Descriptor system techniques in solving $\mathcal{H}_2$-optimal fault detection problems

Andras Varga

German Aerospace Center (DLR)

DAE'10 Workshop – Banff, Canada, October 25-29, 2010

Outline

- approximate fault detection and isolation problem
- $\mathcal{H}_2$-optimal model-matching approach
- enhanced model-matching procedure
- computational issues
- illustrative example
- conclusions

Fault model

Additive LTI fault model

$$\mathbf{y}(\lambda) = G_u(\lambda)\mathbf{u}(\lambda) + G_d(\lambda)\mathbf{d}(\lambda) + G_w(\lambda)\mathbf{w}(\lambda) + G_f(\lambda)\mathbf{f}(\lambda),$$

where $\lambda = s$ for continuous-time (Laplace transform)
$\lambda = z$ for discrete-time (Z-transform)

$y(t) \in \mathbb{R}^p$	–	system output (measurable)
$u(t) \in \mathbb{R}^{m_u}$	–	control input (measurable)
$d(t) \in \mathbb{R}^{m_d}$	–	disturbance input (unknown)
$w(t) \in \mathbb{R}^{m_w}$	–	noise input (unknown)
$f(t) \in \mathbb{R}^{m_f}$	–	fault input (unknown)

Note: No restrictions on the *transfer-function matrices* (TFMs) $G_u(\lambda)$, $G_d(\lambda)$, $G_w(\lambda)$, $G_f(\lambda)$ (improper OK!)

Exact fault detection and isolation problem (EFDIP)

Determine a stable and proper residual generator

$$\mathbf{r}(\lambda) = Q(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix}$$

and a stable and proper diagonal filter specification $M_r(\lambda)$ such that $\forall\ u(t)$, $d(t)$, and for $w(t) \equiv 0$

$$r(\lambda) = M_r(\lambda) f(\lambda)$$

Algebraic conditions for EFDIP

Residual generation system:

$$r(\lambda) = R_u(\lambda)\mathbf{u}(\lambda) + R_d(\lambda)\mathbf{d}(\lambda) + R_w(\lambda)\mathbf{w}(\lambda) + R_f(\lambda)\mathbf{f}(\lambda)$$

where

$$[\, R_u(\lambda) | R_d(\lambda) | R_w(\lambda) | R_f(\lambda) \,] := Q(\lambda) \left[\begin{array}{cccc} G_u(\lambda) & G_d(\lambda) & G_w(\lambda) & G_f(\lambda) \\ I_{m_u} & 0 & 0 & 0 \end{array} \right]$$

Synthesis goal: Choose appropriate $M_r(\lambda)$ to ensure that

$$R_u(\lambda) = 0, \quad R_d(\lambda) = 0, \quad R_f(\lambda) = M_r(\lambda)$$

$$\Leftrightarrow Q(\lambda) \left[\begin{array}{c|c|c} G_u(\lambda) & G_d(\lambda) & G_f(\lambda) \\ I_{m_u} & 0 & 0 \end{array} \right] = [\, 0 \; 0 \; M_r(\lambda) \,]$$

Approximate fault detection and isolation problem (AFDIP)

Determine a stable and proper residual generator

$$\mathbf{r}(\lambda) = Q(\lambda) \begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix}$$

and a stable and proper diagonal filter specification $M_r(\lambda)$ such that $\forall\ u(t), d(t), w(t)$

$$\mathbf{r}(\lambda) \approx M_r(\lambda)\mathbf{f}(\lambda)$$

Synthesis goals: Choose appropriate $M_r(\lambda)$ to ensure that

$$R_u(\lambda) = 0, \quad R_d(\lambda) = 0, \quad R_w(\lambda) \approx 0, \quad R_f(\lambda) \approx M_r(\lambda)$$

with $R_f(\lambda)$ and $R_w(\lambda)$ stable and proper.

Interpretation of $d(t)$ and $w(t)$

Disturbance input $d(t)$**:** includes all additive effects from which exact decoupling of the residuals is presumably possible and is targeted in the detector synthesis.

Noise input $w(t)$**:** contains everything else, i.e., proper random noise, or “ordinary” disturbances in excess of those which may be exactly decoupled, or fictive inputs which model the effect of parametric uncertainties in the process model.

Advantage: This distinction between $d(t)$ and $w(t)$ allows to address the solution of both exact and approximate fault detection problems using a unique computational framework.

Solvability conditions

Theorem 1: An $M_r(\lambda)$ exists such that EFDIP is solvable iff

$$rank\,[\,G_f(\lambda)\;G_d(\lambda)\,] = m_f + rank\,G_d(\lambda) \qquad (1)$$

Corollary 1: If $m_d = 0$, an $M_r(\lambda)$ exists such that EFDIP is solvable iff

$$rank\,G_f(\lambda) = m_f \qquad (2)$$

Theorem 2: An $M_r(\lambda)$ exists such that AFDIP is solvable iff (1) is fulfilled.

Corollary 2: If $m_d = 0$, an $M_r(\lambda)$ exists such that AFDIP is solvable iff (2) is fulfilled.

$\mathcal{H}_2$-optimal model-matching approach

Solve $\mathbf{r}(\lambda) \approx M_r(\lambda)\mathbf{f}(\lambda)$ by minimizing the $\mathcal{H}_2$-norm of

$$\mathcal{R}(\lambda) := F(\lambda) - Q(\lambda)G(\lambda),$$

with

$$F(\lambda) = [\, M_r(\lambda)\ O\ O\ O\,],$$

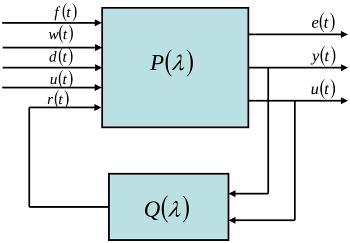
$$G(\lambda) = \left[\begin{array}{cccc} G_f(\lambda) & G_w(\lambda) & G_d(\lambda) & G_u(\lambda) \\ 0 & 0 & 0 & I_{m_u} \end{array} \right]$$

Approach: Rewrite $\mathcal{R}(\lambda)$ as the transfer function matrix of a generalized plant $P(\lambda)$ with $Q(\lambda)$ as feedback controller and determine the optimal $Q(\lambda)$ using standard $\mathcal{H}_2$ synthesis tools (e.g., `h2syn` of ML).

$\mathcal{H}_2$-optimal synthesis setting

Underlying equations:

$$\begin{aligned} \mathbf{e}(\lambda) &= \mathbf{r}(\lambda) - M_r(\lambda)\mathbf{f}(\lambda) \\ \mathbf{y}(\lambda) &= G_f(\lambda)\mathbf{f}(\lambda) + G_w(\lambda)\mathbf{w}(\lambda) + \\ & \quad G_d(\lambda)\mathbf{d}(\lambda) + G_u(\lambda)\mathbf{u}(\lambda) \\ \mathbf{r}(\lambda) &= Q(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix} \end{aligned}$$



Generalized plant:

$$P(\lambda) = \left[\begin{array}{c|c} P_{11}(\lambda) & P_{12}(\lambda) \\ \hline P_{21}(\lambda) & P_{22}(\lambda) \end{array}\right] := \left[\begin{array}{cccc|c} -M_r(\lambda) & 0 & 0 & 0 & I \\ \hline G_f(\lambda) & G_w(\lambda) & G_d(\lambda) & G_u(\lambda) & 0 \\ 0 & 0 & 0 & I & 0 \end{array}\right]$$

Difficulties with standard tools

Main limitations

1. Technical assumptions may prevent computation of a solution even if exists!
   - proper system
   - stabilizability of realization of $P(\lambda)$
   - lack of zeros $P_{21}(\lambda)$ on the extended imaginary axis
2. Choice of appropriate $M_r(\lambda)$ not supported!

Proposed enhanced general approach

1. No technical assumptions
2. Choice of appropriate $M_r(\lambda)$ fully supported

Modified $\mathcal{H}_2$-optimal model-matching

Choose appropriate $M(\lambda)$ (i.e., stable, proper, diagonal, invertible) and determine stable and proper $Q(\lambda)$ to minimize $\|\mathcal{R}(\lambda)\|_2$, where

$$\mathcal{R}(\lambda) = M(\lambda)F(\lambda) - Q(\lambda)G(\lambda),$$

with

$$F(\lambda) = [\, M_r(\lambda)\ O\ O\ O\,],$$

$$G(\lambda) = \left[\begin{array}{cccc} G_f(\lambda) & G_w(\lambda) & G_d(\lambda) & G_u(\lambda) \\ 0 & 0 & 0 & I_{m_u} \end{array} \right]$$

and $M_r(\lambda)$ a given reference model (i.e., stable, proper, diagonal, invertible).

Enhanced $\mathcal{H}_2$-optimal model-matching (1)

Step 1: $R_u(\lambda) = 0$, $R_d(\lambda) = 0 \Rightarrow Q(\lambda) = Q_1(\lambda)N_l(\lambda)$, where $Q_1(\lambda)$ is free (to be determined) and

$$N_l(\lambda)\begin{bmatrix} G_d(\lambda) & G_u(\lambda) \\ 0 & I_{m_u} \end{bmatrix} = 0$$

$N_l(\lambda)$ can be chosen stable and proper (e.g., a rational left nullspace basis) such that

$$[\, N_f(\lambda)\ N_w(\lambda)\,] := N_l(\lambda)\begin{bmatrix} G_f(\lambda) & G_w(\lambda) \\ 0 & 0 \end{bmatrix}$$

are proper and stable, and $[\, N_f(\lambda)\ N_w(\lambda)\,]$ has full row rank.

Solvability check: $rank\, N_f(\lambda) = m_f$

Enhanced $\mathcal{H}_2$-optimal model-matching (2)

Choose appropriate $M(\lambda)$ (i.e., stable, proper, diagonal, invertible) and determine stable and proper $Q_1(\lambda)$ to minimize $\|\mathcal{R}_1(\lambda)\|_2$, where

$$\mathcal{R}_1(\lambda) = M(\lambda)F(\lambda) - Q_1(\lambda)G(\lambda),$$

with

$$F(\lambda) := [\, M_r(\lambda) \;\; O \,],$$

$$G(\lambda) := [\, N_f(\lambda) \;\; N_w(\lambda) \,],$$

and $M_r(\lambda)$ a given reference model (i.e., stable, proper, diagonal, invertible).

Note: $\|\mathcal{R}_1(\lambda)\|_2 = \|\mathcal{R}(\lambda)\|_2$.

Enhanced $\mathcal{H}_2$-optimal model-matching (3)

Step 2.1: Compute a *quasi*-co-outer-inner factorization

$$G(\lambda) = [\, G_{o,1}(\lambda) \; 0 \,] \left[\begin{array}{c} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{array} \right] := G_o(\lambda) G_i(\lambda),$$

where

- $G_i(\lambda)$ is inner (i.e., $G_i^*(s) := G_i^T(-s)$) or $G_i^*(z) := G_i^T(1/z)$)
- $G_{o,1}(\lambda)$ is invertible (with possible zeros on the boundary of the stability domain).

Enhanced $\mathcal{H}_2$-optimal model-matching (4)

Step 2.2: Choose $Q_1(\lambda) = Q_2(\lambda)G_{o,1}^{-1}(\lambda)$ and define

$$\mathcal{R}_2(\lambda) = \mathcal{R}_1(\lambda)G_i^*(\lambda) = \left[\; M(\lambda)F_1(\lambda) - Q_2(\lambda) \;\middle|\; M(\lambda)F_2(\lambda) \;\right],$$

where

$$\begin{array}{rcl} F_1(\lambda) & := & [\,M_r(\lambda)\ O\,]G_{i,1}^*(\lambda) \\ F_2(\lambda) & := & [\,M_r(\lambda)\ O\,]G_{i,2}^*(\lambda) \end{array}$$

Updated $\mathcal{H}_2$ synthesis problem: Choose appropriate $M(\lambda)$ and determine stable and proper $Q_2(\lambda)$ to minimize $\|\mathcal{R}_2(\lambda)\|_2 = \|\mathcal{R}_1(\lambda)\|_2$.

Enhanced $\mathcal{H}_2$-optimal model-matching (4)

Step 3: Take

$$Q_2(\lambda) = M(\lambda)[F_1(\lambda)]_+,$$

where $[\cdot]_+$ denotes the stable part and $M(\lambda)$ is a stable, proper, diagonal and invertible TFM chosen to ensure that

$$Q(\lambda) := M(\lambda)[F_1(\lambda)]_+ G_{o,1}^{-1}(\lambda) N_l(\lambda)$$

is proper and stable, and $[\, M(\lambda)F_1(\lambda) - Q(\lambda) \;\; M(\lambda)F_2(\lambda)\,]$ is strictly proper.

Solution of the modified $\mathcal{H}_2$ synthesis problem:

$$\|\mathcal{R}(\lambda)\|_2 = \|\mathcal{R}_2(\lambda)\|_2 = \|[\, M(\lambda)F_1(\lambda) - Q_2(\lambda) \;\; M(\lambda)F_2(\lambda)\,]\|_2$$

Enhanced $\mathcal{H}_2$-optimal model-matching (5)

Expressions for $R_f(\lambda)$ and $R_w(\lambda)$:

$$[\, R_f(\lambda) \; R_w(\lambda) \,] = M(\lambda)[\, M_r(\lambda) \;\; 0 \,] \left[\begin{array}{c} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{array} \right] = M(\lambda) M_r(\lambda) G_{i,1}(\lambda)$$

$\Rightarrow R_f(\lambda)$ and $R_w(\lambda)$ are stable and proper.

Integrated general computational algorithm

Key features:

- exploiting properties of intermediary results in the successive steps
- using detector updating techniques $\rightarrow$ least order detector
- relying on descriptor representations and computational techniques

Computational issues (1)

Underlying regular descriptor system representation:

$$\begin{array}{rcl} E\lambda x(t) & = & Ax(t)+B_u u(t)+B_d d(t)+B_w w(t)+B_f f(t) \\ y(t) & = & Cx(t)+D_u u(t)+D_d d(t)+D_w w(t)+D_f f(t) \end{array}$$

$$\begin{array}{rcl} G_u(\lambda) & = & C(\lambda E - A)^{-1}B_u + D_u \\ G_d(\lambda) & = & C(\lambda E - A)^{-1}B_d + D_d \\ G_w(\lambda) & = & C(\lambda E - A)^{-1}B_w + D_w \\ G_f(\lambda) & = & C(\lambda E - A)^{-1}B_f + D_f \end{array}$$

or, equivalently

$$\left[\; G_u(\lambda) \;\; G_d(\lambda) \;\; G_w(\lambda) \;\; G_f(\lambda) \;\right] := \left[\begin{array}{c|cccc} A-\lambda E & B_u & B_d & B_w & B_f \\ \hline C & D_u & D_d & D_w & D_f \end{array}\right]$$

Computational issues (2)

Step 1: Use rational nullspace method (V, 2008) based on orthogonal pencil manipulation algorithms. The resulting realizations have the form

$$\left[\; N_l(\lambda) \quad N_f(\lambda) \quad N_w(\lambda) \;\right] = \left[\begin{array}{c|ccc} \widetilde{A} - \lambda\widetilde{E} & \widetilde{B}_{yu} & \widetilde{B}_f & \widetilde{B}_w \\ \hline \widetilde{C} & \widetilde{D}_{yu} & \widetilde{D}_f & \widetilde{D}_w \end{array}\right],$$

where $\widetilde{E}$ is invertible (thus all TFMs are proper) and the pair $(\widetilde{A}, \widetilde{E})$ has only finite generalized eigenvalues which can be arbitrarily placed.

Note: To guarantee $[\, N_f(\lambda)\; N_w(\lambda)\,]$ is full row rank, use $W(\lambda)N_l(\lambda)$ instead $N_l(\lambda)$ (via minimal dynamic covers).

Computational issues (3)

Step 2.1: The quasi-co-outer–inner factorization

$$[\, N_f(\lambda)\ N_w(\lambda)\,] = [\, G_{o,1}(\lambda)\ 0\,] G_i(\lambda)$$

is computed using the general orthogonal transformations based numerically reliable algorithm of (Oara & V, 2000; Oara, 2005). The invertible quasi-co-outer factor $G_{o,1}(\lambda)$ is obtained in the form

$$G_{o,1}(\lambda) = \left[\begin{array}{c|c} \widetilde{A} - \lambda\widetilde{E} & \overline{B}_o \\ \hline \widetilde{C} & \overline{D}_o \end{array}\right], \quad G_i^*(\lambda) = \left[\begin{array}{c|c} A_i - \lambda E_i & B_i \\ \hline C_i & D_i \end{array}\right]$$

Computational issues (4)

Step 2.2: To compute $\overline{N}_l(\lambda) := G_{o,1}^{-1}(\lambda)N_l(\lambda)$, we can solve the linear rational system of equations

$$G_{o,1}(\lambda)\overline{N}_l(\lambda) = N_l(\lambda)$$

Explicit solution as a descriptor system realization:

$$\overline{N}_l(\lambda) = \left[\begin{array}{cc} 0 & I \end{array}\right] \left[\begin{array}{cc} \widetilde{A} - \lambda\widetilde{E} & \overline{B}_o \\ \widetilde{C} & \overline{D}_o \end{array}\right]^{-1} \left[\begin{array}{c} \widetilde{B}_{yu} \\ \widetilde{D}_{yu} \end{array}\right]$$

Computational issues (5)

Step 3: To compute a suitable $M(\lambda)$ which guarantees that:

1. the final detector $Q(\lambda) := M(\lambda)[F_1(\lambda)]_+\overline{N}_l(\lambda)$ is proper and stable, and
2. $\|\mathcal{R}(\lambda)\|_2 = \|[\, M(\lambda)(F_1(\lambda) - [F_1(\lambda)]_+) \;\; M(\lambda)F_2(\lambda)\,]\|_2$ is finite

we can solve (strict) proper coprime factorizations problems for each row of the TFM $[\, F_1(\lambda) - [F_1(\lambda)]_+ \;\; F_2(\lambda)\,]$ using state-space algorithms described in (V,1998).

Illustrative example (1)

Parametric model with uncertainties recast as input noise:

$$A(\delta_1,\delta_2) = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.5(1+\delta_1) & 0.6(1+\delta_2) \\ 0 & -0.6(1+\delta_2) & -0.5(1+\delta_1) \end{bmatrix}$$

$$B_u = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_d = 0, \;\; B_f = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D_u = 0, \; D_d = 0, \; D_f = 0.$$

$$\Rightarrow \qquad A \leftarrow A(0,0), \quad B_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D_w = 0$$

Problem: $P_{21}(\lambda)$ has zeros at ∞ (strictly proper system).

Illustrative example (2)

Step 1:

$$N_l(s) = [\, I \;\; -G_u(s)\,] = \left[\begin{array}{c|cc} A-sI & 0 & -B_u \\ \hline C & I & -D_u \end{array}\right]; \;\; \delta(N_l(s)) = 3$$

$\Rightarrow N_f(s) = G_f(s)$, $N_w(s) = G_w(s)$

Step 2:

$N_f(s)$ invertible; $[\, N_f(s) \; N_w(s)\,]$ has two zeros at ∞

$\Rightarrow G_{o,1}(s)$ with zeros $\{\infty, \infty, -1.134\}$; $G_i(s)$ with pole -1.134

$\Rightarrow \overline{N}_l(s) = G_{o,1}^{-1}(s) N_l(s)$ with poles $\{\infty, \infty, -1.134\}$; $\delta(\overline{N}_l(s)) = 5$

$\Rightarrow F_1(s)$ and $F_2(s)$ proper with pole 1.134

Illustrative example (3)

Step 3:

For $M_r(s) = I_2 \Rightarrow M(s) = \begin{bmatrix} \frac{10}{s+10} & 0 \\ 0 & \frac{10}{s+10} \end{bmatrix}$

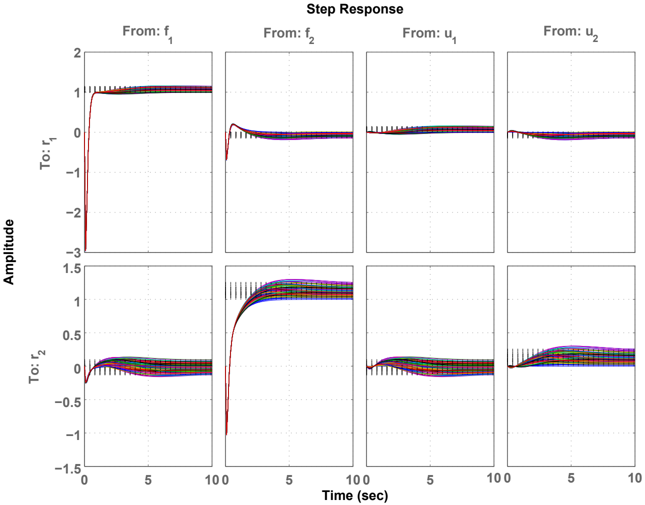
Resulting FDI filter:

$$Q(s) = M(s)F_1(\infty)G_{o,1}^{-1}(s)[\, I \;\; -G_u(s)\,]$$

Resulting least order: $\delta(Q(s)) = 3$

Sum of order of factors: 2 + 0 + 5 + 3 = 10!

Parametric step responses (original system)



Conclusions

- an integrated algorithm proposed to solve $\mathcal{H}_2$-optimal FDI filter synthesis problems in the most general setting
- all technical assumptions of standard tools completely avoided
- underlying algorithms based on descriptor system representations and rely on orthogonal matrix pencil reductions
- similar approach has been recently developed (V, 2010) for solving $\mathcal{H}_\infty$-optimal FDI filter synthesis problems
- software tools available for MATLAB in the DESCRIPTOR SYSTEMS Toolbox (V,2000) and in the current version of the FAULT DETECTION Toolbox (V,2006,2009).

Key references to descriptor techniques

- A. Varga.
  On computing nullspace bases – a fault detection perspective.
  Proc. IFAC 2008 World Congress, Seoul, Korea., pp. 6295–6300, 2008.
- C. Oară and A. Varga.
  Computation of general inner-outer and spectral factorizations.
  IEEE Trans. Automat. Control, 45:2307–2325, 2000.
- A. Varga.
  Computation of coprime factorizations of rational matrices.
  Lin. Alg. & Appl., 271:83–115, 1998.

Recent works on optimal synthesis of FDI filters

A. Varga.
Integrated algorithm for solving $\mathcal{H}_2$-optimal fault detection and isolation problems.
Proc. SYSTOL'10, Nice, France, 2010.

A. Varga.
Integrated algorithm for solving $\mathcal{H}_\infty$-optimal fault detection and isolation problems.
Submitted to IFAC 2011 World Congress, Milano, Italy.