A Quasisymmetric function for Generalized Permutahedron

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Hopf Monoids

A Hopf monoid $(\mathsf{H}, \mu, \Delta)$ is given by:

- For each finite set I, a set $\mathsf{H}[I]$.
- For each $I = S \sqcup T$, maps

$$\mathsf{H}[S] \times \mathsf{H}[T] \xrightarrow{\mu_{S,T}} \mathsf{H}[I] \quad \text{and} \quad \mathsf{H}[I] \xrightarrow{\Delta_{S,T}} \mathsf{H}[S] \times \mathsf{H}[T].$$

  with associativity and compatibility axioms.

Notation:

$$x \cdot y := \mu_{S,T}(x, y) \qquad \text{for } x \in \mathsf{H}[S],\ y \in \mathsf{H}[T],$$

$$(z|_S, z/_S) := \Delta_{S,T}(z) \qquad \text{for } z \in \mathsf{H}[I].$$

Consequences: iterations of μ and Δ are well-defined

$$\mathsf{H}[R]\times\mathsf{H}[S]\times\mathsf{H}[T] \xrightarrow{\mu_{R,S,T}} \mathsf{H}[I] \quad \text{and} \quad \mathsf{H}[I] \xrightarrow{\Delta_{R,S,T}} \mathsf{H}[R]\times\mathsf{H}[S]\times\mathsf{H}[T]$$

and compatible with respect to differing decompositions.

For any $x \in \mathsf{H}[I]$,

$$\Delta_{R,S,T}(x) = (x|_R)((x|_{R\cup S})/_R)((x|_{R\cup S\cup T})/_{R\cup S})$$

The Hopf monoid of matroids

$\mathsf{M}[I]$:= the set of matroids with ground set I.

M is a Hopf monoid with

$$\begin{aligned} \mu(m_1, m_2) &:= \text{direct sum of } m_1 \text{ and } m_2, \\ \Delta_{S,T}(m) &:= (m|_S, m/_s) \end{aligned}$$

$$\begin{aligned} m|_S &:= \text{restriction of } m \text{ to } S, \\ m/_S &:= \text{contraction of } S \text{ from } m. \end{aligned}$$

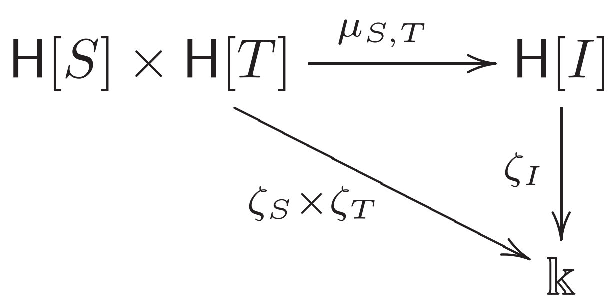
Characters

Let H be a Hopf monoid. A character ζ consists of maps

$$\zeta_I : \mathsf{H}[I] \longrightarrow \Bbbk$$

such that for each $I = S \sqcup T$, $x \in \mathsf{H}[S]$, $y \in \mathsf{H}[T]$,

$$\zeta_S(x) \cdot \zeta_T(y) = \zeta_I(\mu_{S,T}(x, y)) :$$



Given a Hopf monoid and a character we define two things:

1. A quasisymmetric function
2. A simplicial complex

Let $F \vDash I$, $F = F^1 | F^2 | \cdots | F^k$ be an ordered set partition of I (Interpret as a flag)

$$\mathsf{H}[I] \quad \xrightarrow{\Delta_F} \quad \mathsf{H}[F^1] \times \cdots \times \mathsf{H}[F^k]$$

$$x \longrightarrow x|_{F^1} \times \cdots \times (x|_{F^1 \cup \cdots \cup F^k})/_{F^1 \cup \cdots \cup F^{k-1}}$$

$$x|_{F^1} \times \cdots \times (x|_{F^1 \cup \cdots \cup F^k})/_{F^1 \cup \cdots \cup F^{k-1}} \xrightarrow{\zeta^{F^1} \times \cdots \times \zeta^{F^k}} \Bbbk$$

$$x \xrightarrow{\zeta^F} \Bbbk$$

Invariants

$$\sigma_\zeta(x) = \{F \vDash I : \zeta^F(x) = 0\}$$
$$\Sigma_\zeta(x) = \{F \vDash I : \zeta^F(x) \neq 0\}$$

Quasisymmetric function:

$$G(x) = \sum_{F \in \sigma_\zeta(X)} M_F \qquad \xrightarrow{\text{type}} \qquad Q(x) = \sum_{\alpha : \text{type}(F) = \alpha} f_\alpha M_\alpha$$

Simplicial Complex:

$\Sigma_\zeta(x)$ Closed under coarsening in the Coxeter complex

Generalized Permutahedron

A Generalized Permutahedron is any polytope $P \in \mathbb{R}^n$ such that every edge is parallel to $e_i - e_j$ for some i, j.

(Any polytope obtained from the permutahedron by parallel transporting some of its edges.)

Includes: permutahedra, associahedra, graphic zonotopes, matroid polytopes

Hopf Monoid of Generalized Permutahedron

Aguiar and Ardila

$\mathsf{GP}[I] :=$ the set of generalized permutahedra in $\mathbb{R}^I$.

GP is a Hopf monoid with

$$(P, Q) \xrightarrow{\mu_{S,T}} P \times Q \qquad P \xrightarrow{\Delta_{S,T}} (P|_S, P/_S)$$

$P_S \times P/_S = P_{S,T} =$ face of P maximized by any vector $v_{S,T} \in \mathbb{R}^I$ constant on S and T with $v_S > v_T$.

Let $\zeta_I : \mathsf{GP}[I] \to \Bbbk$ be

$$\zeta_I(P) \ := \begin{cases} 1 & \text{if } P \text{ is a point,} \\ 0 & \text{otherwise.} \end{cases}$$

Quasisymmetric invariant for GP

Given $P \in \mathbb{R}^n$, $F \models I$

$F \longrightarrow$ class of linear functionals ω_F,

constant on each F^i and $\omega_{F^i} < \omega_{F^{i+1}}$

Let P_F be the face of P minimized by any element of ω_F

$$G(P) = \sum_{F : P_F \text{is a point}} M_F$$

$$Q(P) = \sum_{\substack{\alpha : \text{type}(F) = \alpha \\ F : P_F \text{is a point}}} f_\alpha M_\alpha$$

Simplicial complex for GP

$F : P_F$ is a point = integral pts in the interior of the normal fan.

Postnikov-Reiner-Williams
Morton-Pachter-Shiu-Sturmfels-Wienand:
P is a generalized permutahedron iff its normal fan is refined by the Braid arrangement.

$\Sigma_\zeta(P)$ = codimensional one skeleton of the normal fan of P as subdivided by the Coxeter complex.

Special Instances

- Graphs (graphical zonotopes)

  - $Q(G)$ = Stanley chromatic symmetric function

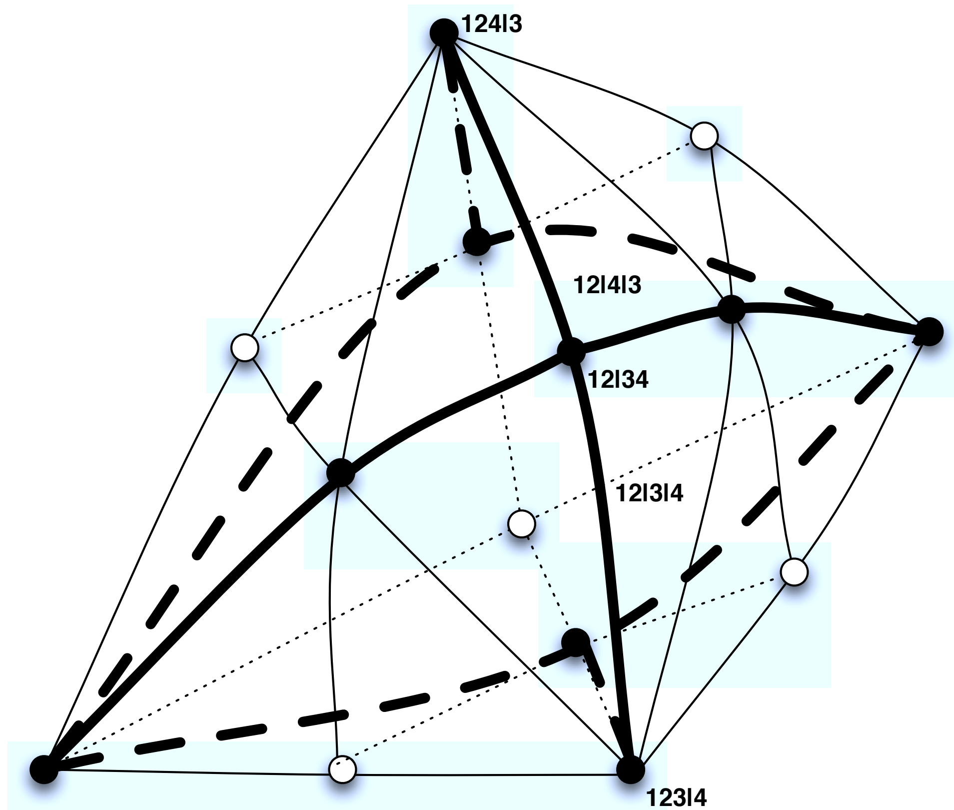
  - $\Sigma(G)$ = Steingrimsson coloring complex

- Matroids (matroid base polytope)

  - $Q(P_M)$ = Billera-Jia-Reiner quasisymmetric function

  - $\Sigma(P_M)$ = A new simplicial complex associated to M

Example: $M = \{13, 14, 23, 24, 34\}$



Positivity Results

- h-positivity of the complex

  - Subdivision of skeleton of polar fan

  - Cohen Macaulay topological condition

- L-positivity of the quasisymmetric function

  - Holds for generating function over ordered set partitions

  - Geometric argument in Coxeter complex

Directed Faces

Aguiar and Mahajan

A directed face of the Coxeter complex is a pair (C, D)
(C = face, D = chamber which contains C).

Define a partial order on directed faces:

$$(F, C) \leq (G, D) \Longleftrightarrow C = D \text{ and } F \leq G$$

M, L bases:

$$L_{(G,D)} = \sum_{H : G \leq H \leq D} M_{(H,D)}$$

Specialize to first index and take type to return to monomial and fundamental basis of Qsym.

L-positivity

For chambers A, B, let $AB =$ walk on chambers from A to B
(covector composition from oriented matroids)

Fix a cone (poset) p in the Coxeter complex.
(i.e. some subcomplex cut our by hyperplanes)

$\exists$ a chamber C s.t. if $F, G \in \hat{p}$ and $FC = GC$ then $F \cap G \in \hat{p}$
(In fact any C whose opposite lies in p will work.)

L-positivity

Let v be a vertex of P

$$G(P) = \sum_{v \in P} \sum_{F \in v^{\perp}} M_F$$

$$\sum_{F \in \hat{p}} M_{(F,FC)} = \sum_{D \in \hat{p}} L_{(Des_p(C,D),D)}$$

where $\mathrm{Des}_p(C, D) = \min \{F \in \hat{p} : FC = D\}$

Connection between $Q(P)$ and $\Sigma(P)$

Let $h_\Sigma(t) = h$-polynomial of Σ
Let $\chi_{Q(P)} = Q(P)(1^{\mathbf{m}})$ polynomial specialization of $Q(P)$

$$\sum_{n\geq 0}((n+1)^d - \chi_{Q(P)}(n+1))t^n = \frac{h_\Sigma(t)}{(1-t)^d}$$

Steingrimsson proves this formula for χ equal to the chromatic polynomial and Σ equal to the coloring complex.

True for any subcomplex of the Coxeter complex.

Alternate formulation:

$$h_\Delta(t) = \frac{A_n(t) - W(t)}{t(1-t)}$$

$A_n(t) = a_0 + a_1 t + \cdots + a_n t^n$ is the Eulerian polynomial
a_i = number of permutations in S_n with i descents

$W(t) = w_0 + w_1 t + \cdots + w_n t^n$
w_l = sum of coefficients of $\chi_{Q(P)}$ of compositions of length l.

This formula + positivity of h vector implies

$$a \trianglerighteq w$$

Bergman Quasisymmetric function

Change of character in Hopf monoid of matroids

Let $\zeta_I : \mathsf{M}[I] \to \Bbbk$ be

$$\zeta_I(P) \;:=\; \begin{cases} 1 & \text{if } M \text{ has no loops,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \Sigma(M) &= \text{Integer points in some subcomplex of normal fan of } P_M \\ &= \text{Order complex of lattice of flats of } M \\ &\quad\; \text{as subdivided by the Coxeter complex} \\ &= \text{Bergman complex of } M \end{aligned}$$