Multivariate Operator Theory

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1 Overview of the Field

While the study of operator theory on Hilbert space has been underway for more than a hundred years, multivariate operator theory - the study of more than one operator at a tim - is of more recent origin. The study of self-adjoint algebras emerged in the late thirties in the works of Murray-von Neumann and Gelfand-Naimark. A couple of decades later, non self-adjoint algebras enjoyed considerable development in the sixties and seventies starting with the work of Kadison-Singer. Studies of multivariate operator theory which emphasized the analogues of analyticity, both commutative and non commutative, had to wait for the most part until the last couple of decades. Since then, however, this area has been pursued rather vigorously with some remarkable successes, both for its own sake and for its connections with other areas of mathematics such as complex and algebraic geometry. Moreover, the techniques and viewpoint from the multivariate case has enriched and contributed to the one variable theory. Although analyticity has played a key role in operator theory from the start, due in part to the analyticity of the resolvent, the algebraic and geometric underpinnings of function theory were less important in the one variable case. However, even natural examples in the multivariate case involve the framework of several complex variables in an essential way and depend on results from the theory of partial differential operators. One obstacle to the systematic development of this side of multivariate operator theory has been the development of an effective framework for the subject. Early researchers considered either n-tuples of operators or representations of algebras. Many researchers now have adopted the language of Hilbert modules.

A *Hilbert module* H over algebras of holomorphic functions such as the polynomials or entire functions depending on n (commuting) complex variables yields a variety of invariants, either with respect to topological isomorphism or the more rigid unitary equivalence. First, such a module defines a compact subset of $\mathbb{C}^n$, known as the Taylor joint spectrum. The Hochschild-type topological-homological localization encodes the refined structure of the joint spectrum with local invariants such as the Fredholm index, the local analytical K-theoretic index, a Hilbert-Samuel polynomial, or, on restricted subsets of the joint spectrum, a Hermitian holomorphic vector bundle with canonical connection and related curvature. All of these invariants, with pertinent examples and applications, were studied in the last two decades by (to name only a few) J. L. Taylor, F.-H. Vasilescu, R. Levi, X. Fang, K. Yan, G. Misra. W. B. Arveson and the four organizers of this workshop. Applications range from a novel proof of the Atiyah-Singer index theorem, of Grauert's finiteness theorem in complex analytic geometry, and to a classification of homogeneous Hermitian holomorphic vector bundles on classical domain of $\mathbb{C}^n$.

Many concrete Hilbert modules obtained by completing the polynomial algebra $A = \mathbb{C}[z_1, ..., z_n]$ with respect to an inner product are essentially reductive in the sense that all of the multiplication operators given by the module structure are essentially normal. Examples are the Hardy and Bergman spaces on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ as well as the symmetric Fock or n-shift space H^{2n}. In the case of the latter module, Arveson raised the question of whether this property is inherited by submodules $[I]$ (or, equivalently, the quotient $H/[I]$) obtained as the closure of a homogeneous ideal I in A and gave a positive answer for ideals generated by monomials. Arveson conjectured that the result was true in general and that, in fact, the cross-commutators were not only compact but were in the Schatten p-class for $p > n$. Arvesons conjecture was refined and extended by Douglas in two ways. First, he conjectured that for the cross-commutators of the operators defined on the quotient module were in fact in the Schatten p-class for p greater than the dimension of the zero variety, $Z(I)$, of I and established that was the case for ideals generated by monomials. Second, he observed that the quotient module defines an extension of the compact operators by $C(Z(I))$ and hence defines an element of the odd K-homology group for the intersection $Z(I)$ of $Z(I)$ with the unit sphere. Moreover, he conjectured that this element corresponds to the fundamental class determined by the almost complex structure on $Z(I)$. Such a result would yield a new kind of index theorem. (One can compare this conjectured result with the corresponding result for the K-homology class defined by a Dirac operator on a spinc-manifold.) Exciting progress has been made recently on these questions, mostly by Guo and Wang. In particular, they establish the original conjecture for all principal homogeneous ideals and for all homogeneous ideals for $n = 2, 3$. Moreover, they establish the refined conjecture involving the dimension of $Z(I)$ for $n = 2, 3$, and verify the index formula for $n = 2$. Finally, they show that the K-homology element is nontrivial for all proper ideals for $n = 3$. Principal tools, among other things, are the Hilbert-Samuel polynomial of M and techniques developed in the rigidity theory of analytic Hilbert modules. There should be interesting connections to recent results of X. Fang and J. Eschmeier on operator theoretic Samuel multiplicities and of Gleason-Richter-Sundberg on the index of invariant subspaces of analytic Hilbert modules. The above result on the essential normality of submodules and quotient modules are equivalent to compactness properties of commutators of the Bergman projector and multiplication by smooth functions in the case of certain weighted Bergman spaces, which, at present, cannot be obtained with the methods from PDE or SCV. In particular, there would seem to be implications from these results for global regularity for the $\overline{\partial}$-bar problem on such spaces over the unit ball. For these reasons and others, it would be extremely interesting to extend these results to higher dimensions or domains other than the unit ball. As one possible, quite fascinating application, one might hope to obtain operator theoretic invariants for the Brieskorn exotic spheres.

In addition to the specific problems described above, there are many basic questions in the field whose answers would have significant application to the general field as well as related areas. As was mentioned earlier, much of the progress which has been made has depended on the development of techniques which often arose in connection with similar basic questions. The rigidity theory developed to classify submodules defined by ideals and the Hilbert-Samuel polynomial used to classify certain Hilbert modules defined by isometries by Fang are two examples as is the functional calculus developed by Taylor and other researchers to define the joint spectrum. Among the basic questions for which there has been significant progress but for which many questions remain are interpolation and division. The seminal work connecting interpolation to operator theory is due to Sarason. His approach was absorbed into dilation theory and furthered the development by Sz.-Nagy and Foias of their canonical model for contraction operators. Work of Arveson extended dilation theory to a much more general context by showing that any (not necessarily commutative) operator algebra lives inside a canonical C*-algebra known as its C*-envelope. Interpolation problems can be reformulated in terms of calculating the C*-envelope of a quotient algebra of the space of analytic functions by the ideal of functions vanishing on the data set. Deep results of Agler for interpolation on an annulus make essential use of these operator theoretic ideas. Direct connections between interpolation and the Nevanlinna-Pick problem for the bidisk were obtained by Agler and McCarthy. More recent work of McCullough and Paulsen show in that for other domains, in both one and several variables, the C*- envelope of even three dimensional abelian quotients can be infinite dimensional and highly non commutative. These results illustrate a principle which is a second side of multivariate operator: understanding commutative phenomena often takes one naturally into the non commutative realm. Such lessons in classical algebraic geometry and theoretical physics are leading to the development of non commutative algebraic geometry in recent years. Connections between the work described in this overview and these latter developments seem likely.

The Sz.-Nagy-Foias machinery for studying a single operator has been extended to multivariate operator theory by Popescu, Davidson-Kribs-Shpigel and others based on the Frazho and Bunce dilation of a row contraction to a canonical model of m shifts on the full Fock space. Again, even when the operators all commute, the natural dilation theory leads to non-commutative C*-algebras. For example, if the model is of a row contraction, then one is lead to Drury's dilation theorem and the Arveson n-shift space mentioned earlier. Arveson showed that the C*-envelope is determined by representation by n-shifts on the symmetric Fock space. In general, calculating the C*-envelope is very difficult and only in the past decade have tools been developed to allow one to do this in general contexts. Muhly and Solel have a very general construction of C*-correspondences based on Pimsner's construction of a C*- algebra from a Hilbert C*-module. Many classical situations fit into this general framework. Nevertheless, explicit calculation of the C*-envelope in concrete situations such as interpolation makes a compelling connections between operator theory and function theory. Recent progress provides an opportunity to make real progress which holds the promise of significant applications to interpolation theory.

The interplay between the ideas and methods from operator theory and functional analysis with methods and ideas from function theory, commutative algebra and algebraic, analytic and complex geometry gives the field a strong interdisciplinary character. Moreover, the results obtained in operator theory have depended on extending and developing the techniques and ideas from the other fields. Further, the questions raised and results obtained in operator theory have cross-fertilized the other areas. Finally, in summary, the overwhelming goal for the workshop was to bring together leading researchers and young mathematicians from multivariate operator theory along with experts from related areas to survey, consolidate and extend the many advances of the past two decades.

2 Outline of the talks

Resolutions and quotients of Hilbert modules

The classical dilation theorem of Sz.-Nagy and Foias shows in particular that, for each contraction T of class $C_{0.}$ on a complex Hilbert space H, there is a short exact sequence $0 \to H^2(\mathbb{D}, E) \xrightarrow{M_\theta} H^2(\mathbb{D}, E_*) \xrightarrow{q} H \to 0$ of $\mathbb{C}[z]$-module maps consisting of an isometric multiplier M_θ and a co-isometry q. Here the module action on H is given by the polynomial functional calculus of T. Using Arveson's model theory for row contractions and an extension of the classical Beurling-Lax-

Halmos theorem due to McCullough and Trent to the case of Nevanlinna-Pick spaces one finds that, for every commuting pure row contraction, or equivalently, every pure co-spherically contractive Hilbert module $T \in L(H)^d$, there exists a resolution

$$(H_d^2 \otimes E_\bullet, M_{\theta_\bullet}) \xrightarrow{q} H \to 0$$

of $\mathbb{C}[z]$-module maps consisting of partially isometric multipliers $M_{\theta_i} : H_d^2 \otimes E_i \to H_d^2 \otimes E_{i-1}$ between vector-valued Drury-Arveson spaces and a co-isometry q. In his talk, Ronald Douglas gave a survey of this module theoretic approach to multivariate operator theory emphasizing in particular the differences between the single and multivariable case. The basic idea is to study general Hilbert modules over natural function algebras via resolutions by canonical modules. To give just one example, a recent result obtained in joint work of Douglas, Foias and Sarkar [6] shows that a pure co-spherically contractive Hilbert module H admits a finite partially isometric resolution of the above form only in the trivial case that H is isometrically isomorphic to $H_d^2 \otimes F$ for some Hilbert space F. A related problem is to find natural conditions under which quotient Hilbert modules are similar to free Hilbert modules of the form $H_d^2 \otimes F$. In addition Ronald Douglas described how one can reduce the structure of such quotient modules to the structure of an associated hermitian holomorphic bundle defined using the multiplier yielding the quotient module [7]. In the talk several problems of this type and their connection to complex analytic geometry were discussed.

Essential normality of homogeneous submodules

Let H_d^2 be the Drury-Arveson space on the open unit ball $\mathbb{B}$ in $\mathbb{C}^d$, that is, the analytic functional Hilbert space with reproducing kernel $K(z,w) = (1 - \langle z, w \rangle)^{-1}$. It is well known that the multiplication tuple $M_z = (M_{z_1}, \dots M_{z_d}) \in L(H_d^2)^d$ consisting of the multiplication operators M_{z_i} with the coordinate functions is q-essentially normal for every $q > d$. Arveson conjectured that q-essential normality is inherited by every quotient tuple $S^M = M_z/M$ of M_z modulo a closed homogeneous submodule $M \subset H_d^2$, that is, modulo every closed subspace $M = \overline{(p)}$ arising as the closure of an ideal $(p) = (p_1, \dots p_r) \subset \mathbb{C}[z]$ generated by finitely many homogeneous polynomials $p_1, \dots, p_r$. A strengthening of the conjecture due to Douglas, supported by many typical examples, says that S^M should even be q-essentially normal for every q larger than the complex dimension of the zero variety $Z(I)$ of the underlying ideal.

Affirmative answers to these conjectures are expected to lead to interesting new connections between multivariate operator theory and complex geometry. A positive answer was given by Arveson for submodules generated by monomials. This result was extended by Douglas to more general classes of analytic Hilbert modules. Guo and Wang showed that the conjectures are true in dimension $d \leq 3$ and for all principal homogeneous submodules generated by a single homogeneous polynomial p. In his talk, Jörg Eschmeier showed that a modification of an operator inequality used by Guo and Wang in the case of principal homogeneous submodules is equivalent to the existence of factorizations of the form $[M_{z_j}^*, P_M] = (N+1)^{-\frac{1}{2}} A_j$, where N is the number operator on H_d^2, and therefore implies that the cross commutators $[S_j^{M*}, S_i^M]$ $(1 \leq i, j \leq d)$ factorize boundedly through $(N+1)^{-1}$. Using recent results on the Fredholm theory of graded Hilbert space tuples [10], one obtains that a proof of the above mentioned operator inequality would immediately yield positive answers to the conjectures of Arveson and Douglas. It turns out that in all cases in which the conjectures are known to be true, the inequality holds and leads to a unified proof of stronger results [9]. Whether the inequality is satisfied in general remains an intriguing open question at this moment. Recent work of Michael Wernet shows that all the results remain true on a much larger class of functional Hilbert spaces.

Operator algebraic geometry

For a homogeneous ideal $I \subset \mathbb{C}[z]$, let $F_I = H^2_d \ominus I$ be the orthogonal complement of I in the Drury-Arveson space H^2_d, and let $A_I \subset L(F_I)$ be the norm-closed unital subalgebra generated by the compression S^I of the multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_d}) \in L(H^2_d)^d$ to F_I. According to Gelu Popescu, A_I is the universal operator algebra generated by a commuting row contraction T satisfying the relations $p(T) = 0$ for all $p \in I$. In his talk, Orr Shalit reported on joint results obtained with Ken Davidson and Christopher Ramsey which show that in quite general situations the operator algebras A_I can be classified in terms of the zero varieties $Z(I)$ of the underlying ideals. In this way intriguing analogues to the well-known classical correspondence between commutative algebra and geometry are obtained. For instance, if $I, J \subset \mathbb{C}[z]$ are radical homogeneous ideals, then A_I and A_j are isometrically isomorphic if and only if there is a unitary operator on $\mathbb{C}^d$ which maps $Z(I)$ onto $Z(J)$. Under suitable extra conditions, satisfied in many cases, A_I and A_J are shown to be isomorphic if and only if there is an invertible linear map A such that $AZ(J) = Z(I)$. It turns out that the complex geometry of the zero varieties of the underlying ideals is very rigid in the sense that in typical situations every biholomorphic map between $Z(I)$ and $Z(J)$ is automatically induced by a linear map.

Toeplitz quantization on symmetric domains

Let $D = G/K$ be an irreducible bounded symmetric domain in its Harish-Chandra realization, with G the connected component of the identity in the group of all biholomorphic self-maps of D and K the stabilizer of the origin. The unweighted Berman kernel on D is given by $K(x, y) = c\, h(x, y)^{-p}$, where c is a normalization constant, p is the genus and h denotes the Jordan determinant. The standard weighted Bergman spaces, that is, the subspaces consisting of all holomorphic functions in $L^2(D, \mu_\nu)$ with $\mu_\nu = c_\nu\, h(z, z)^{\nu - p}\, dz$, possess the reproducing kernels $K_\nu(z, w) = h(z, w)^{-\nu}$ for $\nu > p - 1$.

The asymptotic expansions of the Toeplitz star product $f \star g = \sum_{j=0}^{\infty} \nu^{-j} C_j(f, g)$, of the Berezin star product $f \odot g = \sum_{j=0}^{\infty} \nu^{-j} \tilde{C}_j(f, g)$ and the Berezin transform $B_\nu = \sum_{j=0}^{\infty} \nu^{-j} Q_j$ yield sequences of G-invariant (bi)-differential operators $C_j, \tilde{C}_j$ and Q_j. The Berezin transform is of central importance in the theory of deformation quantization of complex Kähler manifolds. Miroslav Englis indicated how a Peter-Weyl type decomposition for G-invariant differential operators can be used to obtain Peter-Weyl decompositions for the Berezin transform, the Berezin and the Toeplitz star product. For the Berezin transform, a result of this type was proved in a paper of Arazy and Orsted [1]. A corresponding expansion for the Berezin star product can be reduced to the the Arazy-Orsted expansion using suitable factorization properties of the star product $\odot$. A Peter-Weyl decomposition for the Toeplitz star product can be derived from a suitable expansion of the inverse Berezin transform B^{-1}. The resulting expansion for the Toeplitz star product $\star$ is new even in the simplest case of the unit disc. Analogous decompositions are also possible in the case of real bounded symmetric domains, where again there is a natural Berezin transform which is closely related to the well-known Segal-Bargmann transformations. The results presented in the talk have been obtained in joint work with Harald Upmeier [8].

In his lecture, Harald Upmeier reported on an extension of the above results, also obtained in collaboration with Miroslav Englis, concerning invariant operators and their asymptotic expansions in the case of compact symmetric spaces such as the Riemann sphere, the projective spaces or the Grassmann manifolds. Since in the compact case Bergman type spaces are finite dimensional, single Toeplitz operators have to be replaced by whole sequences of Toeplitz matrices in this case.

Classification of analytic Hilbert modules

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and denote by $A(\Omega)$ the associated "disk algebra", that is the algebra of analytic functions in Ω which are continuous on $\overline{\Omega}$. The class of topological Hilbert modules H over the algebra $A(\Omega)$ comprises as particular cases all commutative n-tuples of operators with joint spectrum contained in Ω, and hence it is far from being classifiable in simple terms. Assume in addition that the fibers

$$H(\lambda) = H/\mathfrak{m}_\lambda H$$

are finite dimensional for all $\lambda \in \Omega$, where $\mathfrak{m}_\lambda \subset A(\Omega)$ stands for the maximal ideal at λ. If the function $\lambda \mapsto \dim H(\lambda)$ is locally constant and $\cap_{\lambda\in\Omega}\mathfrak{m}_\lambda H = 0$, then differential geometric constructions lead to curvature type invariants of H (modulo unitary, analytic equivalence). This dictionary and refined classification constitutes the heart of the famous Cowen-Douglas theory. The talk by Biswas focused on analytic Hilbert modules H with merely finite dimensional fibers $H(\lambda),\ \lambda \in \Omega$. Without aiming at obtaining complete unitary invariants, he has shown how methods of algebraic geometry and function theory of several complex variables naturally lead to higher curvature type invariants for H. His main idea was to adapt Grothendieck's duality between coherent analytic sheaves and families of linear subspaces of a vector space to the Hilbert space setting. The main technical difficulties being related to division problems with L^2-bounds, a well charted territory in several complex variables, see [3].

A related subject was exposed by Rongwei Yang in his lecture. He was dealing with analytic Hilbert submodules H of the Hardy space of a polydisk $\mathbb{D}^n$. By applying the classical one variable model theory (due to Sz.-Nagy and Foiaş) to the localization of H to the fibers of the canonical projection map $\mathbb{D}^n \longrightarrow \mathbb{D}^{n-1}$ he was able to obtained new unitary invariants for H. In particular, differential topology invariants were attached in this setting to the analytic submodule H, via the invertibility of the linear pencil $w_0 + w_1 M_1 + ... + w_n M_n$, where M_j denotes the multiplication with the coordinate function z_j on H.

A good portion of Marcus Carlsson's lecture was devoted to the classification of analytic submodules of finite codimension in a classical Hilbert space of vector valued analytic functions, such as the Bergman, Hardy or Dirichlet spaces. The quintessential example being a finite codimension submodule H of the Bergman space $L^2_a(\Omega)$ associated to a bounded, strictly pseudoconvex domain with smooth boundary. In which case $H = I \cdot L^2_a(\Omega)$, where $I \subset \mathcal{O}(\Omega)$ is an ideal of finite codimension, such that $\mathrm{supp}[\mathcal{O}(\Omega)/I] \subset \Omega$. This simple phenomenon was recently generalized by Carlsson to submodules of finite codimension of a vector valued Bergman space.

Applications of multivariate operator theory to function theory

One of the most interesting and rich component of the workshop was concerned with novel applications of multivariate operator theory to classical function theory. We include below details of three talks given during the workshop.

In his lecture, John McCarthy has presented new results, obtained jointly with Jim Agler and Nicholas Young, on matrix monotone functions. In 1934, K. Löwner published a very influential paper studying functions on an open interval $E \subseteq \mathbb{R}$ that are matrix monotone. That is functions f with the property that whenever S and T are self-adjoint matrices whose spectra are in E then

$$S \leq T \qquad \Rightarrow \qquad f(S) \leq f(T).$$

Roughly speaking, Löwner showed that if one fixes a dimension n and wants the inequality to hold for n-by-n self-adjoint matrices, then certain matrices derived from the values of f must all be positive semi-definite. As n increases, the conditions become more restrictive. In the limit as $n \to \infty$ (equivalently, if one passes to self-adjoint operators on an infinite dimensional Hilbert space), then a necessary and sufficient condition is that the function f must have an analytic continuation to a function F that maps the upper half-plane Π to itself.

McCarthy and collaborators have extended Löwner's results to functions of d variables applied to d-tuples of commuting self-adjoint operators. A few definitions are necessary for stating their main results. Given two d-tuples $S = (S^1, \ldots, S^d)$ and $T = (T^1, \ldots, T^d)$, we shall say that $S \leq T$ if and only if $S^r \leq T^r$ for every $1 \leq r \leq d$. We shall let $CSAM_n^d$ denote the set of d-tuples of commuting self-adjoint n-by-n matrices.

Definition: Let E be an open set in $\mathbb{R}^d$, and f be a real-valued C^1 function on E. We say f is locally M_n-monotone on E if, whenever S is in $CSAM_n^d$ with $\sigma(S) = \{x_1, \ldots, x_n\}$ consisting of n distinct points in E, and $S(t)$ is a C^1 curve in $CSAM_n^d$ with $S(0) = S$ and $\frac{d}{dt} S(t)|_{t=0} \geq 0$, then $\frac{d}{dt} f(S(t))|_{t=0}$ exists and is ≥ 0.

Definition: Let E be an open subset of $\mathbb{R}^d$. The set $Ł_n^d(E)$ consists of all real-valued C^1-functions on E that have the following property: whenever $\{x_1, \ldots, x_n\}$ are n distinct points in E, there exist positive semi-definite n-by-n matrices $A^1, \ldots, A^d$ so that

$$\begin{aligned} A^r(i,i) &= \left. \frac{\partial f}{\partial x^r} \right|_{x_i} \\ \text{and} \quad f(x_j) - f(x_i) &= \sum_{r=1}^{d} (x_j^r - x_i^r) A^r(i,j) \qquad \forall\, 1 \leq i,j \leq n. \end{aligned}$$

Theorem (Agler-McCarthy-Young) *Let E be an open set in $\mathbb{R}^d$, and f a real-valued C^1 function on E. Then f is locally M_n-monotone if and only if f is in $Ł_n^d(E)$.*

Theorem (Agler-McCarthy-Young) *Let E be an open set in $\mathbb{R}^2$, and f a real-valued C^1 function on E. Then f is locally operator monotone (i.e. M_n-monotone for all n) if and only if f has an analytic extension that maps Π^2 to Π.*

For rational functions, we can replace "local" by "global".

Theorem (Agler-McCarthy-Young) *Let F be a rational function of two variables. Let Γ be the zero-set of the denominator of F. Assume F is real-valued on $\mathbb{R}^2 \setminus \Gamma$. Let E be an open rectangle in $\mathbb{R}^2 \setminus \Gamma$. Then F is globally operator monotone on E if and only if F is in $Ł(E)$.*

The lecture by Jim Agler aimed at generalizing a classical result of C. Carathéodory on analytic functions that map the disk to the disk. Let B_d be the open unit ball in $\mathbb{C}^d$. Assume that ϕ is an analytic function from B_d to the unit disk.

We shall say that ϕ has an angular gradient at a point $b \in \partial B$ if there exists $\omega \in \mathbb{C}$ and $\eta \in \mathbb{C}^d$ such that

$$n.t. \lim_{l \to b} \frac{f(l) - \omega - \eta \cdot (l - b)}{\|l - b\|} \;=\; 0,$$

where the left-hand side means that l tends to b non-tangentially from within B_d. We call ω the non-tangential gradient at b.

Theorem (Agler-McCarthy-Young) *If ϕ has a non-tangential gradient ω at b, then $n.t. \lim_{l \to b} \nabla \phi(l)$ exists and equals η.*

Theorem (Agler-McCarthy-Young) *If $b \in \partial B_d$ satisfies*

$$\liminf_{l \to b} \frac{1 - |\phi(l)|}{1 - \|l\|} \; < \; \infty,$$

then ϕ has a directional derivative at b in every direction that points into B_d.

In dimension $d = 1$, the hypothesis of the second theorem implies that of the first; but if $d \geq 2$, this no longer holds.

The talk by Brett Wick was focused on multivariate aspects of the Corona Problem on a Besov space. Quite specifically, the space $B_2^\sigma(\mathbb{B}_n)$ roughly consists of those analytic functions f whose derivatives of order $\frac{n}{2} - \sigma$ lie in the classical Hardy space $H^2(\mathbb{B}_n)$. More precisely, let $d\lambda_n(z) := (1 - |z|^2)^{-(n+1)} dV(z)$, with $dV(z)$ denoting the Lebesgue measure on the unit ball $\mathbb{B}_n$. For $0 \leq \sigma < \infty$, the Besov–Sobolev space $B_2^\sigma(\mathbb{B}_n)$ is the collection of functions that are analytic on $\mathbb{B}_n$ such that, for any integer $m \geq 0$, with $m + \sigma > \frac{n}{2}$, we have the following norm being finite:

$$\|f\|^2_{B_2^\sigma(\mathbb{B}_n)} := \sum_{j=0}^{m-1} |f^{(j)}(0)|^2 + \int_{\mathbb{B}_n} |(1 - |z|^2)^{m+\sigma} f^{(m)}(z)|^2 d\lambda_n(z).$$

The parameter σ recovers many of the important spaces in function and operator theory on the unit ball $\mathbb{B}_n$. When $\sigma = 0$, this corresponds to the Dirichlet space of analytic functions, the value $\sigma = \frac{n}{2}$ yields the classical Hardy space on the ball, and $\sigma = \frac{n+1}{2}$ yields the Bergman space.

When $\sigma = \frac{1}{2}$, $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ is the celebrated Drury–Arveson space. In a sense, this space of analytic function is a universal space from the point of view of applications in operator theory. Using it, one can prove a generalization of the famous von Neumann Inequality for contractions, analogues of Beurling's Theorem on invariant subspaces, and Commutant Lifting Theorems for the multiplier algebra of this space. Surprisingly, many of the corresponding results for the spaces associated with $H^\infty(\mathbb{B}_n)$, the multiplier algebra of $H^2(\mathbb{B}_n)$, do *not* hold, placing the multiplier algebra of the Drury–Arveson space in a distinguished position. Thus, one can see that the Corona Theorems obtained in [4] are the proper generalization of Carleson's famous Theorem, to the unit ball $\mathbb{B}_n$.

For the Besov–Sobolev space $B_2^\sigma(\mathbb{B}_n)$, one defines the *multiplier algebra* $\mathcal{M}_2^\sigma(\mathbb{B}_n)$ as the collection of analytic functions φ that are pointwise multipliers of $B_2^\sigma(\mathbb{B}_n)$. Namely, $\varphi f \in B_2^\sigma(\mathbb{B}_n)$ for all $f \in B_2^\sigma(\mathbb{B}_n)$, and then norms $\mathcal{M}_2^\sigma(\mathbb{B}_n)$ by

$$\|\varphi\|_{\mathcal{M}_2^\sigma(\mathbb{B}_n)} := \sup_{f \in B_2^\sigma(\mathbb{B}_n)} \frac{\|\varphi f\|_{B_2^\sigma(\mathbb{B}_n)}}{\|f\|_{B_2^\sigma(\mathbb{B}_n)}}.$$

Theorem (Ş. Costea, E. Sawyer, Wick, [4]) *Let $\sigma \geq 0$ and $1 < p < \infty$. Given $g_1, \ldots, g_N$ in $\mathcal{M}_p^\sigma(\mathbb{B}_n)$ satisfying*

$$1 \geq \sum_{j=1}^N |g_j(z)|^2 \geq \delta > 0 \quad \forall z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,p,\delta}$ such that for each $h \in B_p^\sigma(\mathbb{B}_n)$, there are $f_1, \ldots, f_N$ in $B_p^\sigma(\mathbb{B}_n)$ satisfying

$$\sum_{j=1}^N \|f_j\|^p_{B_p^\sigma(\mathbb{B}_n)} \leq C_{n,\sigma,p,\delta} \|h\|^p_{B_p^\sigma(\mathbb{B}_n)} \quad \text{and} \quad \sum_{j=1}^N g_j(z) f_j(z) = h(z) \quad \forall z \in \mathbb{B}_n.$$

When $0 \leq \sigma \leq \frac{1}{2}$ and $p = 2$, these spaces of analytic functions have a complete Nevanlinna–Pick kernel, following the terminology of Agler and McCarthy. Consequently, this Hilbert space result for $B_2^\sigma(\mathbb{B}_n)$ can be lifted to the corresponding result for the multiplier algebra $\mathcal{M}_2^\sigma(\mathbb{B}_n)$. This leads to the following theorem giving the first positive (non-trivial) Corona result for multiplier algebras in several complex variables.

Theorem (Ş. Costea, E. Sawyer, Wick, [4]) *Let* $0 \leq \sigma \leq \frac{1}{2}$. *Given* $g_1, \dots, g_N$ *in* $\mathcal{M}_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$1 \geq \sum_{j=1}^{N} |g_j(z)|^2 \geq \delta > 0 \quad \forall z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,\delta}$ *and there are functions* $f_1, \dots, f_N$ *in* $\mathcal{M}_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$\sum_{j=1}^{N} \|f_j\|_{\mathcal{M}_2^\sigma(\mathbb{B}_n)} \leq C_{n,\sigma,\delta} \quad \text{and} \quad \sum_{j=1}^{N} g_j(z) f_j(z) = 1 \quad \forall z \in \mathbb{B}_n.$$

The above theorem is also true in the case of matrix-valued Corona Data, though for convenience, only the scalar case is stated. Namely, the constant $C_{n,\sigma,\delta}$ is *independent* of N (as listed). When $\sigma = \frac{1}{2}$, the space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ is the Drury–Arveson space, and we obtain the Corona Theorem for its multiplier algebra of analytic functions on $\mathbb{B}_n$.

In the case of one complex variable, $n = 1$, some related results were known. In particular, Carleson's famous Corona Theorem is the case $\sigma = \frac{1}{2}$. Tolokonnikov obtained the result for $0 \leq \sigma < \frac{1}{2}$. And recent work by Trent gave another proof of $\sigma = 0$, [12]. The results in [4] not only give new proofs of these results, but also addresses the situation $n > 1$. These new results are a significant breakthrough in the study of multi-variable Corona Problems.

Multivariate moment problems

The transition from one to several variables in moment problems is not smooth. Due to the fact that not all positive polynomials on $\mathbb{R}^n, n > 1$, are sums of squares, there is no effective way of solving a full or truncated moment problem for positive measures. One venue, preferred by physicists and statisticians, is to add to the moment constraints the maximum entropy assumption. This selects among all solutions the most unbiased one, in a precise sense. The talk by Calin Ambrozie was focused on the theoretical aspects of maximal entropy solutions to truncated moment problems. His main techniques were derived from infinite dimensional duality between convex cones (the so-called Fenchel duality), obtaining in this way exponential of polynomials as solutions. A more constructive approach to truncated moment problems, in a more restrictive setting, made the subject of Florian-Horia Vasilescu's talk. He focused on flat (i.e. constant rank) extensions of Hankel type matrices of moments, reflecting in their structure the condition that the unknown measure/distribution is supported by an algebraic curve in $\mathbb{R}^2$. A great deal of his theory extends to the non-commutative, free-* setting.

Non-commutative operator theory

Operators generally do not commute, and there is a lot of recent work on multivariate operator theory where the algebras studied are not abelian.

Gelu Popescu spoke about part of his program to develop the theory of holomorphic functions on a domain consisting of n-tuples of operators on Hilbert space. His early work concentrated on the analogue of the n-ball, the set of all row contractions. Many theorems about analytic functions on the ball in $\mathbb{C}^n$ have natural analogues for power series in indeterminates lying in this non-commutative ball. In his talk at BIRS, Popescu defined a family of domains that have rotational symmetry like the Reinhardt domains, but here they are defined in terms of completely positive maps. He was particularly interested in questions of isomorphism of the operator algebras of all operator holomorphic functions on these domains. He obtained a rigidity theorem based on Cartan's theorem in several complex variables. (Shalit had a similar result in the commutative context mentioned earlier.) This led to an analogue of Thulen's theorem, showing that the list of domains with non-trivial automorphisms is limited to a small list. The final results showed that isomorphism of the algebras is determined by biholomorphic equivalence of the underlying operator domains.

Paul Muhly reported on work with Baruch Solel. Inspired by an old paper of Joseph Taylor on general functional calculus in several variables, and recent work of Dan Voiculescu on what he calls free analysis, Muhly considers these questions in the context of Hardy algebras of C*-correspondences. Like Popescu's work, these are algebras of operators that are non-commutative analogues of $H^\infty(\mathbb{D})$. He considers elements of these algebras as functions on a non-commutative ball, and asks about the structure of the ball and these functions. One of their results is the existence of a completely positive definite Szego kernel on the ball, forming a non-commutative reproducing kernel Hilbert space.

We give below more details about Muhly's talk, as it also naturally interacts with the talk of Vinnikov. *Fully matricial sets and functions* arise quite naturally when one tries to build a complex function theory based on free algebras of various sorts. This was recognized first by J. Taylor (Adv. Math. **3** (1972). See section 6.) and has arisen anew in the work of Voiculescu on free analysis questions. He coined the terms "fully matricial sets and functions". But they are implicit in a lot of other work that has been evolving in recent years and they are becoming more and more explicitly studied. (See, in particular, the recent work by Helton, McCullough, Klepp, Putinar, Vinnikov and others on dimension free linear matrix inequalities and the work of Ball, Davidson, Katsoulis, Pitts, Popescu and others on free holomorphic functions.) Muhly's talk focused on sets and functions that arise as follows: Let E be a W^*-correspondence over a W^*-algebra M and let $H^\infty(E)$ be the Hardy algebra we build from (E, M) as described in Math. Ann. **330** (2004). Then for each normal representation σ of M on a Hilbert space H_σ there is a natural W^*-correspondence over $\sigma(M)'$, called the *σ-dual correspondence of E*, and denoted E^σ. This is a space of intertwining operators between σ and the representation induced by E in the sense of Rieffel, $\sigma^E \circ \varphi$, where φ gives the left action of M on E. The unit ball in the space of adjoints of E^σ, $\mathbf{D}(E^\sigma)^*$, is part of a fully matricial family of sets. This is because $E^{n\sigma}$ is in a very natural way $M_n(E^\sigma)$ for each positive integer n. The importance of $\mathbf{D}(E^\sigma)^*$ lies in the fact that as σ runs over the space of normal representations of M, the points in $\mathbf{D}(E^\sigma)^*$ label (most of) the ultra-weakly continuous, completely contractive representations of $H^\infty(E)$ in $B(H_\sigma)$. For $\eta^* \in \mathbf{D}(E^\sigma)^*$, $\eta^* \times \sigma$ denotes the representation of $H^\infty(E)$ determined by η^*. Each each element $F \in H^\infty(E)$ gives rise to a function $\widehat{F}_\sigma : \mathbf{D}(E^\sigma)^* \to B(H_\sigma)$ defined by $\widehat{F}_\sigma(\eta^*) := (\eta^* \times \sigma)(F)$. It is an easy calculation to see that for each σ, the collection $\{\widehat{F}_{n\sigma}\}_{n\geq 1}$ forms a fully matricial function on the fully matricial set $\{\mathbf{D}(E^{n\sigma})^*\}_{n\geq 1}$.

When $M = \mathbf{C}$, $E = \mathbf{C}^d$ and σ is the one dimensional representation of $\mathbf{C}$, then $H^\infty(E)$ is the free semigroup algebra $\mathcal{L}_d$ and $\mathbf{D}(E^{n\sigma})^*$ is the space of row contractions of d $n \times n$ matrices. An $F \in H^\infty(E)$ has a representation in terms of a series indexed by the free semigroup on d generators, and the function $\widehat{F}_{n\sigma}$ is obtained by replacing the d generators by the d components of a row contraction. Thus $\widehat{F}_{n\sigma}$ lies in a certain completion of the algebra of d generic $n \times n$ matrices.

The basic question addressed by Muhly and Solel is: Given a family of functions $\{\Phi_\sigma\}$, where σ runs over all the normal representations of M and where $\Phi_\sigma : \mathbf{D}(E^\sigma)^* \to B(H_\sigma)$, when does there exist an element $F \in H^\infty(E)$ so that $\widehat{F}_\sigma = \Phi_\sigma$ for all σ. One solution is formulated in terms of certain intertwining spaces and is connected to Taylor's original analysis, as well as to recent work by D. Kalyuzhnyĭ-Verbovetzkiĭ and V. Vinnikov. The second solution is based on their generalization of the Nevannlina-Pick Theorem and Lyapunov analysis.

Although the results of Muhly and Solel are formulated in terms of general W^*-algebras, W^*-correspondences and normal representations, they contain the situations when M and E are finite dimensional and σ is finite dimenionsional as special cases. These cases yield very interesting finite dimensional matrix balls expressed as $\mathbf{D}(E^\sigma)^*$. The functions we study lie in completions of spaces of polynomial maps studied in invariant theory.

Elias Katsoulis discussed joint work with E. Kakariadis. He considered the action of an isometric automorphism on the tensorr algebra associated to C*-correspondence. When the automorphism is the restriction of a $*$-automorphism of the enveloping C*-algebra, there is an associated semicrossed product sitting inside the semicrossed product of the C*-algebra. The study of such semicrossed products goes back to work of Arveson, and later to the universal construction of Peters, in the case of a single self map on a compact space. The associated algebras were eventually shown to be (algebraically) isomorphic if and only if the two maps are topologically conjugate by Davidson and Katsoulis [5]. More recent work has dealt with families of maps and endomorphisms of nonself-adjoint operatopr algebras. Arveson's famous work reshaping dilation theory [2] suggests that one should identify the C*-envelope, the minimal C*-algebra containing a nonself-adjoint algebra. Katsoulis and Kakariadis identify this C*-envelope for the semicrossed product of a tensor algebra by a $*$-extendable endomorphism.

Wing Suet Li talked about Horn's conjecture. For complex selfadjoint $n \times n$ matrices A, B, C, with $A + B + C = 0$, A. Horn conjectured in 1962 a set of inequalities that would characterize the possible eigenvalues of these matrices. The conjecture was proved to be correct by A. Klyachko and A. Knutson and T. Tao in the late 1990s, using techniques from algebraic geometry, representation theory and very intricate combinatorics. One of the key ingredients to show that these inequalities are necessary is to establish that the intersection of certain Schubert varieties is nonempty. Recently, W.-S. Li, H. Bercovici, B. Collins, K. Dykema, and D. Timotin were able to construct an explicit element in the intersection of three given Schubert varieties when the intersection contains a unique element. This element is constructed generically as the result of a (finite) sequence of lattice operations. This sequence of operations can be applied to appropriately defined analogous of Schubert varieties in the Grassmannian, associated with a finite von Neumann algebra. The arguments requires a good understanding of a combinatorial structure that is closely related to the honeycombs, developed by A. Knutson and T. Tao. Earlier work of W.-S. Li with H. Bercovici showed that the eigenvalues of selfadjoint elements a, b, c with $a + b + c = 0$ in the factor $\mathcal{R}^\omega$ are characterized by a system of inequalities analogous to the classical Horn inequalities, as one would naturally expect. The present result shows that these inequalities are also true for an arbitrary finite factor. In particular, if x, y, z are selfadjoint elements of a finite factor and $x + y + z = 0$, then there exist selfadjoint $a, b, c \in \mathcal{R}^\omega$ such that $a + b + c = 0$ and a (resp. b, c) has the same eigenvalues as x (resp. y, z).

Victor Vinnikov's talk was focused on his recent work with Dmitry S. Kaliuzhnyi-Verbovetskyi on rational functions over free variables, their difference-differential calculus and realization formulas as determinants of linear pencils of matrices. Their approach complements that of Muhly-Solel, and it is pertinent to recent advances in free probability theory.

3 Outcome of the Meeting

As was perhaps suggested in the Overview section of the report, Multivariate Operator Theory (MVOT) is less a community than a collection of several sub communities, each focused on topics in which algebras of Hilbert space operators occur either concretely or as part of the framework for the research. In most instances the goals of the various groups are quite distinct and the techniques and conceptual templates may have little, if any, overlap. In many instances the other communities, such as several complex variables, harmonic analysis to name just two, have been around for almost a century, which is the case also for single operator theory.

As the subject of MVOT has matured, techniques and ideas from one part can and should enrich and leaven others. Considerable progress at accomplishing this goal was realized at the meeting as was detailed in the previous section on talks. Moreover, people working on similar problems but in different communities got to meet and know each other. One can expect rich collaborations to flow from these events.

As was pointed out in the overview, one could divide MVOT into the commutative and non-commutative areas and to too large an extent these two communities have remained apart. However, researchers have begun to realize deep, non obvious connections between the two. In a fascinating juxtaposition of talks, three approaches to the concept of functions of non commuting variables were given starting with the obvious notion of polynomials in non-commuting variables. A clearer understanding of this notion should have applications in free probability theory and theoretical physics. Moreover, there could be some surprising insights gained into ordinary algebraic geometry.

In a similar vein, ideas from classical interpolation theory lead to new results in the extension of functions as well as abstract frameworks in the context of non selfadjoint and C*-algebras. Several talks explored these connections and generalizations. Related to these questions are problems and structures which arise in the single operator theory motivated by systems theory. Looking at these matters in the context of MVOT provides new insights as well as frames questions about the extension of these questions to the context of several variables.

As was pointed out in the overview section, concepts and techniques from complex geometry enter the picture naturally in the MVOT context even showing that geometrical notions went unnoticed in the single variable case. One situation in which this has become apparent is in the efforts related to the corona problem.

All in all, the most recent results in the subject were reviewed among the varied groups of researchers which is expected to result in considerable cross-fertilization and adoption of a broader set of tools and methods in exploring the topic and its connections with other areas.

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