Model reduction of nonlinear circuit equations

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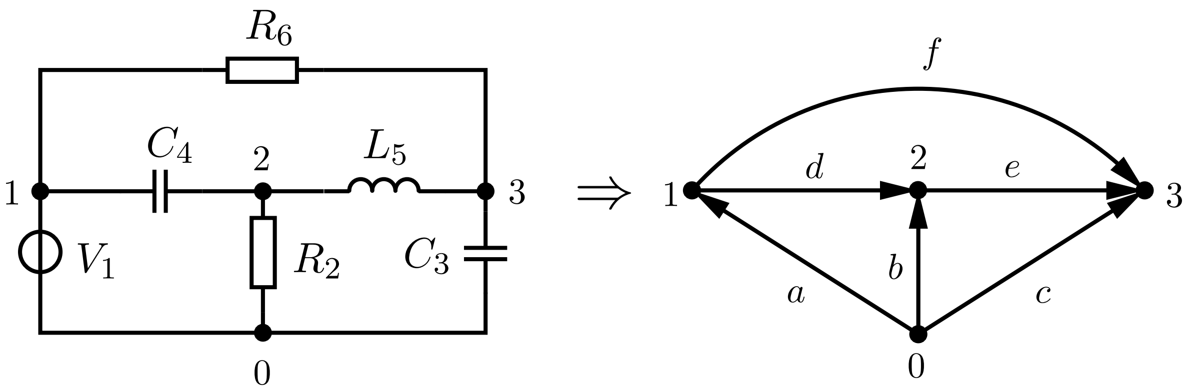
SyreNe
System Reduction for Nanoscale IC Design

BIRS Workshop, Banff, Canada, October 25-29, 2010

Outline

- Differential-algebraic equations in circuit simulation
- Model order reduction problem
- PAssivity-preserving Balanced Truncation method for Electrical Circuits (PABTEC)
  - Decoupling of linear and nonlinear parts
  - Model reduction of linear equations
  - Recoupling
- Numerical examples
- Conclusion

Modified Nodal Analysis (MNA)



$$\mathbf{A} = [A_{\mathcal{R}},\ A_{\mathcal{C}},\ A_{\mathcal{L}},\ A_{\mathcal{V}},\ A_{\mathcal{I}}]$$

Kirchhoff's current law: $\mathbf{A}j = 0, \quad j = \left[\, j_{\mathcal{R}}^T,\ j_{\mathcal{C}}^T,\ j_{\mathcal{L}}^T,\ j_{\mathcal{V}}^T,\ j_{\mathcal{I}}^T \,\right]^T$

Kirchhoff's voltage law: $\mathbf{A}^T\eta = v, \quad v = \left[\, v_{\mathcal{R}}^T,\ v_{\mathcal{C}}^T,\ v_{\mathcal{L}}^T,\ v_{\mathcal{V}}^T,\ v_{\mathcal{I}}^T \,\right]^T$

Branch constitutive relations:

resistors: $j_{\mathcal{R}} = g(v_{\mathcal{R}}), \qquad \mathcal{G}(v_{\mathcal{R}}) = \dfrac{\partial g(v_{\mathcal{R}})}{\partial v_{\mathcal{R}}}$

capacitors: $j_{\mathcal{C}} = \dfrac{dq(v_{\mathcal{C}})}{dt}, \qquad \mathcal{C}(v_{\mathcal{C}}) = \dfrac{\partial q(v_{\mathcal{C}})}{\partial v_{\mathcal{C}}}$

inductors: $v_{\mathcal{L}} = \dfrac{d\phi(j_{\mathcal{L}})}{dt}, \qquad \mathcal{L}(j_{\mathcal{L}}) = \dfrac{\partial \phi(j_{\mathcal{L}})}{\partial j_{\mathcal{L}}}$

MNA circuit equations

Consider a linear DAE system

$$\mathcal{E}(x)\,\dot{x} = \mathcal{A}\,x + f(x) + \mathcal{B}\,u$$
$$y = \mathcal{B}^T x$$

with

$$\mathcal{E}(x) = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}(A_{\mathcal{C}}^T\eta)A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L}(j_{\mathcal{L}}) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix},$$

$$f(x) = \begin{bmatrix} -A_{\mathcal{R}}g(A_{\mathcal{R}}^T\eta) \\ 0 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} j_{\mathcal{I}} \\ v_{\mathcal{V}} \end{bmatrix}, \quad x = \begin{bmatrix} \eta \\ j_{\mathcal{L}} \\ j_{\mathcal{V}} \end{bmatrix}, \quad y = -\begin{bmatrix} v_{\mathcal{I}} \\ j_{\mathcal{V}} \end{bmatrix},$$

η	– node potentials,
$j_{\mathcal{L}},\ j_{\mathcal{V}},\ j_{\mathcal{I}}$	– currents through inductors, voltage and current sources,
$v_{\mathcal{V}},\ v_{\mathcal{I}}$	– voltages at voltage and current sources,
$A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}, A_{\mathcal{V}}, A_{\mathcal{I}}$	– incidence matrices of resistors, capacitors, inductors, voltage and current sources

Index

Assumptions

- $A_{\mathcal{V}}$ has full column rank (= no V-loops)
- $[\,A_{\mathcal{C}},\, A_{\mathcal{L}},\, A_{\mathcal{R}},\, A_{\mathcal{V}}\,]$ has full row rank (= no I-cutsets)
- $\mathcal{C},\ \mathcal{L},\ \mathcal{G}$ are symmetric, positive definite

Index characterization

[Estévez Schwarz/Tischendorf'00]

Index = 0 $\Leftrightarrow$ no voltage sources and every node has a capacitive path to a reference node

Index = 1 $\Leftrightarrow$ no CV-loops except for C-loops and no LI-cutsets

Index = 2, otherwise

Model reduction problem

Given a large-scale system

$$\mathcal{E}(x)\,\dot{x} = \mathcal{A}\,x + f(x) + \mathcal{B}\,u$$
$$y = \mathcal{C}\,x$$

with $x \in \mathbb{R}^n$ and $u,\, y \in \mathbb{R}^m$,

find a reduced-order system

$$\widetilde{\mathcal{E}}(\widetilde{x})\,\dot{\widetilde{x}} = \widetilde{\mathcal{A}}\,\widetilde{x} + \widetilde{f}(\widetilde{x}) + \widetilde{\mathcal{B}}\,u$$
$$\widetilde{y} = \widetilde{\mathcal{C}}\,\widetilde{x}$$

with $\widetilde{x} \in \mathbb{R}^r$, $u, \widetilde{y} \in \mathbb{R}^m$, $r \ll n$.

- preservation of passivity and stability
- small approximation error $\|\widetilde{y} - y\| \le tol\,\|u\|$ for all $u \in \mathcal{U}$

  $\hookrightarrow$ need for computable error bounds
- numerically stable and efficient methods

Model reduction techniques

- Linear circuit equations
  - Krylov subspace methods (moment matching)

    SyPVL for RC, RL, LC circuits [Freund et al.'96,'97]

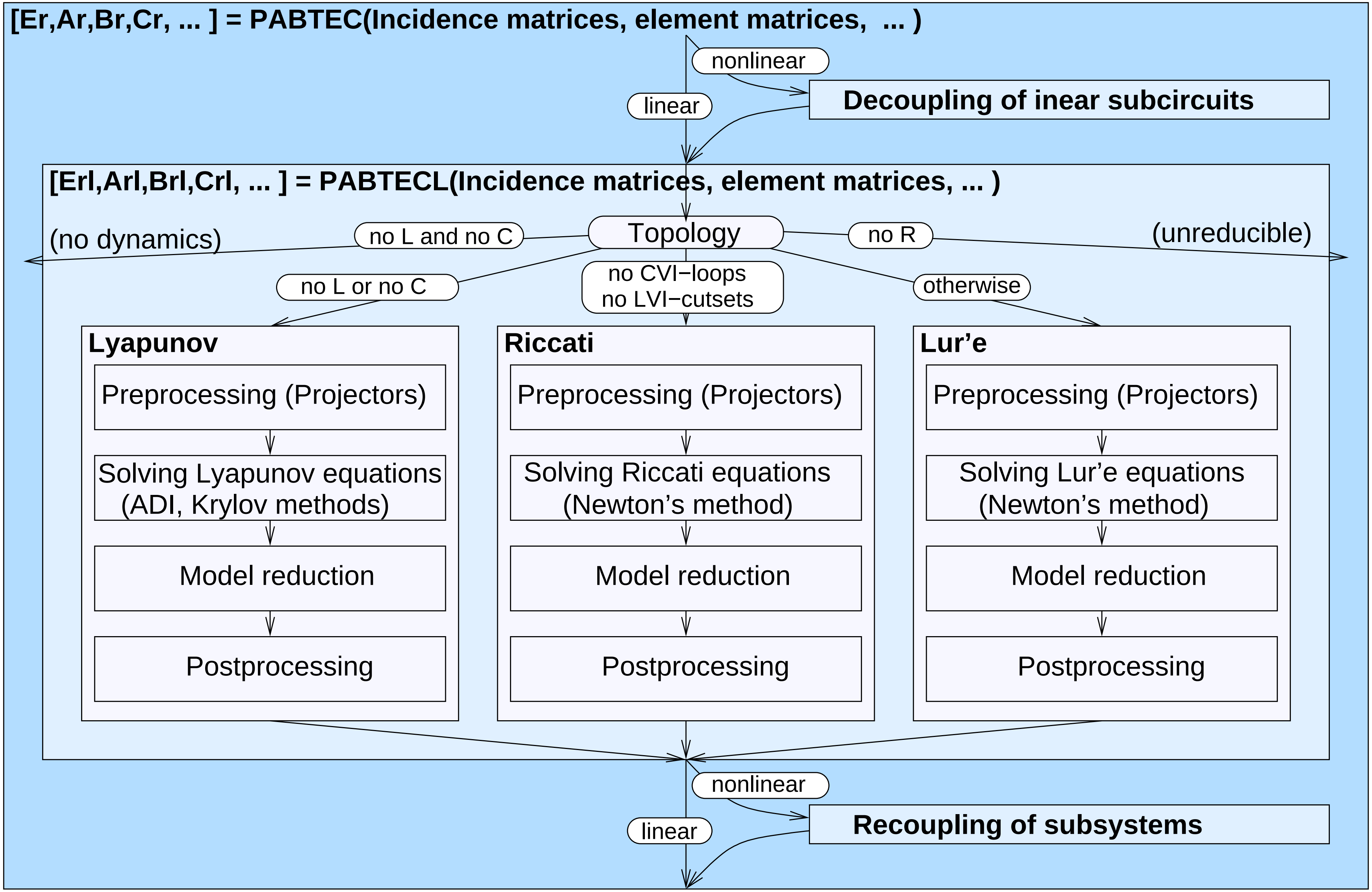
    PRIMA, SPRIM for RLC circuits [Odabasioglu et al.'96,'97; Freund'04,'05]

    Positive real interpolation [Antoulas'05, Sorensen'05, Ionutiu et al.'08]
  - Balancing-related model reduction methods

    LyaPABTEC for RC, RL circuits [Reis/S.'10]

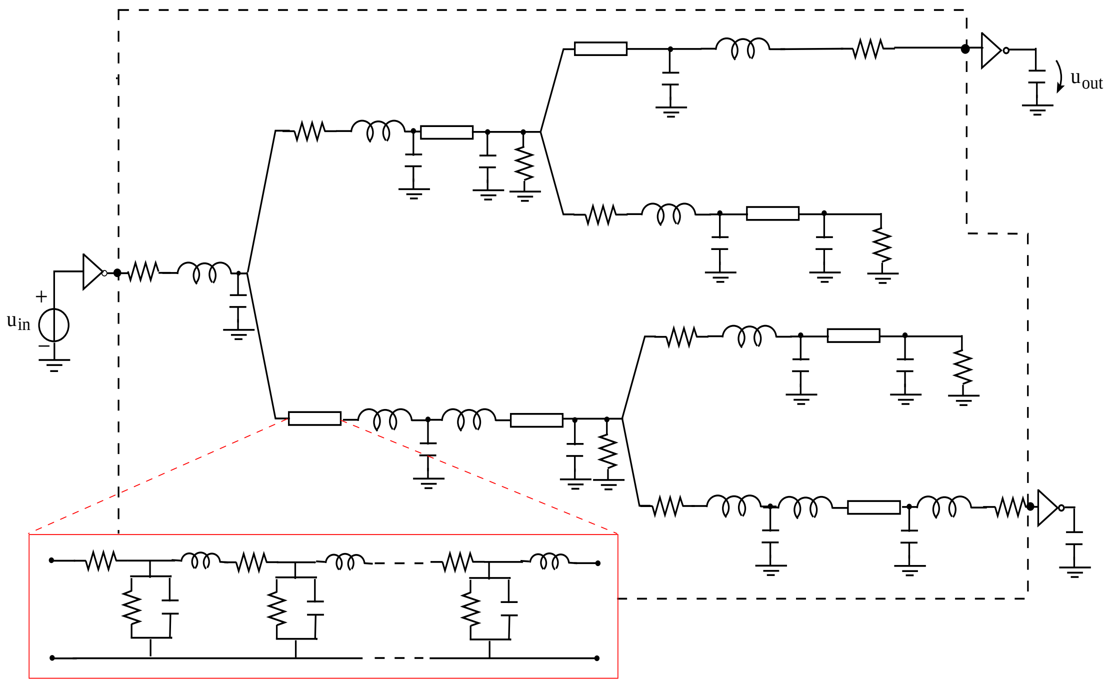
    PABTEC for RLC circuits [Reis/S.'09]
- Nonlinear circuit equations
  - Proper orthogonal decomposition (POD) [Verhoeven'08]
  - Trajectory piece-wise linear approach (TPWL) [Rewieński'03]
  - (Quadratic) bilinearization + balanced truncation [Benner/Breiten'10]

PABTEC Tool



[Er,Ar,Br,Cr, ...] = PABTEC(Incidence matrices, element matrices, ...)
nonlinear
Decoupling of inear subcircuits
linear
[Erl,Arl,Brl,Crl, ...] = PABTECL(Incidence matrices, element matrices, ...)
(no dynamics)
no L and no C
Topology
no R
(unreducible)
no L or no C
no CVI−loops
no LVI−cutsets
otherwise
Lyapunov
Preprocessing (Projectors)
Solving Lyapunov equations
(ADI, Krylov methods)
Model reduction
Postprocessing
Riccati
Preprocessing (Projectors)
Solving Riccati equations
(Newton's method)
Model reduction
Postprocessing
Lur'e
Preprocessing (Projectors)
Solving Lur'e equations
(Newton's method)
Model reduction
Postprocessing
nonlinear
Recoupling of subsystems
linear

Decoupling



Large linear RLC circuits arise in

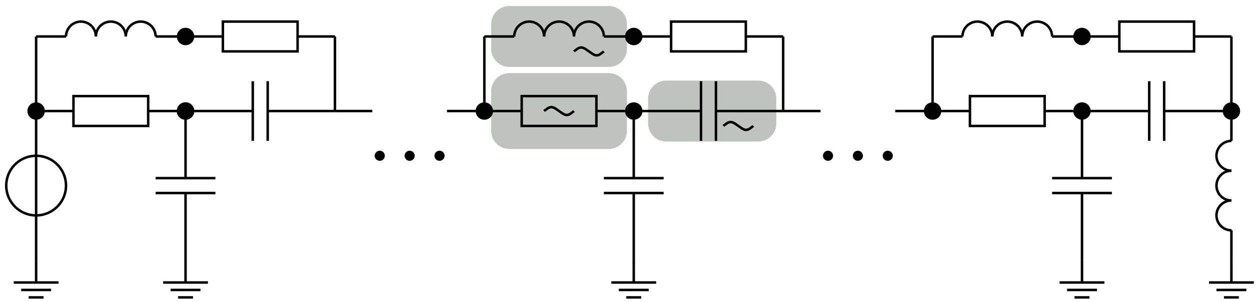
- modelling transmission lines and pin packages
- modelling circuits elements by Maxwell's equations via partial element equivalent circuits (PEEC)

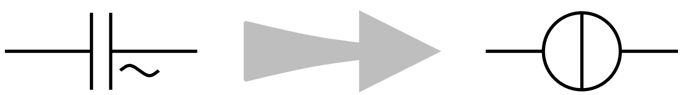
Assume that

$$A_{\mathcal{C}} = [A_{\mathcal{C}_l}, \;\; A_{\mathcal{C}_n}], \qquad A_{\mathcal{L}} = [A_{\mathcal{L}_l}, \;\; A_{\mathcal{L}_n}], \qquad A_{\mathcal{R}} = [A_{\mathcal{R}_l}, \;\; A_{\mathcal{R}_n}],$$

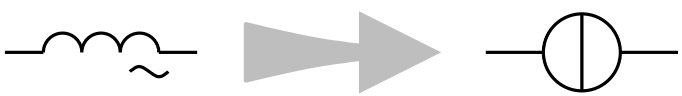
$$\mathcal{C}(A_{\mathcal{C}}^T\eta) = \begin{bmatrix} \mathcal{C}_l & 0 \\ 0 & \mathcal{C}_n(A_{\mathcal{C}_n}^T\eta) \end{bmatrix}, \;\; \mathcal{L}(j_{\mathcal{L}}) = \begin{bmatrix} \mathcal{L}_l & 0 \\ 0 & \mathcal{L}_n(j_{\mathcal{L}_n}) \end{bmatrix}, \;\; g(A_{\mathcal{R}}^T\eta) = \begin{bmatrix} \mathcal{G}_l A_{\mathcal{R}_l}^T\eta \\ g_n(A_{\mathcal{R}_n}^T\eta) \end{bmatrix}.$$

Replacement of nonlinear elements





:-(LI-cutsets may arise



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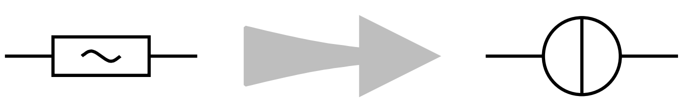
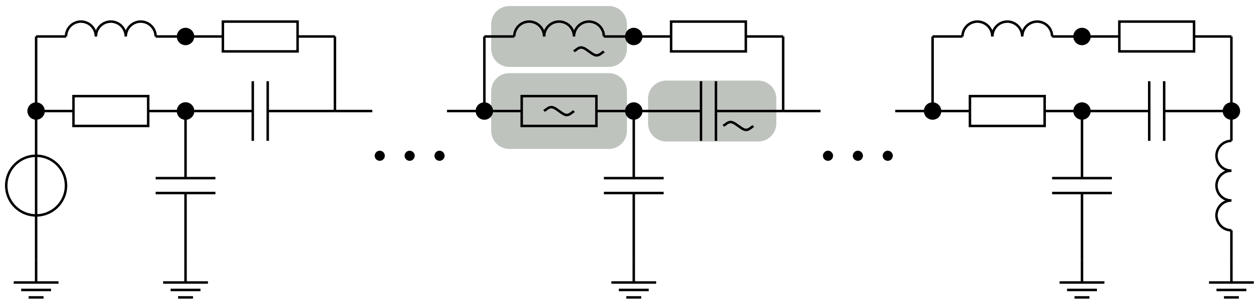
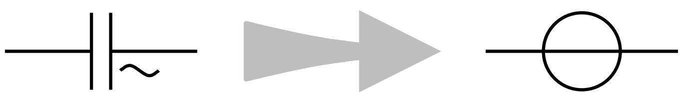


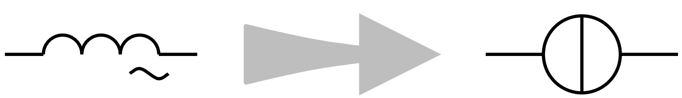
:-(LI- or I-cutsets may arise

Replacement of nonlinear elements





:-| additional variables



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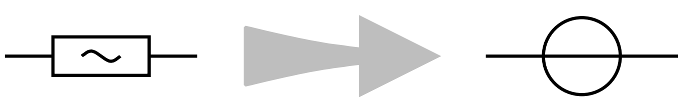
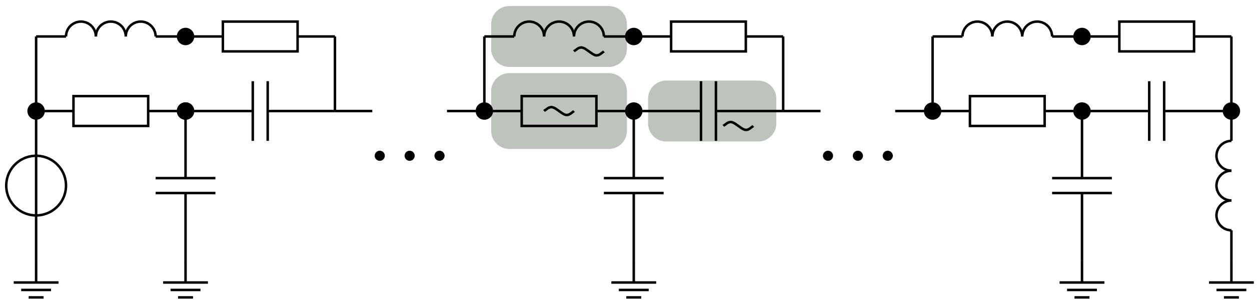
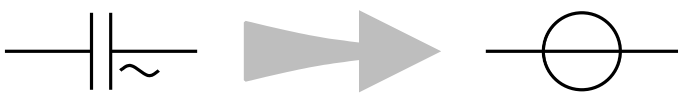


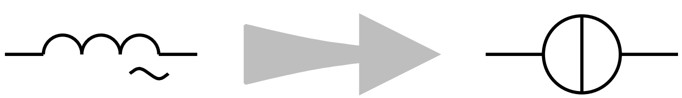
:-(CV- or V-loops may arise

Replacement of nonlinear elements





:-| additional variables



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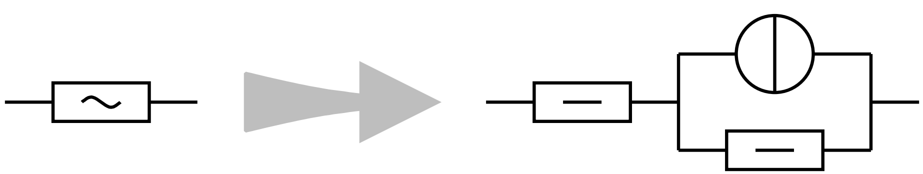


:-| additional variables

Decoupled system

Linear RLC equations:

$$E\,\dot{x}_l = A\,x_l + B\,u_l$$
$$y_l = B^T x_l$$

with

$$E = \begin{bmatrix} A_C\mathcal{C}A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_R G A_R^T & -A_L & -A_V \\ A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix},$$

$$A_C = \begin{bmatrix} A_{\mathcal{C}_l} \\ 0 \end{bmatrix}, \quad A_L = \begin{bmatrix} A_{\mathcal{L}_l} \\ 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} A_{\mathcal{R}_l} & A_{\mathcal{R}_n,1} & A_{\mathcal{R}_n,2} \\ 0 & -I & I \end{bmatrix}, \quad A_I = \begin{bmatrix} A_{\mathcal{I}} & A_{\mathcal{R}_n,2} & A_{\mathcal{L}_n} \\ 0 & I & 0 \end{bmatrix},$$

$$A_V = \begin{bmatrix} A_{\mathcal{V}} & A_{\mathcal{C}_n} \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} \mathcal{G}_l & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_2 \end{bmatrix}, \quad \begin{array}{l} x_l^T = \left[\, \eta^T \;\; \eta_z^T \,\middle|\, j_{\mathcal{L}_l}^T \,\middle|\, j_{\mathcal{V}}^T \;\; j_{\mathcal{C}_n}^T \,\right], \\ u_l^T = \left[\, j_{\mathcal{I}}^T \;\; j_z^T \,\middle|\, j_{\mathcal{L}_n}^T \,\middle|\, u_{\mathcal{V}}^T \;\; u_{\mathcal{C}_n}^T \,\right]; \end{array}$$

Nonlinear equations:

$$\mathcal{C}_n(v_{\mathcal{C}_n})\frac{d}{dt}u_{\mathcal{C}_n} = j_{\mathcal{C}_n}, \quad \mathcal{L}_n(j_{\mathcal{L}_n})\frac{d}{dt}j_{\mathcal{L}_n} = A_{\mathcal{L}_n}^T\eta,$$

$$j_z = (G_1 + G_2)G_1^{-1}g_n(A_{\mathcal{R}_n}^T\eta) - G_2A_{\mathcal{R}_n}^T\eta$$

Balanced truncation

System $\boldsymbol{G} = (\,E,\ A,\ B,\ C\,)$ is balanced if the controllability and observability Gramians X and Y satisfy

$$X = Y = \mathrm{diag}(\sigma_1, \ldots, \sigma_n).$$

Idea: balance the system, i.e., find an equivalence transformation

$$(\,\hat{E},\ \hat{A},\ \hat{B},\ \hat{C}\,) = (\,W_bET_b,\ \ W_bAT_b,\ \ W_bB,\ \ CT_b\,)$$

$$= \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1,\ C_2] \right)$$

such that $\hat{X} = \hat{Y} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and truncate the states corresponding to small σ_j $\hookrightarrow$ $\widetilde{\boldsymbol{G}} = (\,E_{11},\ A_{11},\ B_1,\ C_1\,)$.

DAEs: $\boldsymbol{G}(s) = C(sE - A)^{-1}B = \boldsymbol{G}_{sp}(s) + \mathbf{P}(s) \ \Rightarrow\ \widetilde{\boldsymbol{G}}(s) = \widetilde{\boldsymbol{G}}_{sp}(s) + \mathbf{P}(s)$

Projected Lur'e equations

- If $\boldsymbol{G} = (E, A, B, C)$ is passive, then there exist matrices $X = X^T \geq 0$, J_c, K_c and $Y = Y^T \geq 0$, J_o, K_o that satisfy the projected Lur'e equations

$$
\begin{aligned}
&(A - BC)XE^T + EX(A - BC)^T + 2P_l BB^T P_l^T = -2K_c K_c^T, \quad X = P_r X P_r^T, \\
&EXC^T - P_l B M_0^T = -K_c J_c^T, \qquad I - M_0 M_0^T = J_c J_c^T,
\end{aligned}
$$

$$
\begin{aligned}
&(A - BC)^T Y E + E^T Y (A - BC) + 2P_r^T C^T C P_r = -2K_o^T K_o, \quad Y = P_l^T Y P_l, \\
&-B^T Y E + M_0^T C P_r = -J_o^T K_o, \qquad I - M_0^T M_0 = J_o^T J_o,
\end{aligned}
$$

  where $M_0 = I - 2 \lim\limits_{s \to \infty} C(sE - A + BC)^{-1} B$, P_r and P_l are the spectral projectors onto the left and right deflating subspaces of the pencil $\lambda E - A + BC$ corresponding to the finite eigenvalues.

- $0 \leq X_{\min} \leq X \leq X_{\max}, \qquad 0 \leq Y_{\min} \leq Y \leq Y_{\max}$

  $X_{\min}$ – controllability Gramian, $\quad Y_{\min}$ – observability Gramian

Passivity-preserving BT method

Given a passive system $\boldsymbol{G} = (\,E,\,A,\,B,\,C\,)$.

1. Compute $P_r,\; P_l,\; M_0$.

2. Compute $X_{\min} = RR^T,\; Y_{\min} = LL^T$ (= solve the Lur'e equations).

3. Compute the SVD $L^TER = [U_1,\,U_2]\begin{bmatrix} \Pi_1 & \\ & \Pi_2 \end{bmatrix}[V_1,\,V_2]^T$.

4. Compute the reduced-order model

$$\widetilde{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{A} = \begin{bmatrix} 2\,W^TAT & \sqrt{2}\,W^TB\,C_\infty \\ -\sqrt{2}\,B_\infty CT & 2I - B_\infty C_\infty \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} W^TB \\ -B_\infty/\sqrt{2} \end{bmatrix}, \qquad \widetilde{C} = [\,CT \quad C_\infty/\sqrt{2}\,]$$

with $I - M_0 = C_\infty B_\infty,\;\; W = LU_1\Pi_1^{-1/2}$ and $T = RV_1\Pi_1^{-1/2}$.

Properties

- $\widetilde{\boldsymbol{G}} = (\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,)$ is passive
- $\widetilde{\boldsymbol{G}} = (\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,)$ is reciprocal ($\widetilde{\boldsymbol{G}}(s) = \Sigma\,\widetilde{\boldsymbol{G}}^T(s)\Sigma$)
- Error bounds: $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} := \sup\limits_{\omega\in\mathbb{R}} \|\boldsymbol{G}(i\omega)\|_2$
  - If $2\,\|I + \boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{\ell_f+1} + \ldots + \pi_{n_f}) < 1$, then

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le 2\,\|I + \boldsymbol{G}\|^2_{\mathbb{H}_\infty}(\pi_{\ell_f+1} + \ldots + \pi_{n_f}).$$

  - If $2\,\|I + \widetilde{\boldsymbol{G}}\|_{\mathbb{H}_\infty}(\pi_{\ell_f+1} + \ldots + \pi_{n_f}) < 1$, then

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le 2\,\|I + \widetilde{\boldsymbol{G}}\|^2_{\mathbb{H}_\infty}(\pi_{\ell_f+1} + \ldots + \pi_{n_f}).$$

Application to circuit equations

$$E=\begin{bmatrix}A_C\mathcal{C}A_C^T & 0 & 0\\ 0 & L & 0\\ 0 & 0 & 0\end{bmatrix},\ A-BC=\begin{bmatrix}-A_RGA_R^T-A_IA_I^T & -A_L & -A_V\\ A_L^T & 0 & 0\\ A_V^T & 0 & -I\end{bmatrix},\ B=\begin{bmatrix}-A_I & 0\\ 0 & 0\\ 0 & -I\end{bmatrix}=C^T$$

- Compute P_r and P_l using the canonical projectors technique [März'96]

$$\hookrightarrow \quad P_r=\begin{bmatrix}H_5(H_4H_2-I) & H_5H_4A_{\mathcal{L}}H_6 & 0\\ 0 & H_6 & 0\\ -A_{\mathcal{V}}^T(H_4H_2-I) & -A_{\mathcal{V}}^TH_4A_{\mathcal{L}}H_6 & 0\end{bmatrix}$$

with $H_1=Z_{CRIV}^TA_LL^{-1}A_L^TZ_{CRIV},\ H_2=\ldots,\ H_3=Z_C^TH_2Z_C,\ H_4=Z_CH_3^{-1}Z_C^T,$

$H_5=Z_{CRIV}H_1^{-1}Z_{CRIV}^TA_LL^{-1}A_L^T-I,\ \ H_6=I-L^{-1}A_L^TZ_{CRIV}H_1^{-1}Z_{CRIV}^TA_L,$

Z_C and Z_{CRIV} are basis matrices for $\ker\, A_C^T$ and $\ker[A_C,\ A_R,\ A_I,\ A_V]^T$.

$$\hookrightarrow \quad P_l=S\,P_r^TS^T \quad \text{with} \quad S=\mathrm{diag}(I_{n_\eta},-I_{n_L},-I_{n_V})$$

Application to circuit equations

$$E=\begin{bmatrix} A_C\mathcal{C}A_C^T & 0 & 0\\ 0 & L & 0\\ 0 & 0 & 0\end{bmatrix},\ A-BC=\begin{bmatrix} -A_RGA_R^T-A_IA_I^T & -A_L & -A_V\\ A_L^T & 0 & 0\\ A_V^T & 0 & -I\end{bmatrix},\ B=\begin{bmatrix} -A_I & 0\\ 0 & 0\\ 0 & -I\end{bmatrix}=C^T$$

- Compute P_r and $P_l = S\,P_r^T S^T$ with $S = \mathrm{diag}(I_{n_R}, -I_{n_L}, -I_{n_V})$.
- Compute $M_0 = I - 2\lim_{s\to\infty} C(sE - A + BC)^{-1}B$

$$\hookrightarrow \quad M_0 = \begin{bmatrix} I - 2A_I^T Z H_0^{-1} Z^T A_I & 2A_I^T Z H_0^{-1} Z^T A_V\\ -2A_V^T Z H_0^{-1} Z^T A_I & -I + 2A_V^T Z H_0^{-1} Z^T A_V\end{bmatrix}$$

where $H_0 = Z^T(A_RGA_R^T + A_IA_I^T + A_VA_V^T)Z$, $\quad Z = Z_C Z'_{RIV-C}$,

Z_{RIV-C} is a basis matrix for $\ker\,[\,A_R,\ A_I,\ A_V\,]^T Z_C$ and

$[Z_{RIV-C}, Z'_{RIV-C}\,]$ is nonsingular.

Application to circuit equations

- Compute M_0, P_r, $P_l = S\,P_r^T S^T$ with $S = \text{diag}(I_{n_R}, -I_{n_L}, -I_{n_V})$.
- Compute $X_{\min} = RR^T$, $Y_{\min} = FF^T$ (solve the projected Lur'e equations)

  If $D_0 = I - M_0 M_0^T$ is nonsingular, then the projected Lur'e equations are equivalent to the projected Riccati equation

$$(A - BC)XE^T + EX(A - BC)^T + 2\,P_l BB^T P_l^T \\ + 2(EXC^T - P_l BM_0^T)D_0^{-1}(EXC^T - P_l BM_0^T) = 0, \quad X = P_r X P_r^T$$

$\hookrightarrow$ compute a low-rank approximation $X_{\min} \approx \widetilde{R}\widetilde{R}^T$, $\widetilde{R} \in \mathbb{R}^{n,k}$, $k \ll n$, using the generalized low-rank Newton method [Benner/St.'10]

$\hookrightarrow$ $Y_{\min} = S\,X_{\min}\,S^T \approx S\widetilde{R}\widetilde{R}^T S^T = \widetilde{F}\widetilde{F}^T$

$\hookrightarrow$ D_0 is nonsingular, if the circuit contains neither CVI-loops except for C-loops nor LIV-cutsets except for L-cutsets

Application to circuit equations

- Compute M_0, P_r, $P_l = S\,P_r^T S^T$ with $S = \text{diag}(I_{n_R}, -I_{n_L}, -I_{n_V})$.
- Compute $X_{\min} \approx \widetilde{R}\widetilde{R}^T$, $Y_{\min} = S\,X_{\min}\,S^T \approx S\widetilde{R}\widetilde{R}^T S^T = \widetilde{F}\widetilde{F}^T$.
- Compute the SVD of $\widetilde{F}^T E \widetilde{R}$

  $\hookrightarrow$ $\widetilde{F}^T E \widetilde{R} = \widetilde{R}^T S E \widetilde{R}$ is symmetric

  $\hookrightarrow$ compute the EVD $\widetilde{R}^T S E \widetilde{R} = [\,U_1, \, U_2\,] \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} [\,U_1, \, U_2\,]^T$

  instead of the SVD

  $\hookrightarrow$ $W = S\widetilde{R}U_1|\Lambda_1|^{-1/2}$ and $T = \widetilde{R}U_1|\Lambda_1|^{-1/2}\text{sign}(\Lambda_1)$ with

  $|\Lambda_1| = \text{diag}(|\lambda_1|, \ldots, |\lambda_{\ell_f}|)$, $\text{sign}(\Lambda_1) = \text{diag}(\text{sign}(\lambda_1), \ldots, \text{sign}(\lambda_{\ell_f}))$

Application to circuit equations

- Compute M_0, P_r, $P_l = S\,P_r^T S^T$ with $S = \text{diag}(I_{n_R}, -I_{n_L}, -I_{n_V})$.
- Compute $X_{\min} \approx \widetilde{R}\widetilde{R}^T$.
- Compute the EVD $\widetilde{R}^T S E \widetilde{R} = [\,U_1,\, U_2\,] \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} [\,U_1,\, U_2\,]^T$ and

  $W = S\widetilde{R}U_1|\Lambda_1|^{-1/2}$, $T = \widetilde{R}U_1|\Lambda_1|^{-1/2}\text{sign}(\Lambda_1)$.
- Compute B_∞ and C_∞ such that $C_\infty B_\infty = I - M_0$

  $\hookrightarrow$ $(I - M_0)\Sigma$ with $\Sigma = \text{diag}(I_{n_I}, -I_{n_V})$ is symmetric

  $\hookrightarrow$ compute the EVD $(I - M_0)\Sigma = U_0\,\Lambda_0\,U_0^T$

  $\hookrightarrow$ $B_\infty = \text{sign}(\Lambda_0)|\Lambda_0|^{1/2}U_0^T\Sigma$ and $C_\infty = U_0|\Lambda_0|^{1/2}$

PABTECL algorithm

- Compute M_0, P_r, $P_l = S\,P_r^T S^T$ with $S = \mathrm{diag}(I_{n_R}, -I_{n_L}, -I_{n_V})$.
- Compute $X_{\min} \approx \widetilde{R}\widetilde{R}^T$.
- Compute the EVD $\widetilde{R}^T S E \widetilde{R} = [\,U_1,\, U_2\,] \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} [\,U_1,\, U_2\,]^T$ and

  $W = S\widetilde{R}U_1|\Lambda_1|^{-1/2}, \quad T = \widetilde{R}U_1|\Lambda_1|^{-1/2}\mathrm{sign}(\Lambda_1)$.
- Compute the EVD $(I - M_0)\Sigma = U_0\Lambda_0U_0^T$ with $\Sigma = \mathrm{diag}(I_{n_\mathcal{I}}, -I_{n_\mathcal{V}})$

  and $B_\infty = \mathrm{sign}(\Lambda_0)|\Lambda_0|^{1/2}U_0^T\Sigma, \quad C_\infty = U_0|\Lambda_0|^{1/2}$.
- Compute the reduced-order model

$$\widetilde{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{A} = \frac{1}{2}\begin{bmatrix} 2\,W^TAT & \sqrt{2}\,W^TBC_\infty \\ -\sqrt{2}\,B_\infty CT & 2I - B_\infty C_\infty \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} W^TB \\ -B_\infty/\sqrt{2} \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} CT, & C_\infty/\sqrt{2} \end{bmatrix}.$$

Recoupling

Linear reduced-order model:

$$\widetilde{E}\,\dot{\widetilde{x}}_l = \widetilde{A}\,\widetilde{x}_l + \left[\widetilde{B}_1,\, \widetilde{B}_2,\, \widetilde{B}_3,\, \widetilde{B}_4,\, \widetilde{B}_5\right] u_l,$$

$$\widetilde{y}_l = \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \\ \widetilde{C}_3 \\ \widetilde{C}_4 \\ \widetilde{C}_5 \end{bmatrix} \widetilde{x}_l \approx y_l = \begin{bmatrix} -A_I^T \eta \\ -A_{\mathcal{R}_n}^T \eta + G_1^{-1} g_n(A_{\mathcal{R}_n}^T \eta) \\ -A_{\mathcal{L}_n}^T \eta \\ -j_{\mathcal{V}} \\ -j_{\mathcal{C}_n} \end{bmatrix}$$

Nonlinear equations:

$$\mathcal{C}_n(v_{\mathcal{C}_n}) \frac{d}{dt} u_{\mathcal{C}_n} = j_{\mathcal{C}_n},$$

$$\mathcal{L}_n(j_{\mathcal{L}_n}) \frac{d}{dt} j_{\mathcal{L}_n} = A_{\mathcal{L}_n}^T \eta,$$

$$j_z = (G_1 + G_2) G_1^{-1} g_n(A_{\mathcal{R}_n}^T \eta) - G_2 A_{\mathcal{R}_n}^T \eta$$

Reduced-order nonlinear system

We obtain the reduced-order system

$$\widetilde{\mathcal{E}}(\widetilde{x})\,\dot{\widetilde{x}} = \widetilde{\mathcal{A}}\,\widetilde{x} + \widetilde{f}(\widetilde{x}) + \widetilde{\mathcal{B}}\,u$$
$$\widetilde{y} = \widetilde{\mathcal{C}}\,\widetilde{x}$$

with

$$\widetilde{\mathcal{E}}(\widetilde{x}) = \begin{bmatrix} \widetilde{E} & 0 & 0 & 0 \\ 0 & \mathcal{L}_n(\widetilde{j}_{\mathcal{L}_n}) & 0 & 0 \\ 0 & 0 & \mathcal{C}_n(\widetilde{u}_{\mathcal{C}_n}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{\mathcal{A}} = \begin{bmatrix} \widetilde{A} + \widetilde{B}_2(G_1 + G_2)\widetilde{C}_2 & \widetilde{B}_3 & \widetilde{B}_5 & \widetilde{B}_2 G_1 \\ -\widetilde{C}_3 & 0 & 0 & 0 \\ -\widetilde{C}_5 & 0 & 0 & 0 \\ -G_1\widetilde{C}_2 & 0 & 0 & -G_1 \end{bmatrix},$$

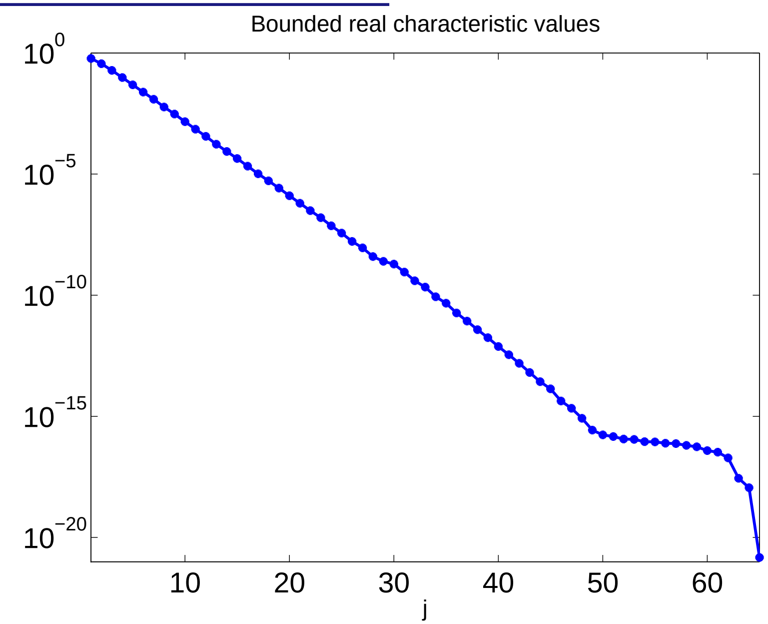
$$\widetilde{\mathcal{B}} = \begin{bmatrix} \widetilde{B}_1 & \widetilde{B}_4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{f}(\widetilde{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_n(\widetilde{u}_{\mathcal{C}_n}) \end{bmatrix}, \qquad \widetilde{x} = \begin{bmatrix} \widetilde{x}_l \\ \widetilde{j}_{\mathcal{L}_n} \\ \widetilde{u}_{\mathcal{C}_n} \\ \widetilde{u}_{\mathcal{R}_n} \end{bmatrix}.$$

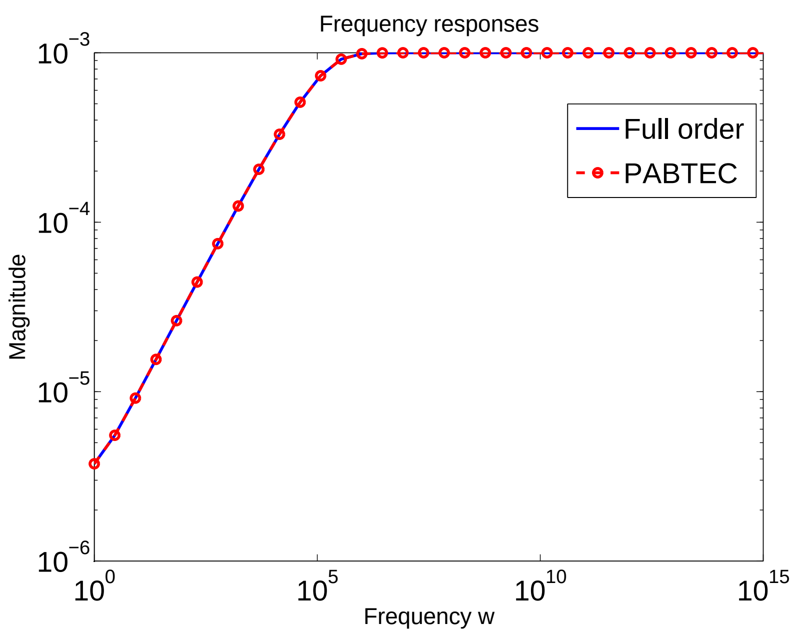
Remark: If $n_{\mathcal{C}_n} = 0$ and $n_{\mathcal{L}_n} = 0$, then passivity is preserved and under some additional topological conditions we have the error bound

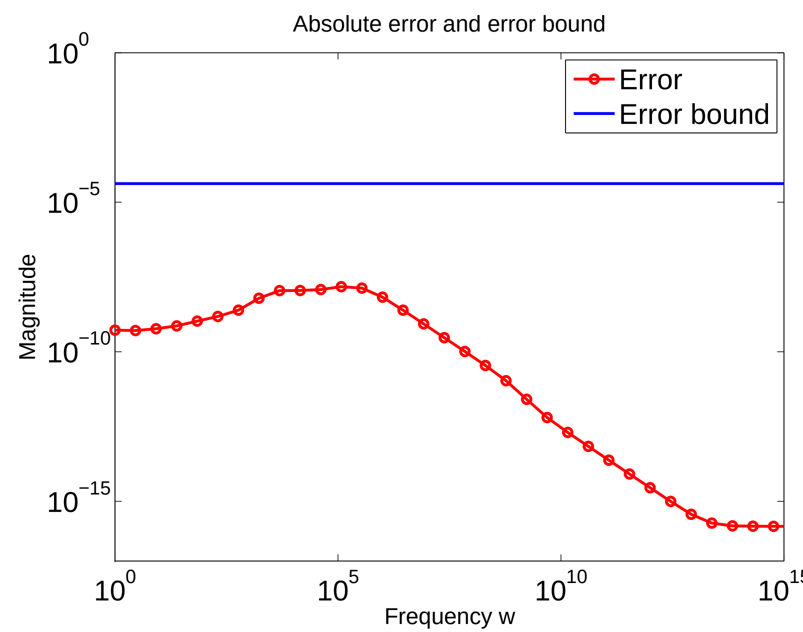
$\|\widetilde{y} - y\|_2 \le c(\pi_{\ell_f} + \ldots + \pi_{n_f})(\|u\|_2 + \|\widetilde{y}\|_2)$. [Heinkenschloss/Reis'09]

Example: linear RLC circuit

- $n = 127\,869$, $m = 1$
  85246 resistors
  42623 inductors
  42623 capacitors
- $X_{\min} \approx \widetilde{R}\widetilde{R}^T, \quad \widetilde{R} \in \mathbb{R}^{n,84}$
- Reduced model: $r = 24$





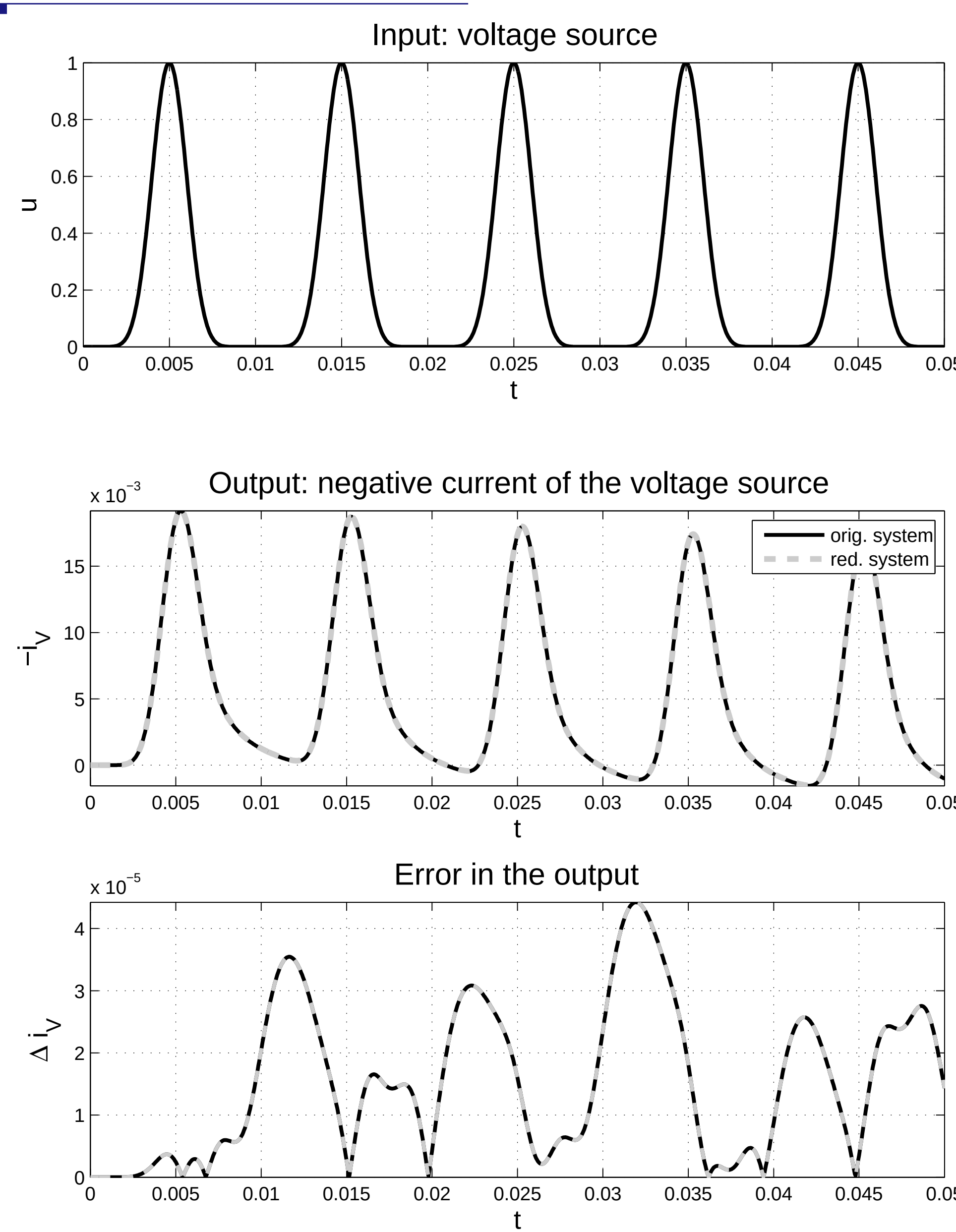


Example: nonlinear circuit

2000 linear capacitors
1990 linear resistors
991 linear inductors
10 nonlinear inductors
10 diodes
1 voltage source

	original system	reduced system
Dimension	4003	203
Simulation time	4557	67

Model reduction time 822
Error in the output 4.4e-05



Conclusions and future work

- Model reduction of nonlinear circuit equations
  - topology based partitioning
  - balancing-related model reduction of linear subsystems with preservation of passivity and computable error bounds
- Exploiting the structure of MNA matrices E, A, B, C
  - use graph algorithms for computing the basis matrices
  - use modern numerical linear algebra algorithms for solving large-scale projected Riccati/Lyapunov equations

$\hookrightarrow$ MATLAB Toolbox PABTEC

- Preservation of passivity and error bounds for general circuits
- Numerical solution of large-scale Lur'e equations