REGULARITY REGIONS OF DIFFERENTIAL ALGEBRAIC EQUATIONS AND LINEARIZATION

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$$f(\,(d(x(t),t))',\, x(t),\;\, t) = 0$$

Even though linearization represents such an important mathematical tool, only few papers deal with the linearization of DAEs, e.g. Campbell (1993).

Outline

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5. Conclusions

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with

$f(y,x,t) \in \mathbb{R}^m,\; d(x,t) \in \mathbb{R}^n,\; y \in \mathbb{R}^n,\; x \in \mathcal{D}_f \subseteq \mathbb{R}^m,\; t \in \mathcal{I}_f \subseteq \mathbb{R}$,

f, f_y, f_x, d, d_x, d_t are continuous, $f_y(y,x,t)$ and $d_x(x,t)$ singular,

$\operatorname{im} d_x$ is a $\mathcal{C}^1$-subspace ($d_x(x,t)$ has constant rank r)

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• DAEs (1) with a properly involved derivative (properly stated leading term): ker f_y is a $\mathcal{C}^1$-subspace ($f_y(y,x,t)$ has constant rank) such that

$$\ker f_y(y,x,t) \oplus \mathrm{im}\, d_x(x,t) = \mathbb{R}^n,\ y \in \mathbb{R}^n,\ x \in \mathcal{D}_f, t \in \mathcal{I}_f$$

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• DAEs (1) with a quasi-proper leading term (here $f_y(y,x,t)$ may change its rank): there is a $\mathcal{C}^1$-subspace $N_A \subseteq \ker f_y$ such that

$$N_A(y,x,t) \oplus \operatorname{im} d_x(x,t) = \mathbb{R}^n,\; y \in \mathbb{R}^n,\; x \in \mathcal{D}_f, t \in \mathcal{I}_f$$

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- standard form DAEs

$$\mathfrak{f}(x'(t), x(t), t) = 0 \quad \Longleftrightarrow \quad \mathfrak{f}((Dx(t))', x(t), t) = 0,$$

if there is a singular incidence or projector matrix $D \in L(\mathbb{R}^m)$ such that $\mathfrak{f}(x^1, x, t) \equiv \mathfrak{f}(Dx^1, x, t)$. Put $N_A = \operatorname{im} D^\perp$.

Outline

For each arbitrary sufficiently smooth function $x_* \in \mathcal{C}(\mathcal{I}_*, \mathbb{R}^m)$, $\mathcal{I}_* \subseteq \mathcal{I}_f$, with values $x_*(t) \in \mathcal{D}_f$, $t \in \mathcal{I}_*$, we may consider the linear DAE

$$\boxed{A_*(t)(D_*(t)x(t))' + B_*(t)x(t) = q(t), \quad t \in \mathcal{I}_*,} \tag{2}$$

with continuous coefficients given by

$$\begin{aligned} A_*(t) &:= f_y((d(x_*(t),t))', x_*(t), t), \\ D_*(t) &:= d_x(x_*(t),t), \\ B_*(t) &:= f_x((d(x_*(t),t))', x_*(t), t), \quad t \in \mathcal{I}_*. \end{aligned}$$

We stress, the reference function x_* is not necessarily a DAE solution!

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The linear DAE (2) is called linearization of the original DAE (1) along x_*.

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The linear DAE (2) inherits from (1) the proper and quasi-proper leading term, respectively. Does it inherit further properties? What about the opposite direction?

Example 1. The semi-explicit DAE (with properly involved derivative)

$$\begin{aligned} x_1'(t) - x_3(t) &= 0, \\ x_2(t)(1 - x_2(t)) - \frac{1}{4} + t^2 &= 0, \\ x_1(t)x_2(t) + x_3(t)(1 - x_2(t)) - t &= 0, \end{aligned}$$

with $m = 3$, $n = 1$, $d(x,t) = x_1$, and

$$f(y,x,t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} -x_3 \\ x_2(1 - x_2) - \frac{1}{4} + t^2 \\ x_1x_2 + x_3(1 - x_2) - t \end{bmatrix}, \quad y \in \mathbb{R},\ x \in \mathbb{R}^3,\ t \in \mathbb{R},$$

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yields the linearizations

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \right)' + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 - 2x_{*2}(t) & 0 \\ x_{*2}(t) & x_{*1}(t) - x_{*3}(t) & 1 - x_{*2}(t) \end{bmatrix} x(t) = q(t).$$

Case $x_{*2}(t) \equiv 0 \Longrightarrow$ index-1 DAE:

$$\begin{aligned} x_1'(t) - x_3(t) &= q_1(t), \\ x_2(t) &= q_2(t), \\ (x_{*1}(t) - x_{*3}(t))\, x_2(t) + x_3(t) &= q_3(t). \end{aligned}$$

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Case $x_{*2}(t) \equiv 1 \Longrightarrow$ index-2 DAE:

$$\begin{aligned} x_1'(t) - x_3(t) &= q_1(t), \\ -x_2(t) &= q_2(t), \\ x_1(t) + (x_{*1}(t) - x_{*3}(t))\, x_2(t) &= q_3(t). \end{aligned}$$

Case $x_{*2}(t) = \frac{1}{2} + t \Longrightarrow$ index-1 DAE with singularities:

$$\begin{aligned} x_1'(t) - x_3(t) &= q_1(t), \\ 2t\, x_2(t) &= q_2(t), \\ (\frac{1}{2} + t)x_1(t) + (x_{*1}(t) - x_{*3}(t))\, x_2(t) + (\frac{1}{2} - t)x_3(t) &= q_3(t). \end{aligned}$$

The inherent ODE reads

$$x_1'(t) = -\frac{1+2t}{1-2t} x_1(t) + q_1(t) + \frac{2}{1-2t} q_3(t) - \frac{1}{(1-2t)t} q_2(t)(x_{*1}(t) - x_{*3}(t)).$$

Example 2. The DAE with quasi-proper leading term ($N_A = \operatorname{im} D^\perp$)

$$x_4(t) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Big(\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{=:D} x(t)\Big)' + x(t) = q(t)$$

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yields linearizations (2) given by the coefficients $D_*(t) = D$ and

$$A_*(t) = \begin{bmatrix}x_{*4}(t)&0&0&0\\0&x_{*4}(t)&0&0\\0&0&x_{*4}(t)&0\\0&0&0&0\end{bmatrix}, \quad B_*(t) = \begin{bmatrix}1&0&0&x'_{*1}(t)\\0&1&0&x'_{*2}(t)\\0&0&1&x'_{*3}(t)\\0&0&0&1\end{bmatrix}.$$

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$\Longrightarrow$ If $x_{*4}(t)$ has no zeros, the resulting linearized DAE has index 4. If $x_{*4}(t)$ vanishes identically, the resulting linearized DAE has index 1.

Example 3. $\alpha(s) = s^2$ for $s > 0$, $\alpha(s) = 0$ for $s \leq 0$, ε is a constant. The DAE with quasi-proper leading term

$$\begin{aligned} x_1'(t) - x_2(t) &= 0, \\ x_2'(t) + x_1(t) &= 0, \\ \alpha(x_1(t))\, x_4'(t) + x_3(t) &= 0, \\ x_4(t) - \varepsilon &= 0, \end{aligned}$$

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yields the linearizations, with $\gamma_*(t) := \alpha_s(x_{*1}(t))x_{*1}'(t)$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha(x_{*1}(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) \right)' + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \gamma_*(t) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) = q(t).$$

The linearization reads in detail

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x_1'(t) - x_2(t) &= q_1(t), \\
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Case $x_*(t) = \begin{bmatrix} sint \\ cost \\ 0 \\ \varepsilon \end{bmatrix} \Longrightarrow$ The linearized DAE has in turn index 2 and index 1 on the intervals $(0, \pi)$, $(\pi, 2\pi)$, and so on.

Observation: Linearizations show astonishing properties, for reference functions being solutions of the given nonlinear DAE but also for arbitrary reference functions.

- They may show a singular flow caused by an inherent ODE with a singularity,
- They may show a lower or a higher index than the original DAE seems to have.
- They may have different index on different subintervals.
- They may become irregular at all.

The so-called regularity regions allow to comprehend what is going on.

Outline

A preliminary note. For the pair of $m \times m$ matrices $\{G, B\}$, we construct a sequence of matrices: Set $G_0 := G$, $B_0 := B$, choose Q_0 to be a projector matrix onto $N_0 := \ker G_0$, $P_0 := I - Q_0$, and, for $i \geq 1$,

$$
\begin{aligned}
G_i :=& G_{i-1} + B_{i-1} Q_{i-1}, \quad r_i := \operatorname{rank} G_i, \\
& Q_i \text{ projector onto } \quad N_i := \ker G_i, \quad N_0 + \cdots + N_{i-1} \subseteq \ker Q_i, \\
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Theorem (Griepentrog/März,1989)

※ *The pencil* $\lambda G + B$ *is regular with Kronecker-index* μ $\Longleftrightarrow$ *the sequence is well defined, and* $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.

※ *The numbers* $r_0, \ldots, r_\mu$ *characterize the structure of the Weierstraß-Kronecker canonical form.*

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⋇ *The numbers* $r_0, \ldots, r_\mu$ *characterize the structure of the Weierstraß-Kronecker canonical form.*

The alternative use of nontrivial subspaces $N_i \subseteq \ker G_i$ yields also a nonsingular G_κ, however then the values r_i loose their structural meaning.

Return to the DAE (1), introduce the basic matrix functions

$$A(x^1, x, t) := f_y(d_x(x,t)x^1 + d_t(x,t),\ x, t),$$
$$B(x^1, x, t) := f_x(d_x(x,t)x^1 + d_t(x,t),\ x, t), \quad x^1 \in \mathbb{R}^m,\ x \in \mathcal{D}_f,\ t \in \mathcal{I}_f,$$

and form pointwise a sequence of continuous matrix functions by

$$G_0 := A d_x,\ B_0 := B,$$
$$Q_0 \text{ projector function onto } N_0 := \ker d_x,\ P_0 := I - Q_0,\ \Pi_0 := P_0,$$

and, for $i \geq 1$, as long as the expressions exist,

$$G_i := G_{i-1} + B_{i-1} Q_{i-1},$$
$$\text{choose a nontrivial } \mathcal{C}\text{-subspace} \quad N_i \subseteq \ker G_i,$$
$$Q_i \text{ projector function onto } N_i, \quad N_0 + \cdots + N_{i-1} \subseteq \ker Q_i,$$
$$P_i := I - Q_i, \quad \Pi_i := \Pi_{i-1} P_i,$$
$$B_i := B_{i-1} P_{i-1} - G_i d_x^-(d_x \Pi_i d_x^-)' d_x \Pi_{i-1},$$

d_x^- denotes a pointwise defined special generalized inverse of d_x

G_1 and $d_x^- \Pi_1 d_x$ depend on the arguments
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Definition

- The DAE (1) with proper leading term is said to be regular on the open connected set $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$, if there is a number $\mu \in \mathbb{N}$, such that a matrix function sequence can be formed on $\mathcal{G}$ up to level μ with $N_i = \ker G_i$, $i = 0, \ldots, \mu - 1$, and $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.
- The DAE (1) with quasi-proper leading term is said to be regular on the open connected set $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$, if it has there a proper reformulationon which is regular.
- The open connected set $\mathcal{G}$ is then named a **regularity region**.
- The number μ is named tractability index, and the ranks $r_0, \ldots, r_\mu$ are said to be characteristic values of the DAE on $\mathcal{G}$.
- A point $(\bar{x}, \bar{t}) \in \mathcal{D}_f \times \mathcal{I}_f$ is a regular point, if there is a neighborhood being a regularity region, and a critical point otherwise.

• If $\mathcal{G}$ is a regularity region of the DAE (1), with characteristics $r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m$, then each open subset $\tilde{\mathcal{G}} \subset \mathcal{G}$ is a regularity region, too, and it has the same characteristics.

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- A regularity region consists of regular points with uniform characteristics.

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• Regularity, in particular the characteristics $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, are invariant with respect to coordinate changes, to refactorizations of the leading term as well as to the special choice of the admissible projector functions Q_i.

Definition

The DAE (1) with quasi-proper leading term is said to be quasi-regular on the open connected set $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$, if there is a number $\kappa \in \mathbb{N}$, such that a matrix function sequence can be formed on $\mathcal{G}$ up to level κ, and G_κ is nonsingular.
The open connected set $\mathcal{G}$ is then called a quasi-regularity region.

A particular class of regular DAEs.
Hessenberg form DAEs are semi-explicit systems of
$m_1 + \ldots + m_{r-1} + m_r = m$ equations, with the special structure

$$\begin{bmatrix} I_{m_1} & & \\ & \ddots & \\ & & I_{m_{r-1}} \\ & & 0 \end{bmatrix} \left(\begin{bmatrix} x_1(t) \\ \vdots \\ x_{r-1}(t) \end{bmatrix} \right)' + b(x_1(t), \ldots, x_r(t), t) = 0. \qquad (4)$$

The partial derivative

$$b_x = \begin{bmatrix} B_{11} & \ldots & B_{1,r-1} & B_{1r} \\ B_{21} & \ddots & \vdots & 0 \\ & \ddots & B_{r-1,r-1} & \\ & & B_{r,r-1} & 0 \end{bmatrix} \begin{matrix} \}m_1 \\ \}m_2 \\ \}m_{r-1} \\ \}m_r \end{matrix}$$

with $B_{ij} := b_{i,x_j}$, shows Hessenberg structure the name comes from.

Theorem

The system (4) is on the open set $\mathcal{G} \subseteq \mathcal{D}_b \times \mathcal{I}_b$ regular with characteristics $r_0 = \cdots = r_{\mu-1} < r_\mu = m,\ \mu = r$ $\Longleftrightarrow$ the matrix function product $B_{r,r-1} \cdots B_{21} B_{1r}$ remains nonsingular on $\mathcal{G}$.

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In general, we do not expect a DAE to be regular on its entire definition domain. It is rather natural that the definition domain decomposes in several maximal regularity regions $\mathcal{G}_1,\ \mathcal{G}_2, \ldots$ the borders of which consist of critical points. Solutions may cross the borders of these regularity regions, and, in particular, undergo bifurcations.

Example 1. The semi-explicit DAE (with properly involved derivative)

$$
\begin{aligned}
x_1'(t) - x_3(t) &= 0,\\
x_2(t)(1 - x_2(t)) - \frac{1}{4} + t^2 &= 0,\\
x_1(t)x_2(t) + x_3(t)(1 - x_2(t)) - t &= 0,
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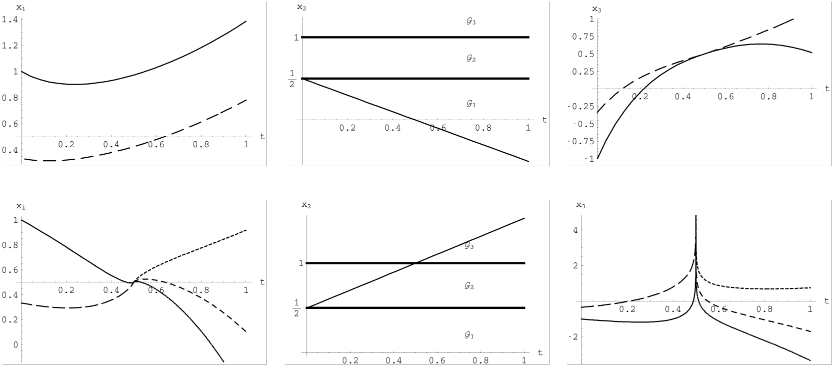
yields $\det G_1(x,t) = (1 - 2x_2)(1 - x_2)$, which has the zeros $x_2 = \frac{1}{2}$ and $x_2 = 1$. This splits the definition domain $\mathcal{D}_f \times \mathcal{I}_f = \mathbb{R}^3 \times \mathbb{R}$ into the three regularity regions

$$\mathcal{G}_1 := \left\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : \ x_2 < \frac{1}{2}\right\},$$
$$\mathcal{G}_2 := \left\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : \ \frac{1}{2} < x_2 < 1\right\},$$
$$\mathcal{G}_3 := \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : \ 1 < x_2\},$$

The DAE is regular with tractability index one on each region $\mathcal{G}_\ell$, $\ell = 1, 2, 3$.

The border points indicate a critical flow behavior in fact. The pictures show two solutions starting at $(1, \frac{1}{2}, -1)$ (solid line), and two solutions starting at $(\frac{1}{3}, \frac{1}{2}, -\frac{1}{3})$ (dashed line):



Example 2. The DAE with quasi-proper leading term ($N_A = \mathrm{im}\, D^{\perp}$, almost proper: $\ker f_y(y,x,t)D = \ker D$, everywhere, except for $x_4 \neq 0$).

$$x_4(t)(\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{=:D} x(t))' + x(t) = q(t)$$

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has the two regularity regions

$$\begin{aligned} \mathcal{G}_+ &:= \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} : \ x_4 > 0\}, \\ \mathcal{G}_- &:= \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} : \ x_4 < 0\}, \end{aligned}$$

The DAE is regular with tractability index 4 and $r_0 = r_1 = r_2 = r_3 = 3, \ r_4 = 4$ on both regions $\mathcal{G}_+$and $\mathcal{G}_-$.

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The DAE is quasi-regular on $\mathbb{R}^4 \times \mathbb{R}$, e.g. with $\kappa = 4$, the critical points are harmless.

Example 3. $\alpha(s) = s^2$ for $s > 0$, $\alpha(s) = 0$ for $s \leq 0$. The DAE (quasi-proper leading term)

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We throw a glance at linear DAEs.

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{I}. \tag{5}$$

and their adjoints

$$-D(t)^*(A(t)^*x(t))' + B(t)^*x(t) = p(t), \quad t \in \mathcal{I}. \tag{6}$$

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Theorem

The DAE (5) is regular on $\mathcal{I}$ with $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.
$\Longleftrightarrow$ *The DAE (6) is regular on $\mathcal{I}$ with $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.*
$\Longleftrightarrow$ *The DAE (5) is regular on each subinterval $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ with the same characteristics $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.*

In consequence, a regular DAE (5) possesses the properties:

1. The dynamical degree of freedom $d = m - \sum_{i=0}^{\mu}(m - r_i)$ is maintained when turning to a subinterval $\tilde{\mathcal{I}} \subseteq \mathcal{I}$.
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In contrast, a quasi-regular DAE (5) possesses the first property, but the second property is no longer given. This has consequences for rigorous solvability relations and the sensitivity analysis, in particular, for the transfer of discontinuities.

Outline

Theorem (Main Linearization Theorem)

The following three assertions are equivalent:

1. *The open connected set $\mathcal{G}$ is a regularity region of the DAE (1).*
2. *Each linearization of the DAE (1) along a sufficiently smooth function x_* with values in $\mathcal{G}$ is a regular linear DAE.*
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Theorem

Let the DAE (1) be quasi-regular on the open connected set $\mathcal{G}$.
$\Longrightarrow$ *Then each linearization of the DAE (1) along a sufficiently smooth function x_* with values in $\mathcal{G}$ is also a quasi-regular linear DAE.*

Outline

★ A nonlinear DAE is regular with index μ and characteristics $r_0, \ldots, r_\mu$, exactly if all its linearizations are so.

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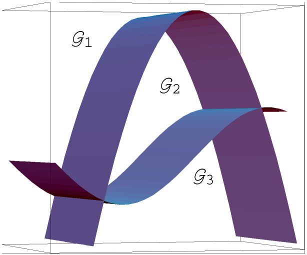
★ The definition domain $\mathcal{D}_f \times \mathcal{I}_f$ of a DAE decomposes into **several regularity regions bordered by critical points**. Solutions and other reference functions may cross or touch the borders and also stay there. A critical point belonging to a quasi-regularity region is harmless, otherwise the DAE shows a singular flow.



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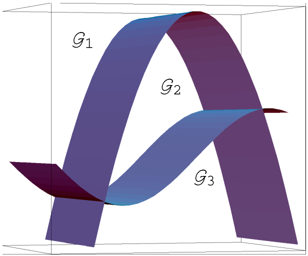

★ One can benefit from the constant rank conditions supporting the regularity notion to detect the critical points and to mark the regularity regions.

• Lamour/März/Tischendorf: Projector based DAE analysis.

... still in preparation...

Thank you for your attention!