# The Dirac operator for graded affine Hecke algebras

Dan Ciubotaru
University of Utah

(joint work with D. Barbasch, P. Trapa)

Banff, July 2010

## The graded affine Hecke algebra $\mathbb{H}$

Let $\Phi = (V_0, R, V_0^\vee, R^\vee)$ be a real reduced root system.

Assume $R \subset V_0 \setminus \{0\}$ spans $V_0$. There is a perfect bilinear pairing $(\cdot,\cdot) : V_0 \times V_0^\vee \to \mathbb{R}$, and a bijection $R \longrightarrow R^\vee$, $\alpha \mapsto \alpha^\vee$. Let $W \subset GL(V_0^\vee)$ (or $GL(V_0)$) be the Weyl group, generated by $\{s_\alpha \mid \alpha \in R\}$.

$R^+ \subset R$, a positive system, and $\Pi$ simple roots in $R^+$. $R^{\vee,+}$ are the positive coroots.

Fix a $W$-invariant inner product $\langle \cdot,\cdot \rangle$ on $V_0^\vee$. Then $W \subset \mathsf{O}(V_0^\vee)$.

Set $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, and $V^\vee = V_0^\vee \otimes_{\mathbb{R}} \mathbb{C}$.

Fix a $W$-invariant "parameter function" $c : R \to \mathbb{R}$, and set $c_\alpha = c(\alpha)$.

**Definition** (Lusztig)**.** *The graded affine Hecke algebra* $\mathbb{H} = \mathbb{H}(\Phi, c)$ *is the complex associative algebra with unit,* $\mathbb{H} = \mathbb{C}[W] \otimes S(V^\vee)$ *where* $\mathbb{C}[W]$ *and* $S(V^\vee)$ *have the usual algebra structure, and*

$$\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = c_\alpha(\alpha, \omega), \quad \alpha \in \Pi,\ \omega \in V^\vee. \tag{1}$$

The center $Z(\mathbb{H}) = S(V^\vee)^W$. The central characters of irreducible $\mathbb{H}$-modules are parameterized by $W$-conjugacy classes in $V$.

If $\nu \in V$, let $\chi_\nu$ be the central character.

## The Casimir element

**Definition.** *If* $\{\omega_i : i = 1, n\}$ *and* $\{\omega^i : i = 1, n\}$ *are dual bases of* $V_0^\vee$ *with respect to* $\langle\ ,\ \rangle$*, define*

$$\Omega = \sum_{i=1}^{n} \omega_i \omega^i \in \mathbb{H}. \tag{2}$$

**Proposition.**

1. *The element* $\Omega$ *is well-defined independent of the choice of bases, and central in* $\mathbb{H}$.
2. *Let* $(\pi, X)$ *is an irreducible* $\mathbb{H}$*-module with central character* $\chi_\nu$*. Then*

$$\pi(\Omega) = \langle \nu, \nu \rangle \operatorname{Id}_X.$$

## Hermitian and unitary representations

The algebra $\mathbb{H}$ has a natural conjugate linear anti-involution:

$$\begin{aligned} t_w^* &= t_{w^{-1}}, \quad w \in W, \\ \omega^* &= -\omega + \sum_{\beta>0} c_\beta(\beta,\omega) t_{s_\beta}, \quad \omega \in V_0^\vee. \end{aligned} \tag{3}$$

This is the right one for the correspondence with unitary $p$-adic group representations.

An $\mathbb{H}$-module $(\pi, X)$ is Hermitian if there exists a Hermitian form $(\ ,\ )_X$ such that:

$$(\pi(h)x, y)_X = (x, \pi(h^*)y)_X, \quad \text{for all } h \in \mathbb{H},\ x, y \in X. \tag{4}$$

If such a form is positive definite, $X$ is unitary.

For every $\omega \in V_0^\vee$, define

$$\widetilde{\omega} = \omega - \frac{1}{2}\sum_{\beta>0} c_\beta(\beta,\omega)t_{s_\beta} \;\in\; \mathbb{H}. \tag{5}$$

Then $\widetilde{\omega}^* = -\widetilde{\omega}$.

If $(\pi, X)$ is Hermitian $\mathbb{H}$-module

$$(\pi(\widetilde{\omega})x, \pi(\widetilde{\omega})x)_X = (\pi(\widetilde{\omega}^*)\pi(\widetilde{\omega})x, x)_X = -(\pi(\widetilde{\omega}^2)x, x)_X. \tag{6}$$

A necessary condition for a Hermitian representation $X$ to be unitary is

$$(\pi(\widetilde{\omega}^2)x, x)_X \leq 0, \quad \text{for all } x \in X,\ \omega \in V_0^\vee. \tag{7}$$

**Definition.** *Let* $\{\omega_i\}, \{\omega^i\}$ *be dual bases of* $V_0^\vee$*. Define*

$$\widetilde{\Omega} = \sum_{i=1}^{n} \widetilde{\omega}_i \widetilde{\omega}^i \ \in \ \mathbb{H}. \tag{8}$$

*The operator* $\widetilde{\Omega}$ *is independent of the bases chosen, and lies in* $\mathbb{H}^W$*.*

A Hermitian $\mathbb{H}$-module $(\pi, X)$ with invariant form $(\ ,\ )_X$ is unitary only if

$$(\pi(\widetilde{\Omega})x, x)_X \leq 0, \quad \text{for all } x \in X. \tag{9}$$

**Theorem.**

$$\widetilde{\Omega} = \Omega - \Omega_W, \ \textit{where}$$

$$\Omega_W = \frac{1}{4} \sum_{\alpha>0, \beta>0} c_\alpha c_\beta \langle \alpha, \beta \rangle t_{s_\alpha} t_{s_\beta}. \tag{10}$$

We have $\Omega_W \in \mathbb{C}[W]^W$, so $\Omega_W$ acts in an irreducible $\sigma \in \widehat{W}$ by a scalar $C(\sigma)$.

**Corollary** (Casimir inequality)**.** *Let* $(\pi, X)$ *be a unitary irreducible* $\mathbb{H}$*-module with c.c.* $\chi_\nu$*,* $\nu \in V$*, and* $\mathrm{Hom}_W[\sigma, X] \neq 0$*. Then:*

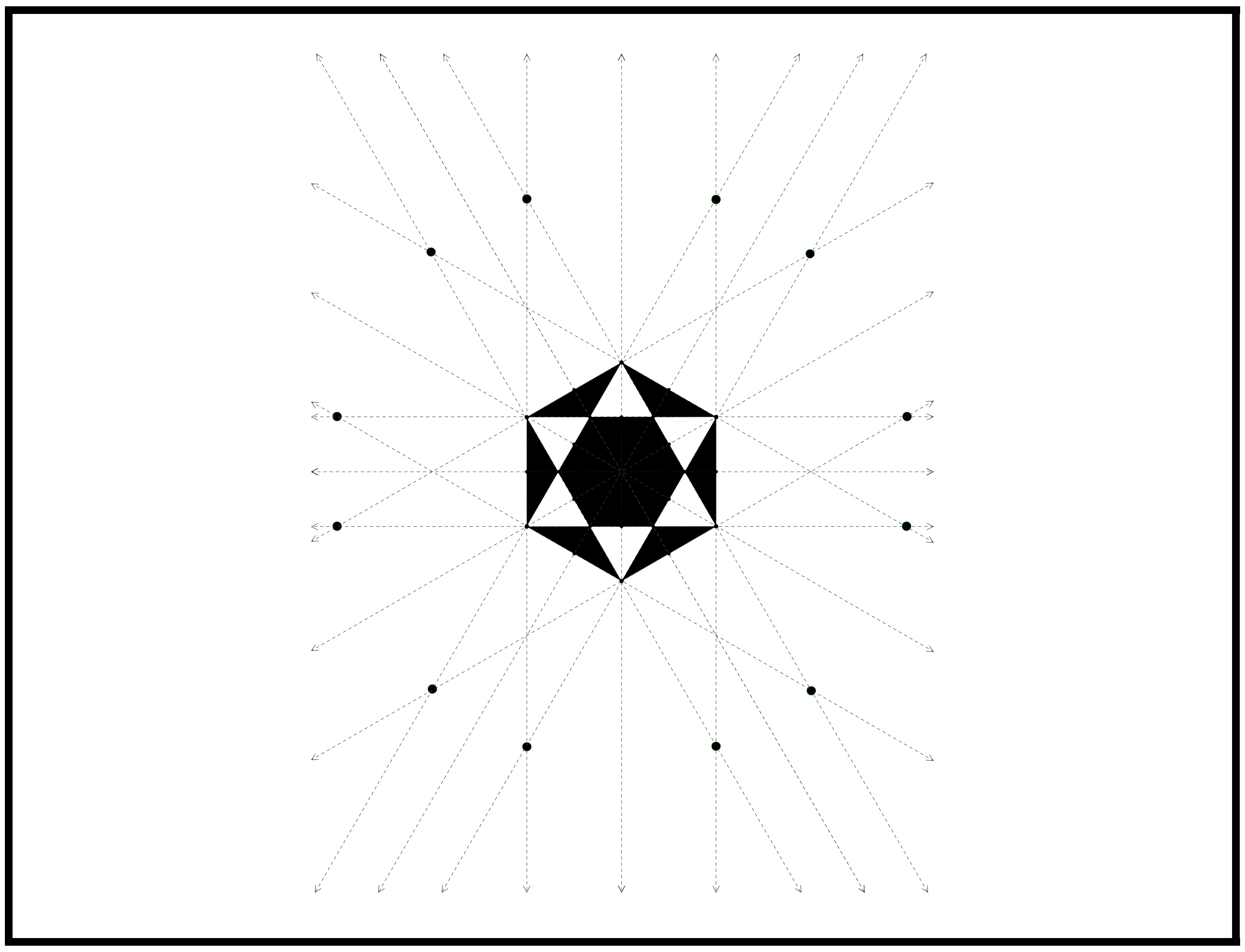
$$\langle \nu, \nu \rangle \leq C(\sigma).$$

The following is a consequence of the Casimir Inequality. For p-adic groups, this is known by other methods, *e.g.* [Casselman] or [Howe-Moore]. Assume $c \equiv 1$.

**Corollary.** *A hermitian irreducible representation* $X$ *with c.c.* $\chi_\nu$ *(*$\nu \in V$*) is unitary only if*

$$\langle \nu, \nu \rangle \leq \langle \rho, \rho \rangle.$$

*When equality holds and the root system is simple, only the trivial and the Steinberg modules are unitary.*



# The Dirac operator

## The Clifford algebra $C(V_0^\vee)$

Denote by $C(V_0^\vee)$ the Clifford algebra defined by $V_0^\vee$ and $\langle\ ,\ \rangle$; $C(V_0^\vee)$ is the associative algebra with unit generated by $V_0^\vee$ with relations:

$$\omega^2 = -\langle \omega, \omega \rangle, \quad \omega\omega' + \omega'\omega = -2\langle \omega, \omega' \rangle. \tag{11}$$

$\mathsf{O}(V_0^\vee)$ acts by algebra automorphisms on $C(V_0^\vee)$, and the action of $-1 \in \mathsf{O}(V_0^\vee)$ induces a grading

$$C(V_0^\vee) = C(V_0^\vee)_{\mathsf{even}} + C(V_0^\vee)_{\mathsf{odd}}.$$

There is an automorphism $\epsilon$: $\epsilon = +1$ on $C(V_0^\vee)_{\mathsf{even}}$ and $-1$ on $C(V_0^\vee)_{\mathsf{odd}}$;

let ${}^t$ be the transpose antiautomorphism

$$\omega^t = -\omega,\ \omega \in V_0^\vee, \quad (ab)^t = b^t a^t,\ a, b \in C(V_0^\vee).$$

The Pin group is

$$\mathsf{Pin}(V_0^\vee) = \{a \in C(V_0^\vee) \mid \epsilon(a) V_0^\vee a^{-1} \subset V_0^\vee,\ a^t = a^{-1}\}. \tag{12}$$

One has:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathsf{Pin}(V_0^\vee) \xrightarrow{p} \mathsf{O}(V_0^\vee) \longrightarrow 1, \tag{13}$$

where the projection $p$ is given by $p(a)(\omega) = \epsilon(a)\omega a^{-1}$.

We call a simple $C(V_0^\vee)$ module $(\gamma, S)$ of dimension $2^{[\dim V_0^\vee/2]}$ a spin module for $C(V_0^\vee)$. When $\dim V_0^\vee$ is even, there is only one such module, but if $\dim V_0^\vee$ is odd, there are two choices.

$(\gamma, S)$ restricts to an irreducible unitary representation of $\mathsf{Pin}(V_0^\vee)$.

**Definition** (Dirac operator)**.** *Fix a spin module $(\gamma, S)$ for $C(V_0^\vee)$, and let $(\pi, X)$ be a $\mathbb{H}$-module. The Dirac operator for $X$ (and $S$) is*

$$D = \sum_{i=1}^{n} \pi(\widetilde{\omega}_i) \otimes \gamma(\omega^i) \in \mathrm{End}_{\mathbb{H}\otimes C(V_0^\vee)}(X \otimes S). \tag{14}$$

For $X$ hermitian with invariant form $(\ ,\ )_X$, $D$ is self-adjoint, *i.e.*

$$(D(x \otimes s), x' \otimes s')_{X \otimes S} = (x \otimes s, D(x' \otimes s'))_{X \otimes S} \tag{15}$$

**Proposition.** *A hermitian* $\mathbb{H}$*-module is unitary only if*

$$(D^2(x \otimes s), x \otimes s)_{X \otimes S} \geq 0, \qquad \text{for all } x \otimes s \in X \otimes S. \tag{16}$$

## The spin cover $\widetilde{W}$

We have $W \subset \mathsf{O}(V_0^\vee)$. We define $\widetilde{W} \subset \mathsf{Pin}(V_0^\vee)$:

$$\widetilde{W} := p^{-1}(\mathsf{O}(V_0^\vee)) \subset \mathsf{Pin}(V_0^\vee),\ \text{where } p \text{ is the projection.}$$

$\widetilde{W}$ is a central extension of $W$,

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{W} \xrightarrow{p} W \longrightarrow 1.$$

Explicitly, $\widetilde{W} \subset C(V_0^\vee)$ is generated by

$$z = -1, \quad \widetilde{s}_\alpha = \check{\alpha}/|\check{\alpha}|, \quad \alpha \in R^+.$$

The analogue of the Coxeter presentation is:

$$\widetilde{W} = \langle z, \widetilde{s}_\alpha,\ \alpha \in \Pi :\ z^2 = 1,\ (\widetilde{s}_\alpha)^2 = z,\ (\widetilde{s}_\alpha \widetilde{s}_\beta)^{m(\alpha,\beta)} = 1 \rangle.$$

Via restriction, we can regard a spin module $(\gamma, S)$ for $C(V_0^\vee)$ as a unitary genuine irreducible $\widetilde{W}$ representation.

We write $\tau$ for the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ into $\mathbb{H} \otimes C(V_0^\vee)$ defined by linearly extending

$$\tau(\widetilde{w}) = t_{p(\widetilde{w})} \otimes \widetilde{w}. \tag{17}$$

The following is an analogue of Parthasarathy's formula.

**Theorem.** *(Barbasch-C.-Trapa)*

$$D^2 = -\Omega \otimes 1 + \tau(\Omega_{\widetilde{W}}). \tag{18}$$

*Here*

$$\Omega_{\widetilde{W}} = \sum_{\alpha>0,\beta>0,\langle\alpha,\beta\rangle\neq 0} c_\alpha c_\beta \frac{\langle\alpha,\beta\rangle}{|\cos(\alpha,\beta)|} \widetilde{s}_\alpha \widetilde{s}_\beta$$

*and acts by a scalar* $C(\widetilde{\sigma})$ *on any irreducible* $\widetilde{W}-$*module.*

## The Dirac inequality

**Corollary.** *Assume that $X$ is irreducible and unitary with central character $\chi_\nu$ with $\nu \in V$. Let $(\widetilde{\sigma}, \widetilde{U})$ be an irreducible representation of $\widetilde{W}$ such that* $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, X \otimes S) \neq 0$. *Then*

$$\langle \nu, \nu \rangle \leq C(\widetilde{\sigma}). \qquad (19)$$

To apply it, we need to know $C(\widetilde{\sigma})$ for $\widetilde{\sigma}$ a genuine $\widetilde{W}$-type.

Example: If $\widetilde{\sigma} = S$ (spin module), then $C(S) = \langle \rho, \rho \rangle$, when $c \equiv 1$.

## A classification of genuine $\widetilde{W}$-types

Assume $c \equiv 1$. Let $\mathrm{Irr}_{\mathsf{gen}}(\widetilde{W}) \subset \mathrm{Irr}(\widetilde{W})$ denote the genuine representations of $\widetilde{W}$.

Let $\mathfrak{g}$ be the Lie algebra for $\Phi$, with Cartan subalgebra $\mathfrak{h} = V$. Let $G$ be the adjoint group.

Let $\mathcal{N}_{\mathsf{sol}}$ be the set of nilpotent adjoint orbits in $\mathfrak{g}$ whose centralizer in $\mathfrak{g}$ is solvable. E.g.: type $A$ these are orbits whose Jordan blocks are all distinct. In general, every distinguished orbit is in $\mathcal{N}_{\mathsf{sol}}$.

If $e$ is nilpotent, let $\nu_e \in V_0$ denote one half of a Jacobson-Morozov "middle" element. Let $A(e)$ denote the A-group, and $\mathrm{Irr}_0 A(e)$ the set of irreducible representations of $A(e)$ of Springer type.

**Theorem** (C.)**.** *There is a surjective map*

$$\Psi : \ \mathrm{Irr}_{\mathsf{gen}}(\widetilde{W}) \longrightarrow G\backslash\mathcal{N}_{\mathsf{sol}} \tag{20}$$

*with the following properties:*

1. *If* $\Psi(\widetilde{\sigma}) = G \cdot e$, *then*

$$C(\widetilde{\sigma}) = \langle \nu_e, \nu_e \rangle, \tag{21}$$

*where* $C(\widetilde{\sigma})$ *is the scalar from the* $D^2$ *formula, and* $\nu_e$ *is the middle element of the corresponding nilpotent orbit.*

*2. (a) If* $e \in \mathcal{N}_{\mathsf{sol}}$ *and* $\phi \in \mathrm{Irr}_0(A(e))$*, and* $S$ *is a spin module, then there exists* $\widetilde{\sigma} \in \Psi^{-1}(G \cdot e)$ *so that*

$$\mathrm{Hom}_W \left(\sigma_{e,\phi}, \widetilde{\sigma} \otimes S\right) \neq 0.$$

*(b) If* $\Psi(\widetilde{\sigma}) = G \cdot e$*, then there exists* $\phi \in \mathrm{Irr}_0(A(e))$ *and a spin module* $S$ *such that*

$$\mathrm{Hom}_W \left(\sigma_{e,\phi}, \widetilde{\sigma} \otimes S\right) \neq 0.$$

*3. If* $e$ *is distinguished, 2) induces a unique bijection*

$$\Psi^{-1}(G \cdot e)/_{\widetilde{\sigma} \sim \widetilde{\sigma} \otimes \mathsf{sgn}} \longleftrightarrow \mathrm{Irr}_0(A(e)).$$

*Here* $\sigma_{e,\phi}$ *is the Weyl group representation associated to* $(e, \phi)$ *by the Springer correspondence.*

**Corollary.** *Suppose* $(\pi, X)$ *is an irreducible unitary* $\mathbb{H}$*-module with central character* $\chi_\nu$ *with* $\nu \in V$.

*(a) Let* $(\widetilde{\sigma}, \widetilde{U})$ *be a representation of* $\widetilde{W}$ *such that* $\mathrm{Hom}_{\widetilde{W}}[\widetilde{U}, X \otimes S] \neq 0$. *Write* $\Psi(\widetilde{\sigma}) = G \cdot e$. *Then*

$$\langle \nu, \nu \rangle \leq \langle \nu_e, \nu_e \rangle. \tag{22}$$

*(b) Suppose* $e \in \mathcal{N}_{\mathsf{sol}}$ *and* $\phi \in \mathrm{Irr}_0(A(e))$ *such that* $\mathrm{Hom}_W[\sigma_{(e,\phi)}, X] \neq 0$. *Then*

$$\langle \nu, \nu \rangle \leq \langle \nu_e, \nu_e \rangle. \tag{23}$$

## Some consequences

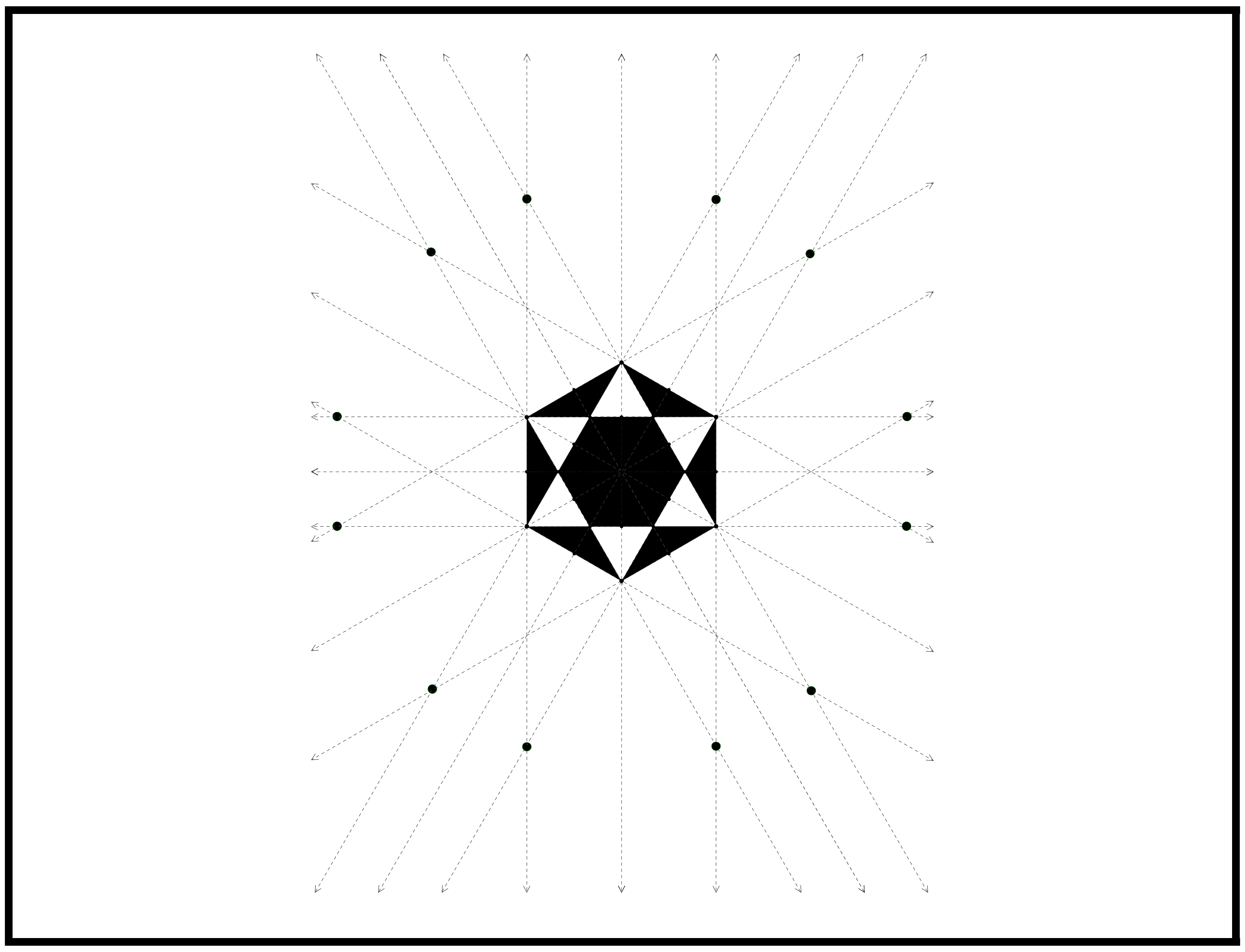
1) If $\mathfrak{g}$ is simple, then there is a unique *subregular* orbit (lies in $\mathcal{N}_{\mathsf{sol}}$). Let $\nu_{\mathsf{sr}} \in V_0$ be one half a middle element for it.

**Corollary.** *If $X$ is irreducible unitary with central character $\chi_\nu$ and $X$ is not trivial or Steinberg, then*

$$\langle \nu, \nu \rangle \leq \langle \nu_{\mathsf{sr}}, \nu_{\mathsf{sr}} \rangle.$$

In particular, for spherical modules, we get the best possible spectral gap for the trivial representation. (At $\nu = \nu_{\mathsf{sr}}$, there *is* a unitary spherical module: the minimal representation.)

Example: The spherical unitary dual of $G_2$.



2) The Kazhdan and Lusztig classification gives for every $(e, \phi)$, $\phi \in \mathrm{Irr}_0 A(e)$, an irreducible tempered $\mathbb{H}$-module $X_t(e, \phi)$ such that:

(a) $X_t(e, \phi)$ has central character $\nu_e$;

(b) $X_t(e, \phi)|_W$ contains the Springer representation $\sigma_{(e,\phi)}$ (with multiplicity one) - as the "lowest $W$-type".

Then inequality (23) implies:

**Corollary.** *An irreducible $\mathbb{H}$-module $X$ that has a lowest $W$-type $\sigma_{(e,\phi)}$, for $e \in \mathcal{N}_{\mathsf{sol}}$, is unitary if and only if $X$ is tempered.*

## Dirac cohomology

Let $(\pi, X)$ be an irreducible $\mathbb{H}$ module with central character $\chi_\nu$.

The kernel $\ker D$ on $X \otimes S$ is invariant under $\widetilde{W}$. The $D^2$ formula implies that if $\widetilde{\sigma}$ occurs in $\ker D$, then

$$\langle \nu, \nu \rangle = C(\widetilde{\sigma}).$$

So the length of $\nu$ is determined by the $\widetilde{W}$ structure of $\ker(D)$.

Define the Dirac cohomology of $X$:

$$H^D(X) := \ker D \big/ (\ker D \cap \operatorname{im} D) \tag{24}$$

For example, if $X$ is unitary, $\ker(D) \cap \operatorname{im}(D) = 0$, and $H^D(X) = \ker(D)$.

**Proposition.** *Let $(\pi, X)$ be an irreducible unitary $\mathbb{H}$ module with central character $\chi_\nu$ with $\nu \in V$. Suppose $(\widetilde{\sigma}, \widetilde{U})$ is an irreducible representation of $\widetilde{W}$ such that $\operatorname{Hom}_{\widetilde{W}}(\widetilde{U}, X \otimes S) \neq 0$. Write $\Psi(\widetilde{\sigma}) = G \cdot e$. Assume further that $\langle \nu, \nu \rangle = \langle \nu_e, \nu_e \rangle$. Then*

$$\operatorname{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X)) \neq 0.$$

*Proof.* Let $x \otimes s$ be an element of the $\widetilde{\sigma}$ isotypic component of $X \otimes S$. By the formula for $D^2$ and for $C(\widetilde{\sigma})$, we have

$$D^2(x \otimes s) = (-\langle \nu, \nu \rangle + \langle \nu_e, \nu_e \rangle)\,(x \otimes s) = 0. \tag{25}$$

Since $X$ is unitary, $\ker D \cap \operatorname{im} D = 0$, and so (25) implies $x \otimes s \in \ker(D) = H^D(X)$. □

**Corollary.** *Let $(\widetilde{\sigma}, \widetilde{U})$ be a $\widetilde{W}$-type such that $\Psi(\widetilde{\sigma}) = G \cdot e$. Then there exists $\phi \in \mathrm{Irr}_0\, A(e)$ such that*

$$\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X_t(e, \phi))) \neq 0.$$

## Vogan's conjecture

The analogue of Vogan's conjecture from real groups (proved for real groups by Huang-Pandzić) is the following.

**Theorem** (Barbasch-C.-Trapa)**.** *Suppose* $(\pi, X)$ *is an* $\mathbb{H}$ *module with central character* $\chi_\nu$ *with* $\nu \in V$*. Suppose that* $H^D(X) \neq 0$*. Let* $(\widetilde{\sigma}, \widetilde{U})$ *be a representation of* $\widetilde{W}$ *such that* $\mathrm{Hom}_{\widetilde{W}}(\widetilde{U}, H^D(X)) \neq 0$*. Write* $\Psi(\widetilde{\sigma}) = G \cdot e$*. Then*

$$\chi_\nu = \chi_{\nu_e}.$$

As in setting of real groups, Vogan's conjecture can be deduced from a purely algebraic statement.

**Theorem.** *Let* $z \in Z(\mathbb{H})$ *be given. Then there exists* $a \in \mathbb{H} \otimes C(V_0^\vee)$ *and a unique element* $\zeta(z)$ *in the center of* $\mathbb{C}[\widetilde{W}]$ *such that*

$$z \otimes 1 = \tau(\zeta(z)) + Da + aD$$

*as elements in* $\mathbb{H} \otimes C(V_0^\vee)$*.*

Proof that the algebraic theorem implies Vogan's conjecture.

*Proof.* Let $\widetilde{x} = x \otimes s \in \ker D \setminus \mathrm{im} D$ be in the isotypic component of a $\widetilde{W}$-type $\widetilde{\sigma}$. Then:

$$(z \otimes 1 - \tau(\zeta(z))\widetilde{x} = (\chi_\nu(z) - \widetilde{\sigma}(\zeta(z)))\widetilde{x};$$
$$(Da + aD)\widetilde{x} = Da\widetilde{x}.$$

Therefore, $\chi_\nu(z) = \widetilde{\sigma}(\zeta(z))$ for all $z \in Z(\mathbb{H})$. Assume $\Psi(\widetilde{\sigma}) = G \cdot e$. We need $\widetilde{\sigma}(\zeta(z)) = \chi_{\nu_e}(z)$ for all $z$. But this follows by invoking the corollary to the previous proposition. □