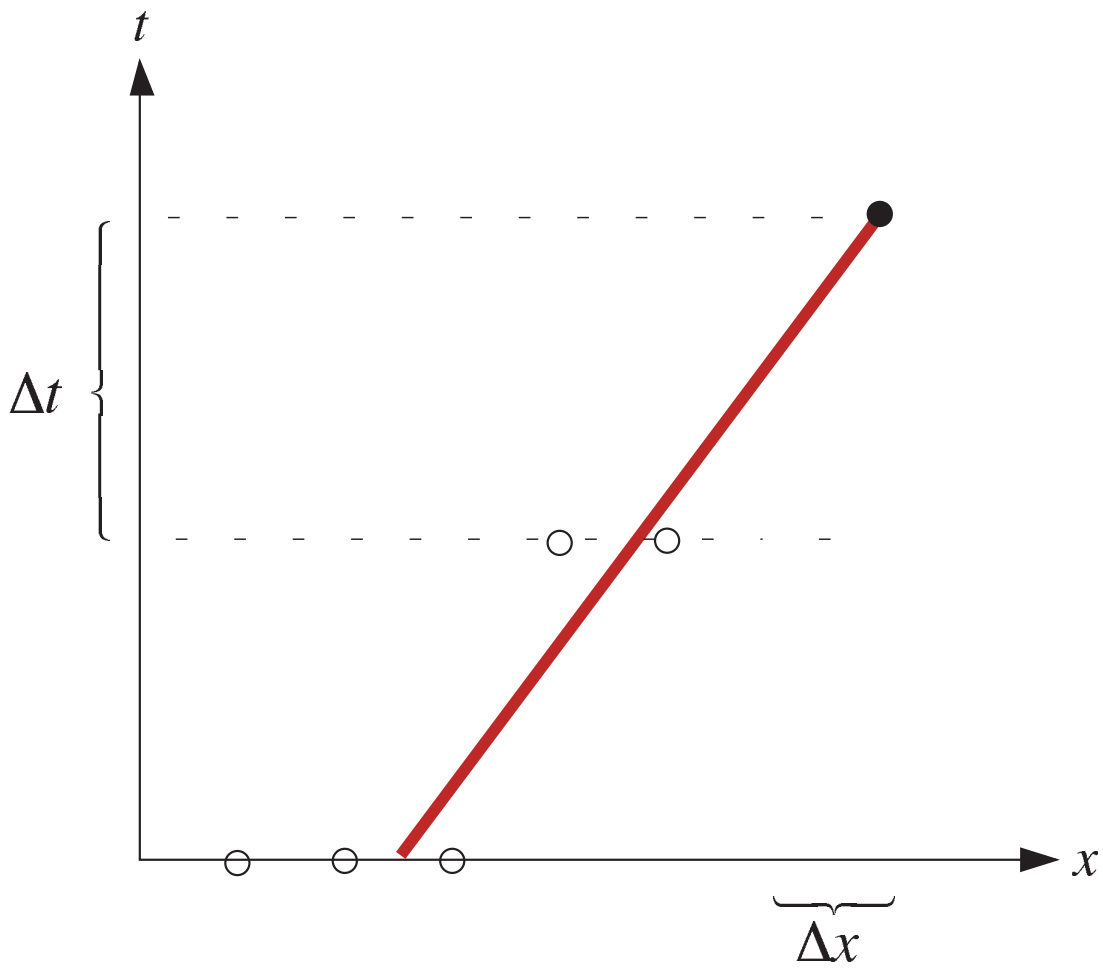
Upstream differencing, constant-windspeed advection





Semi-Lagrangian CFL

Linear interpolation to departure point, constant-windspeed advection



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  - Trajectory morphology *not* simpler than the advected fields.

Reference

Durran, D.R., 2010: *Numerical Methods for Fluid Dynamics: With Applications to Geophysics. 2nd Ed.* Springer.