Quantum Sheaf Cohomology and Brute Force Techniques

Josh Guffin

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- A Kähler manifold X
- A hermetian holomorphic bundle $\mathcal{E}$ satisfying
  - $\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(T_X)$
  - $\det \mathcal{E}^\vee \cong \omega_X$
  - $\mathrm{rk}\ \mathcal{E} \leq 8$ (if $\mathcal{E}$ is not a deformation of T_X
- Quantum Sheaf Cohomology

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- Comes from a subset of operators in the $g = 0$ *twisted* NLSM
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- (2,2) NLSM, topologically A-twisted (an SCFT)
- Two *scalar* supersymmetry charges $Q, \overline{Q}$
- BPS bounds on operators: for $\mathcal{O}$ of conformal weight $(h, \overline{h})$,

$$h \geq 0$$
$$\overline{h} \geq 0$$

- Saturated when $\mathcal{O}$ is in the kernel of Q or $\overline{Q}$:

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- An operator $\mathcal{O}$ is *chiral* if $\mathcal{O} \in \ker Q \cap \ker \overline{Q}$
- Q and $\overline{Q}$ are linear and obey Liebniz
- Operator Product Expansion: in a basis for all operators

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- One scalar supersymmetry charge, $\overline{Q}$
- *Right-moving* BPS bound on operators

$$\overline{h} \geq 0$$

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- Family of half-chiral rings
- Parametrize the family by $\underline{\alpha}$ with $\underline{\alpha} = 0$ the (2,2) point.

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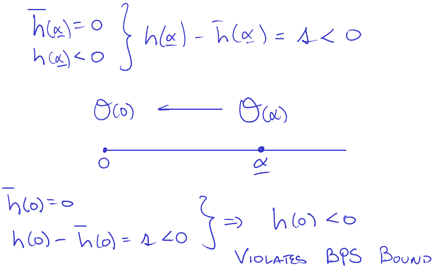
- Conformal weights satisfy $h(\underline{\alpha}) - \overline{h}(\underline{\alpha}) = s \in \mathbb{Z}$



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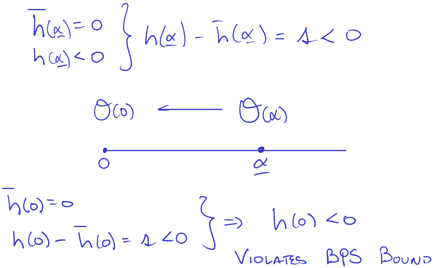

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- Set of $h = 0$ operators in $\ker \overline{Q}$ as a vector space is

$$\bigoplus_{p,q} H^p(X, \Lambda^q \mathcal{E}^\vee)$$

with product structure coming from the QFT

- Would like to describe (0,2) topological rings
- Techniques exist only for X a toric variety or subvariety
- Brute-force method
  - Toric varieties
  - Bundle must be a deformation of the tangent bundle
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- Goal: write down generators and find relations in

$$\bigoplus_{p,q} H^p(X, \Lambda^q \mathcal{E}^\vee)$$

- Compute correlation functions and deduce relations from them

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_\beta q^\beta \qquad q^\beta := e^{i \int_\beta \omega}$$

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- Compute $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_\beta$ by

$$H^p(X, \Lambda^q \mathcal{E}^\vee) \to H^p(\overline{M}_\beta, \Lambda^q \mathcal{F}^\vee) \quad \text{(Eric's map)}$$

$$H^{p_1}(\overline{\mathcal{M}}_\beta, \Lambda^{q_1} \mathcal{F}_\beta^\vee) \otimes \cdots \otimes H^{p_s}(\overline{\mathcal{M}}_\beta, \Lambda^{q_s} \mathcal{F}_\beta^\vee) \xrightarrow{\bullet} H^{n_\beta}(\overline{\mathcal{M}}_\beta, \Lambda^{n_\beta} \mathcal{F}_\beta^\vee)$$

- Here $n_\beta = \dim \overline{\mathcal{M}}_\beta$, $\mathcal{F}_\beta$ is the induced sheaf on $\overline{\mathcal{M}}_\beta$, and

$$H^{n_\beta}(\overline{\mathcal{M}}_\beta, \Lambda^{n_\beta} \mathcal{F}_\beta^\vee) \cong H^{n_\beta}(\overline{\mathcal{M}}_\beta, \omega_{\overline{\mathcal{M}}_\beta}) \cong \mathbb{C}$$

"The trace"

- We require:

explicit cohomology theory	$\longrightarrow$	Čech complex
generators $\mathcal{O}_a \in H^*(X, \Lambda^* \mathcal{E}^\vee)$	$\longrightarrow$	Euler sequence on X
$\overline{\mathcal{M}}_\beta$	$\longrightarrow$	Morrison/Plesser[MRP95]
$\mathcal{F}_\beta$	$\longrightarrow$	Katz/Sharpe [KS06]
images $\widetilde{\mathcal{O}}_a \in H^p(\overline{\mathcal{M}}_\beta, \Lambda^q \mathcal{F}_\beta^\vee)$	$\longrightarrow$	Euler sequence on $\overline{\mathcal{M}}_\beta$
$\bullet$/trace	$\longrightarrow$	Lots of computer time

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- For every toric variety X, the Euler sequence

$$0 \longrightarrow \mathcal{O}_X^r \xrightarrow{E_0} \bigoplus_\rho \mathcal{O}_X(D_\rho) \longrightarrow T_X \longrightarrow 0$$

induces unobstructed deformations as

$$0 \longrightarrow \mathcal{O}_X^r \xrightarrow{E} \bigoplus_\rho \mathcal{O}_X(D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0$$

- Dualizing,

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \bigoplus_\rho \mathcal{O}_X(-D_\rho) \xrightarrow{E^t} \mathcal{O}_X^r \longrightarrow 0$$

induces the long exact sequence containing

$$\cdots \longrightarrow H^0(X, \bigoplus_\rho \mathcal{O}_X(-D_\rho)) \longrightarrow H^0(X, \mathcal{O}_X^r)$$
$$\longrightarrow H^1(X, \mathcal{E}^\vee) \longrightarrow H^1(X, \bigoplus_\rho \mathcal{O}_X(-D_\rho)) \longrightarrow \cdots$$

and when $\dim X \geq 2$,

$$H^1(X, \mathcal{E}^\vee) \cong H^0(X, \mathcal{O}_X^r) \cong \mathbb{C}^r \cong H^1(X, \Omega_X^1)$$

- To find $\widetilde{\mathcal{O}_a} \in H^1(\overline{\mathcal{M}}_\beta, \mathcal{F}^\vee)$,

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow \bigoplus_{\widetilde{\rho}} \mathcal{O}_{\overline{\mathcal{M}}_\beta}(-D_{\widetilde{\rho}}) \xrightarrow{F^t} \mathcal{O}^r_{\overline{\mathcal{M}}_\beta} \longrightarrow 0$$

leading via the induced long-exact sequence to

$$H^1(\overline{\mathcal{M}}_\beta, \mathcal{F}^\vee) \cong H^0(\overline{\mathcal{M}}_\beta, \mathcal{O}^r_{\overline{\mathcal{M}}_\beta}) \cong \mathbb{C}^r$$

so compute by constructing the isomorphism on Čech cochains

$$H^1(\overline{\mathcal{M}}_\beta, \mathcal{F}^\vee) \cong \mathbb{C}^r \cong H^1(X, \mathcal{E}^\vee)$$

- Explicitly construct generators as Čech cochains for each $\overline{\mathcal{M}}_\beta$
- Teach a computer to cup/wedge and trace

$$H^{p_1}(\overline{\mathcal{M}}_\beta, \Lambda^{q_1}\mathcal{F}_\beta^\vee) \otimes \cdots \otimes H^{p_s}(\overline{\mathcal{M}}_\beta, \Lambda^{q_s}\mathcal{F}_\beta^\vee) \xrightarrow{\bullet} H^{n_\beta}(\overline{\mathcal{M}}_\beta, \Lambda^{n_\beta}\mathcal{F}_\beta^\vee) \xrightarrow{\cong} H^{n_\beta}(\overline{\mathcal{M}}_\beta, \omega_{\overline{\mathcal{M}}_\beta}) \xrightarrow{\cong} \mathbb{C}$$

- Simplest example, $X = \mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}_X^2 \xrightarrow{E} \mathcal{O}_X(1,0)^2 \oplus \mathcal{O}_X(0,1)^2 \longrightarrow T_X \longrightarrow 0$$

where

$$E = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & y_0 \\ 0 & y_1 \end{pmatrix}$$

- $X = \mathbb{P}^1 \times \mathbb{P}^1$ unobstructed: parametrize the 6-dimensional family of deformations as

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where

$$E = \begin{pmatrix} x_0 & \epsilon_1 x_0 + \epsilon_2 x_1 \\ x_1 & \epsilon_3 x_0 \\ \gamma_1 y_0 + \gamma_2 y_1 & y_0 \\ \gamma_3 y_0 & y_1 \end{pmatrix}$$

- $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) \cong \mathbb{C}^2$, find Čech reps of $\binom{1}{0}$ and $\binom{0}{1}$: $\psi, \widetilde{\psi}$
- Compute two-point functions in degree (0,0) sector

$$\langle\psi\psi\rangle = \langle\psi\psi\rangle_{0,0} = \frac{1}{\phi}(\epsilon_1 + \gamma_1\epsilon_2\epsilon_3)$$

$$\langle\psi\widetilde{\psi}\rangle = \langle\psi\widetilde{\psi}\rangle_{0,0} = \frac{1}{\phi}(\gamma_2\gamma_3\epsilon_2\epsilon_3 - 1)$$

$$\langle\widetilde{\psi}\widetilde{\psi}\rangle = \langle\widetilde{\psi}\widetilde{\psi}\rangle_{0,0} = \frac{1}{\phi}(\gamma_1 + \epsilon_1\gamma_2\gamma_3)$$

Here

$$\phi = (\gamma_1 + \gamma_2\gamma_3\epsilon_1)(\epsilon_1 + \gamma_1\epsilon_2\epsilon_3) - (\gamma_2\gamma_3\epsilon_2\epsilon_3 - 1)^2$$

- No other instanton sectors contribute

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- Moduli space: $\overline{\mathcal{M}}_{i,j} = \mathbb{P}^{2i+1} \times \mathbb{P}^{2j+1}$
- On each $\overline{\mathcal{M}}_{i,j}$, find Čech reps of image of $\psi, \widetilde{\psi}$ in $H^1(\overline{\mathcal{M}}_{i,j}, \mathcal{F}^\vee)$.
- Four-point functions arise from total degree 1;

$$\langle \psi\psi\psi\psi \rangle = \langle \psi\psi\psi\psi \rangle_{1,0} q + \langle \psi\psi\psi\psi \rangle_{0,1} \widetilde{q}$$

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$$\langle \psi\psi\psi\widetilde{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[(\gamma_2\gamma_3\epsilon_2\epsilon_3 - 1)^2 + \gamma_2\gamma_3 (\epsilon_1 + \gamma_1\epsilon_2\epsilon_3)^2 \right]$$

$$\langle \psi\psi\widetilde{\psi}\widetilde{\psi} \rangle_{1,0} = \frac{1}{\phi^2} (\gamma_2\gamma_3\epsilon_2\epsilon_3 - 1) [2(\gamma_1 + \gamma_2\gamma_3\epsilon_1) - \gamma_1 (1 - \gamma_2\gamma_3\epsilon_2\epsilon_3)]$$

$$\langle \psi\widetilde{\psi}\widetilde{\psi}\widetilde{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[(\gamma_1 + \gamma_2\gamma_3\epsilon_1)^2 + \gamma_2\gamma_3 (\gamma_2\gamma_3\epsilon_2\epsilon_3 - 1)^2 \right]$$

$$\langle \widetilde{\psi}\widetilde{\psi}\widetilde{\psi}\widetilde{\psi} \rangle_{1,0} = \frac{-1}{\phi^2} (\gamma_1 + \epsilon_1\gamma_2\gamma_3) \Big[\gamma_1 (\gamma_1 + \gamma_2\gamma_3\epsilon_1) - 2\gamma_2\gamma_3 (\gamma_2\gamma_3\epsilon_2\epsilon_3 - 1) \Big]$$

- Compute up to total degree 3
- Deduce relations:

$$\psi * \psi + \epsilon_1(\psi * \widetilde{\psi}) - \epsilon_2\epsilon_3\,(\widetilde{\psi} * \widetilde{\psi}) = q$$
$$\widetilde{\psi} * \widetilde{\psi} + \gamma_1(\psi * \widetilde{\psi}) - \gamma_2\gamma_3(\psi * \psi) = \widetilde{q}.$$

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$$\begin{aligned} \psi * \psi - (\epsilon_1 - \epsilon_2)\psi * \widetilde{\psi} &= e^{it_1} \\ \widetilde{\psi} * \widetilde{\psi} &= e^{it_2}. \end{aligned}$$

- Consider a projective variety X, $\dim_{\mathbb{C}} X = 3$, with a $\mathcal{E}$ a generic deformation of T_X.
- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{\text{twisted}}$ gives the holomorphic dependence on bundle deformation parameters of the low-energy superpotential W
- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{[\ell]}$ gives dependence of W linear in q
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- Consider a generic quintic hypersurface $X \subset \mathbb{P}^4$
- For all 2875 lines $\ell \subset X$, a generic deformation $\mathcal{E}$ has balanced splitting type:

$$\mathcal{E}|_\ell \cong \mathcal{O}_X^{\oplus r}$$

- The sheaf $\mathcal{F}$ on $\overline{\mathcal{M}}_{0,3}(X, [\ell])$ has no cohomology
- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{[\ell]} = 0$ on an *open subset* of the family of deformations, but is non-zero at the (2,2) point ($\mathcal{E} = T_X$)

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