A generalization of the MAC scheme on non conforming meshes

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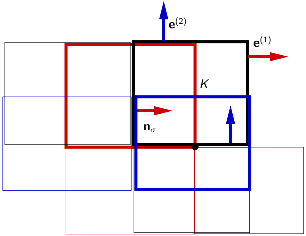
approximation of

$$-\Delta \mathbf{u} + \nabla p = f \text{ in } \Omega, \quad \mathbf{u} = 0 \text{ on } \partial\Omega$$
$$\operatorname{div}(\mathbf{u}) = 0 \text{ in } \Omega$$

- $\mathcal{T}$: Cartesian rectangular mesh of Ω, $\mathcal{E}$: edges of $\mathcal{T}$
- Discretization of $\mathbf{u}$, and p by piecewise constant functions



$p_{\mathcal{T}} \in X_{\mathcal{T}}$, $p_{\mathcal{T}} = p_K$ in K, $K \in \mathcal{T}$ (black cell)

$\mathbf{n}_\sigma = \mathbf{e}^{(1)}$ for $\sigma \in \mathcal{E}_{\mathrm{ver}}$ and $\mathbf{n}_\sigma = \mathbf{e}^{(2)}$ for $\sigma \in \mathcal{E}_{\mathrm{hor}}$

$\mathbf{u}_{\mathcal{T}} = (u_{\mathcal{T}}^{(1)}, u_{\mathcal{T}}^{(2)}) \in \mathbf{H}_{\mathcal{T}}$

$u_{\mathcal{T}}^{(1)} = u_\sigma$, first component of $\mathbf{u}_{\mathcal{T}}$ in the red cell

$u_{\mathcal{T}}^{(2)} = u_\sigma$, second component of $\mathbf{u}_{\mathcal{T}}$ in the blue cell

$u_\sigma \in \mathbb{R}$ is an approximate value for $\mathbf{u} \cdot \mathbf{n}_\sigma$

$u_\sigma = 0$ if $\sigma \subset \partial\Omega$

$$
\begin{aligned}
&\mathbf{u}_{\mathcal{T}} \in \mathbf{H}_{\mathcal{T}}, \quad p_{\mathcal{T}} \in X_{\mathcal{T}} \\
&\langle \mathbf{u}_{\mathcal{T}}, \mathbf{v} \rangle_{\mathcal{T}} - \int_\Omega p_{\mathcal{T}} \operatorname{div}_{\mathcal{T}} \mathbf{v} dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx, \ \forall \mathbf{v} \in \mathbf{H}_{\mathcal{T}} \\
&\int_\Omega q \operatorname{div}_{\mathcal{T}} \mathbf{u}_{\mathcal{T}} dx = 0, \ \forall q \in X_{\mathcal{T}}
\end{aligned}
$$

discrete divergence operator :

For $\mathbf{v} \in \mathbf{H}_{\mathcal{T}}$

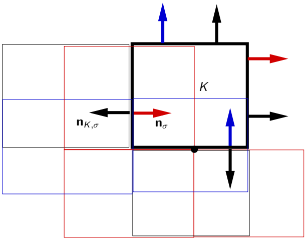
$\operatorname{div}_{\mathcal{T}} \mathbf{v}$ constant on $K \in \mathcal{T}$ defined by

$$\operatorname{div}_K \mathbf{v} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| v_\sigma \mathbf{n}_\sigma \cdot \mathbf{n}_{K,\sigma}$$

$\mathbf{n}_{K,\sigma} \perp \sigma$ outward to K

discrete pressure term

$$\int_\Omega p_{\mathcal{T}} \operatorname{div}_{\mathcal{T}} \mathbf{v} dx = \sum_{K \in \mathcal{T}} |K| p_K \operatorname{div}_K \mathbf{v}$$



$$
\begin{aligned}
&\mathbf{u}_{\mathcal{T}} \in \mathbf{H}_{\mathcal{T}}, \quad p_{\mathcal{T}} \in X_{\mathcal{T}} \\
&\langle \mathbf{u}_{\mathcal{T}}, \mathbf{v} \rangle_{\mathcal{T}} - \int_\Omega p_{\mathcal{T}} \operatorname{div}_{\mathcal{T}} \mathbf{v} dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx, \ \forall \mathbf{v} \in \mathbf{H}_{\mathcal{T}} \\
&\int_\Omega q \operatorname{div}_{\mathcal{T}} \mathbf{u}_{\mathcal{T}} dx = 0, \ \forall q \in X_{\mathcal{T}}
\end{aligned}
$$

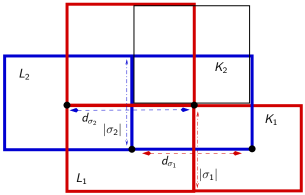
discrete viscous terms

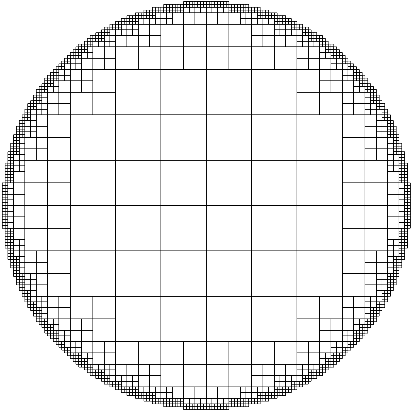
for $\mathbf{u}, \mathbf{v} \in \mathbf{H}_{\mathcal{T}}$,

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{T}} = \langle u_1, v_1 \rangle_{\mathcal{T}_1} + \langle u_2, v_2 \rangle_{\mathcal{T}_2}$$

where, for $i = 1, 2$

$$
\begin{aligned}
\langle u_i, v_i \rangle_{\mathcal{T}_i} &= -\int_\Omega \Delta_{\mathcal{T}_i} u_i \ v_i \\
&= \sum_{\sigma_i = (K_i, L_i)} |\sigma_i| d_{\sigma_i} \frac{u_{K_i} - u_{L_i}}{d_{\sigma_i}} \frac{v_{K_i} - v_{L_i}}{d_{\sigma_i}}
\end{aligned}
$$



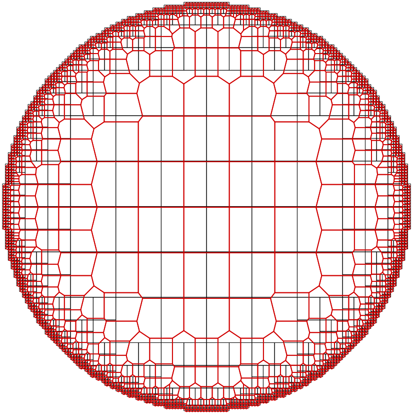


Pressure grid : again approximate

$$\mathrm{div}\mathbf{u} = 0$$

by

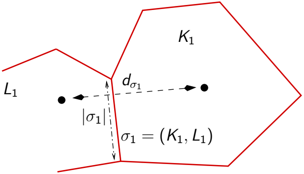
$$\mathrm{div}_K\mathbf{u} = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}|\sigma|u_\sigma\mathbf{n}_\sigma\cdot\mathbf{n}_{K,\sigma} = 0$$

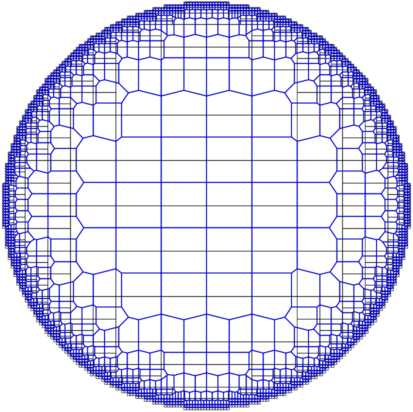


approximation of diffusion term

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{T}} = \langle u_1, v_1 \rangle_{\mathcal{T}_1} + \langle u_2, v_2 \rangle_{\mathcal{T}_2}$$

$$\langle u_1, v_1 \rangle_{\mathcal{T}_1} = \sum_{\sigma_1=(K_1,L_1)} |\sigma_1| d_{\sigma_1} \frac{u_{K_1} - u_{L_1}}{d_{\sigma_1}} \frac{v_{K_1} - v_{L_1}}{d_{\sigma_1}}$$

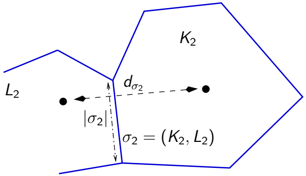




approximation of diffusion term

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{T}} = \langle u_1, v_1 \rangle_{\mathcal{T}_1} + \langle u_2, v_2 \rangle_{\mathcal{T}_2}$$

$$\langle u_2, v_2 \rangle_{\mathcal{T}_2} = \sum_{\sigma_2=(K_2,L_2)} |\sigma_2| d_{\sigma_2} \frac{u_{K_2} - u_{L_2}}{d_{\sigma_2}} \frac{v_{K_2} - v_{L_2}}{d_{\sigma_2}}$$



$(\mathcal{T}_n)_{n\in\mathbb{N}}$ sequence of grids, such that $h_n \to 0$ as $n \to +\infty$
$(\mathbf{u}_n, p_n)$ satisfying the discrete equations on mesh $\mathcal{T}_n$

1. Estimate on the discrete H_0^1 norm of the components of $\mathbf{u}_n$:
$$\mathbf{v} = \mathbf{u}_n \quad \text{and} \quad \|\mathbf{v}\|^2_{L^2(\Omega)^d} \le C\langle \mathbf{v}, \mathbf{v}\rangle_{\mathcal{T}_n} \quad \text{imply} \quad \langle \mathbf{u}_n, \mathbf{u}_n\rangle_{\mathcal{T}_n} \le C$$
2. $L^2(\Omega)$ estimate on p_n : $$p_n = \mathrm{div}\mathbf{v},\ v_\sigma = \frac{1}{|\sigma|}\int_\sigma \mathbf{v}\cdot\mathbf{n}_\sigma$$
3. Compactness : classical, consequence of Kolmogorov :
$$\mathbf{u}_n \to \mathbf{u} \in H_0^1(\Omega)^2 \quad \text{in} \quad L^2(\Omega)^2 \quad \text{and} \quad p_n \rightharpoonup p \quad \text{in} \quad L^2(\Omega)$$
4. Passing to the limit in velocity terms : $$\langle \mathbf{u}_n, \varphi_n\rangle_{\mathcal{T}_n} \to \int_\Omega \nabla\mathbf{u} : \nabla\varphi dx$$
5. Passing to the limit in divergence terms
$$\int_\Omega p_n \mathrm{div}_{\mathcal{T}}\boldsymbol{\varphi}_n dx \to \int_\Omega p\,\mathrm{div}\boldsymbol{\varphi} dx,\ \int_\Omega \varphi_n \mathrm{div}_{\mathcal{T}}\mathbf{u}_n dx \to \int_\Omega \varphi\,\mathrm{div}\mathbf{u} dx = 0$$
6. Strong convergence of p_n in $L^2(\Omega)$: convergence of $$\int_\Omega p_n^2 dx$$

$\varphi \in C_c^\infty(\Omega)$, $\varphi_K =$ mean value of φ on K

$$
\begin{aligned}
0 = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} |\sigma| u_{K,\sigma} \varphi_K \quad &= \sum_{\sigma=K|L} |\sigma| u_\sigma (\varphi_K - \varphi_L) \\
&= \sum_{\sigma=K|L} |K_\sigma| u_\sigma \, |\sigma| \frac{(\varphi_K - \varphi_L)}{|K_\sigma|} \\
&= -\sum_{i=1}^{d} \int_\Omega u_\mathcal{T}^{(i)} \partial_\mathcal{T}^{(i)} \varphi \ dx
\end{aligned}
$$

where

$$\partial_\mathcal{T}^{(i)} \varphi(\mathbf{x}) = |\sigma| \frac{(\varphi_L - \varphi_K)}{|K_\sigma|} \quad \text{for} \quad \mathbf{x} \in K_\sigma \quad \text{and} \quad \mathbf{n}_\sigma = \mathbf{e}^{(i)}, \ \sigma = K|L$$

conclusion thanks to weak convergence of $\partial_\mathcal{T}^{(i)} \varphi$ and strong convergence of $u_\mathcal{T}^{(i)}$

$$\begin{aligned} & \mathbf{u}_{\mathcal{T}} \in \mathbf{H}_{\mathcal{T}} \\ & \langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{T}} + b_{\mathcal{T}}(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \int_{\Omega} p_{\mathcal{T}} \mathrm{div}_{\mathcal{T}} \mathbf{v}, \forall \mathbf{v} \in \mathbf{H}_{\mathcal{T}} \\ & \int_{\Omega} q \ \mathrm{div}_{\mathcal{T}} \mathbf{u} dx = 0, \ \forall q \in X_{\mathcal{T}} \end{aligned} \qquad (\mathrm{NS}_{\mathcal{T}})$$

$$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_{\mathcal{T}}, \ b_{\mathcal{T}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma = \{K, L\}}} |\sigma| u_{\sigma} (\mathbf{\Pi}_L \mathbf{v} - \mathbf{\Pi}_K \mathbf{v}) \cdot \frac{\mathbf{\Pi}_K \mathbf{w} + \mathbf{\Pi}_L \mathbf{w}}{2}$$

with

$$\mathbf{\Pi}_K \mathbf{v} = \begin{pmatrix} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_{K,\mathrm{ver}}} |K \cap K_{\sigma}| \ v_{\sigma} \\ \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_{K,\mathrm{hor}}} |K \cap K_{\sigma}| \ v_{\sigma} \end{pmatrix}$$

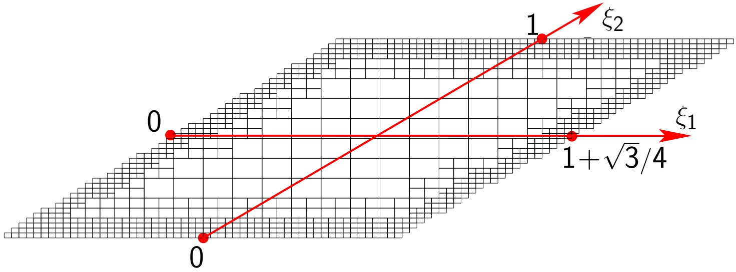
properties

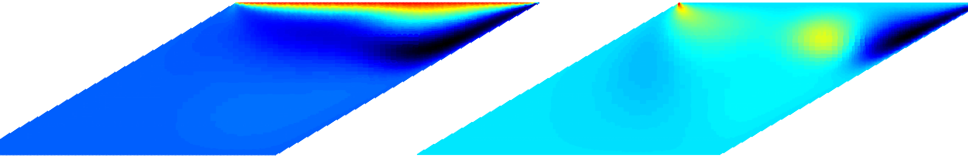
if $\mathrm{div}_{\mathcal{T}} \mathbf{u} = 0$
$b_{\mathcal{T}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b_{\mathcal{T}}(\mathbf{u}, \mathbf{w}, \mathbf{v})$
and $b_{\mathcal{T}}(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$

$\mathbf{v} = \mathbf{u}$ in $(\mathrm{NS})_{\mathcal{T}}$ yields discrete H_0^1 estimate in $\mathbf{u}$

Passing to the limit in $\displaystyle \sum_{i,j=1}^{d} \int_{\Omega} u_{\mathcal{T}}^{(j)} \ \ \partial_{\mathcal{T}}^{(j)} \mathbf{\Pi}_{\mathcal{T}}^{(i)} u_{\mathcal{T}}^{(i)} \ \ P_{\mathcal{T}}^{(i)} \varphi \ dx$

The 30° *inclined driven cavity : the locally refined mesh*





$u^{(1)}$ $u^{(2)}$

Velocity components, $\mathcal{D}_2^m$

	Generalized MAC scheme				*Demirdzic 92*
	$\mathrm{card}(\mathcal{D}_0^{\mathrm{m}})$ 924	$\mathrm{card}(\mathcal{D}_1^{\mathrm{m}})$ 3698	$\mathrm{card}(\mathcal{D}_2^{\mathrm{m}})$ 14796	$\mathrm{card}(\mathcal{D}_3^{\mathrm{m}})$ 59190	102400 $(= 320^2)$
$\min\left(u^{(1)}(\xi_2)\right)$	$-1.86 \cdot 10^{-1}$	$-1.99 \cdot 10^{-1}$	$-1.99 \cdot 10^{-1}$	$-1.98 \cdot 10^{-1}$	$-1.98 \cdot 10^{-1}$
ξ_2	$7.50 \cdot 10^{-1}$	$7.83 \cdot 10^{-1}$	$7.87 \cdot 10^{-1}$	$7.81 \cdot 10^{-1}$	$7.82 \cdot 10^{-1}$
$\min\left(u^{(2)}(\xi_1)\right)$	$-2.14 \cdot 10^{-2}$	$-2.25 \cdot 10^{-2}$	$-2.06 \cdot 10^{-2}$	$-2.01 \cdot 10^{-2}$	$-1.99 \cdot 10^{-2}$
ξ_1	$3.42 \cdot 10^{-1}$	$3.24 \cdot 10^{-1}$	$3.146 \cdot 10^{-1}$	$3.19 \cdot 10^{-1}$	$3.17 \cdot 10^{-1}$
$\max\left(u^{(2)}(\xi_1)\right)$	$1.36 \cdot 10^{-2}$	$1.36 \cdot 10^{-2}$	$1.25 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$
ξ_1	$7.85 \cdot 10^{-1}$	$8.04 \cdot 10^{-1}$	$8.26 \cdot 10^{-1}$	$8.24 \cdot 10^{-1}$	$8.26 \cdot 10^{-1}$

Max and min of velocity components along the centerlines ξ_1 and ξ_2.

A second extension to more general grids

$$u \in H_{\mathcal{T}}, \quad p \in X_{\mathcal{T}}$$
$$\langle u, v\rangle_{\mathcal{T}} - \int_\Omega p \,\mathrm{div}_{\mathcal{T}} v dx + b_{\mathcal{T}}(u,u,v) = \int_\Omega \mathbf{f}\cdot \mathbf{\Pi} v dx, \ \forall v \in H_{\mathcal{T}}$$
$$\int_\Omega q \ \mathrm{div}_{\mathcal{T}} u dx = 0, \ \forall q \in X_{\mathcal{T}}$$

$H_{\mathcal{T}} = \{u = (u_\sigma)_{\sigma\in\mathcal{E}}\}$ set of normal velocities

$X_{\mathcal{T}} = \{p = (p_K)_{K\in\mathcal{T}}\}$

2nd order reconst. $\mathbf{\Pi}_\sigma u$ such that $\mathbf{\Pi}_\sigma u \cdot \mathbf{n}_\sigma = u_\sigma$

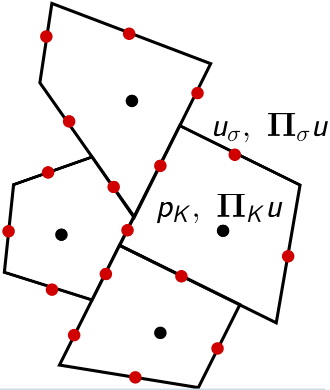
1st order reconst. $\mathbf{\Pi}_K u$

using SUSHI scheme :

$$\langle u, v\rangle_{\mathcal{T}} = \sum_{i=1}^{d} \langle \mathbf{\Pi}_{\mathcal{E}}^{(i)} u, \mathbf{\Pi}_{\mathcal{E}}^{(i)} v\rangle_{\mathcal{T}}$$

$$\mathrm{div}_K u = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}_K} |\sigma| u_\sigma \mathbf{n}_\sigma \cdot \mathbf{n}_{K,\sigma}$$

$$b_{\mathcal{T}}(u,v,w) = \sum_{\substack{\sigma\in\mathcal{E} \\ \sigma=\{K,L\}}} |\sigma| u_\sigma (\mathbf{\Pi}_L v - \mathbf{\Pi}_K v) \cdot \frac{\mathbf{\Pi}_K w + \mathbf{\Pi}_L w}{2}$$



$$u^{n+1} \in H_{\mathcal{T}},\ u = \theta u^{n+1} + (1-\theta)u^n,\ \theta \in [\frac{1}{2}, 1],\ p \in X_{\mathcal{T}}$$
$$\int_\Omega \frac{\mathbf{\Pi} u - \mathbf{\Pi} u^n}{\theta \delta t} \cdot \mathbf{\Pi} v dx$$
$$+\langle u, v\rangle_{\mathcal{T}} - \int_\Omega p \mathrm{div}_{\mathcal{T}} v dx + b_{\mathcal{T}}(u,u,v) = \int_\Omega \mathbf{f}^{n+1} \cdot \mathbf{\Pi} v dx,\ \forall v \in H_{\mathcal{T}}$$
$$\int_\Omega q\ \mathrm{div}_{\mathcal{T}} u dx = 0,\ \forall q \in X_{\mathcal{T}}$$

MAC scheme in space, estimate with $v = u$ and sum on n (it works since $\theta \geq \frac{1}{2}$)
θ-scheme in time
Compactness in time ?

B a Banach, $(B_n)_{n\in\mathbb{N}}$ family of finite dimensional subspaces of B
$\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ two norms on B_n such that

- If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

$X_n = B_n$ with norm $\|\cdot\|_{X_n}$, $Y_n = B_n$ with norm $\|\cdot\|_{Y_n}$.
Let $T > 0$, $k_n > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- for all n, $u_n(\cdot, t) = u_n^{(p)} \in B_n$ for $t \in ((p-1)k_n, pk_n)$
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), X_n)$,
- $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), Y_n)$.

Then $\exists u \in L^1((0,T), B)$ s.t., up to a subsequence, $u_n \to u$ in $L^1((0,T), B)$.

Application to incompressible NS extended MAC scheme
$B = L^2(\Omega)$, $B_n = H_{\mathcal{T}_n}$

$$\|w\|_{X_n} = \|w\|_{1,\mathcal{T}_n} \text{ and } \|w\|_{Y_n} = \sup_{v\in H_{\mathcal{T}_n}\setminus\{0\}} \frac{1}{\|v\|_{1,\mathcal{T}_n}} \int_\Omega w\ v\ dx$$