LINEAR-QUADRATIC PROBLEMS FOR DESCRIPTOR SYSTEMS WITH INTERMEDIATE POINTS AND A SMALL PARAMETER

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Feedback Control for Problems with Intermediate Points

It is widely known that an optimal control in a feedback form for linear-quadratic problems can be found by using the matrix differential Riccati equation. The latter allow us to avoid solving boundary value problems.

In the case of singular control problems, non-standard operator Riccati equations arise. Different classes of such equations have been considered by S.L. Campbell, J.D. Cobb, L. Pandolfi, G.A. Kurina, D.J. Bender, A.J. Laub, P. Kunkel, V. Mehrmann, R. März.

The recent results on feedback solutions of optimal control problems with DAE constraints are contained in

[1] G.A. Kurina, R. März. SIAM J. Control Optim. 2007. Vol. 46. No.4. P.1277-1298.

Finite dimensional case

We consider the non-standard quadratic cost functional

$$J(u,y) = \frac{1}{2}\sum_{j=1}^{N+1}\left\langle y(t_j)-\xi_j, F_j(y(t_j)-\xi_j)\right\rangle +$$

$$+\frac{1}{2}\int_0^T \left\langle \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} W(t) & S(t) \\ S(t)' & R(t) \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \right\rangle dt \qquad (1)$$

to be minimized with respect to trajectories of the system

$$\frac{d(Ex(t))}{dt} = A(t)x(t) + B(t)u(t) + f(t), \quad Ex(0) = x^0, \qquad (2)$$

$$y(t) = C(t)x(t). \qquad (3)$$

The prime denotes the transposition.

Finite dimensional case

We will assume that t_j are fixed, $F_j, W(t), R(t)$ are symmetric, $R(t) > 0$, admissible controls $u(\cdot)$ are piecewise continuous functions on $[0, T]$ ensuring the solvability of a state equation with given conditions for the state variable.

Finite dimensional case

Theorem 1. Let the operator–function $K(\cdot)$ be a solution of the problem

$$E'\frac{dK(t)}{dt} = -K(t)'A(t) - A(t)'K(t) +$$
$$+(C(t)'S(t) + K(t)'B(t))R(t)^{-1}(S(t)'C(t) + B(t)'K(t)) -$$
$$-C'(t)W(t)C(t), \quad t \neq t_j, \tag{4}$$
$$E'(K(t_j - 0) - K(t_j + 0)) = C'(t)F_jC(t), \quad j = 1, \ldots, N, \tag{5}$$
$$E'K(T) = C'(T)F_{N+1}C(T), \tag{6}$$

the function $\varphi(\cdot)$ be a solution of the problem

$$E'\frac{d\varphi(t)}{dt} = -(A(t) - B(t)R(t)^{-1}(B(t)'K(t) + S(t)'C(t)))'\varphi(t) -$$
$$-K(t)'f(t), \quad t \neq t_j, \tag{7}$$
$$E'(\varphi(t_j + 0) - \varphi(t_j - 0)) = C'(t_j)F_j\xi_j, \quad j = 1, \ldots, N, \tag{8}$$
$$E'\varphi(T) = -C'(T)F_{N+1}\xi_{N+1}. \tag{9}$$

Finite dimensional case

Let $x_*(\cdot)$ be a solution of the initial value problem

$$\frac{d(Ex(t))}{dt} = (A(t) - B(t)R(t)^{-1}((S(t)'C(t) + B(t)'K(t)))x(t) -$$

$$-B(t)R(t)^{-1}B(t)'\varphi(t) + f(t), \quad Ex(0) = x^0, \tag{10}$$

$$y_* = Cx_*. \tag{11}$$

Then

$$u_*(t) = -R(t)^{-1}((S(t)'C(t) + B(t)'K(t))x_*(t) + B(t)'\varphi(t)) \tag{12}$$

is an optimal control for the problem (1)–(3)

Finite dimensional case

and the minimal value of the functional (1) is

$$J(u_*,y_*) = \frac{1}{2}\sum_{j=1}^{N+1}\langle \xi_j, F_j\xi_j\rangle +$$

$$+\left\langle x^0, \varphi(0) + \frac{1}{2}K(0)'\widetilde{E}^{-1}x^0\right\rangle +$$

$$+\frac{1}{2}\int_0^T \langle \varphi(t), 2f(t) - B(t)R(t)^{-1}B(t)'\varphi(t)\rangle\, dt. \qquad (13)$$

Coefficients in the problem condition are continuous however an optimal control is discontinuous in general case.

We use the following notations. $\widetilde{E}^{-1}$ is the inverse operator to the operator

$$(I-Q)E(I-P) : ImE' \to ImE. \tag{14}$$

Q is the orthogonal projector onto $KerE'$ corresponding to the decomposition

$$X = KerE' \oplus ImE, \tag{15}$$

P is the orthogonal projector onto $KerE$ corresponding to the decomposition

$$X = KerE \oplus ImE', \tag{16}$$

The last theorem is generalized for the case of Hilbert space, when the operator E is normally solvable, i.e., ImE is a closed set, and $A(t) = A$ is unbounded linear operator, acting from $D(A) \subset X$ into X, $\overline{D(A)} = X$ (the bar denotes here the closure). Let $F_j = E^* G_j E, \; j = 1, \ldots, N+1$, where the operators G_j are self–adjoint, $C(t) = I$, the operators $F_j \, (j = 1, \ldots, N+1)$, $W(t)$ and $R(t)$ are self-adjoint, moreover $\begin{bmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{bmatrix} \geq 0$, $R(t)$ is positive and has a bounded inverse for every $t \in [0, T]$. The superscript $*$ denotes here the adjoint operator, and the argument t is usually dropped for sake of brevity almost everywhere.

Infinite dimensional case

Theorem 2 (with A. Favini). Let $K(t) \in L(X)$, $t \in [0,T]$, be a solution of the problem

$$\langle E^* \tfrac{dK}{dt} y, z\rangle = -\langle Ay, Kz\rangle - \langle Ky, Az\rangle + \langle R^{-1}(S^* + B^*K)y, (S^* + B^*K)z\rangle - \\ -\langle Wy, z\rangle, \quad t \neq t_j, \quad y, z \in D(A), \tag{17}$$

$$E^*(K(t_j - 0) - K(t_j + 0)) = F_j, \quad j = 1, \ldots, N, \tag{18}$$

$$E^* K(T) = F_{N+1}, \tag{19}$$

Infinite dimensional case

$\varphi(t) \in X$, $t \in [0,T]$, be a solution of the problem

$$\langle E^* \frac{d\varphi}{dt}, y\rangle = -\langle \varphi, (A - BR^{-1}(B^*K + S^*))y\rangle - \langle K^* f(t), y\rangle, \qquad t \neq t_j, \quad y \in D(A), \tag{20}$$

$$E^*(\varphi(t_j+0) - \varphi(t_j-0)) = F_j\xi_j, \quad j = 1,\dots,N, \tag{21}$$

$$E^*\varphi(T) = -F_{N+1}\xi_{N+1}. \tag{22}$$

Let $x_*(\cdot)(x_*(t) \in D(A))$ be a solution of the initial value problem

$$\frac{d(Ex)}{dt} = Ax - BR^{-1}((S^* + B^*K)x + B^*\varphi) + f(t), \quad Ex(0) = x^0, \tag{23}$$

and

$$u_* = -R^{-1}((S^* + B^*K)x_* + B^*\varphi). \tag{24}$$

Then $u_*(\cdot)$ is an optimal control for the considered problem and the minimal value of the performance index is

$$J(u_*, x_*) =$$

$$= \frac{1}{2}\sum_{j=1}^{N+1}\langle \xi_j, F_j \xi_j\rangle + \frac{1}{2}\langle \widetilde{E}^{-1}(I-Q)x^0, K(0)^* x^0\rangle + Re\langle x^0, \varphi(0)\rangle +$$
$$+ \int_0^T \left(Re\,\langle f(t), \varphi(t)\rangle - \frac{1}{2}\langle \varphi(t), B(t)R^{-1}(t)B(t)^* \varphi(t)\rangle \right) dt. \tag{25}$$

Asymptotic Solution

- Linear-quadratic problems with differential-algebraic constraints and with intermediate points and a small parameter in a performance index (joint work with post-graduate student Smirnova E.V.)

Asymptotic Solution

- Linear-quadratic problems with differential-algebraic constraints and with intermediate points and a small parameter in a performance index (joint work with post-graduate student Smirnova E.V.)
- Singularly perturbed problems for discontinuous systems (joint work with post-graduate student Hoai Thi Nguen)

Asymptotic Solution

At first, we will construct asymptotic expansions of solutions by substituting postulated asymptotic expansions into the problem conditions and then defining a series of optimal control problems in order to find the expansions terms. This method has been called the ”direct scheme”.
The applications of the direct scheme and the survey of the publications, devoted to optimal control problems with a small parameter, are presented in [2].

[2] M. G. Dmitriev and G. A. Kurina, ”Singular perturbations in control problems”, Avtomatika i Telemehanika, no. 1, pp. 3-51, 2006 (in Russian).

Asymptotic Solution

- The estimates are obtained for the proximity of the asymptotic approximate solutions to the exact one.

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- The asymptotic expansion of the optimal feedback control is constructed for singularly perturbed problems with discontinuous coefficients, using the asymptotics of a corresponding matrix differential Riccati equation with discontinuous coefficients.

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- The nice property is proved, namely, the values of the minimized functional do not increase when higher-order approximations to the optimal control are used.
- The asymptotic expansion of the optimal feedback control is constructed for singularly perturbed problems with discontinuous coefficients, using the asymptotics of a corresponding matrix differential Riccati equation with discontinuous coefficients.
- The numerical examples are given in order to illustrate the proposed methods.

Problem Formulation

We consider a linear-quadratic optimal control problem P_ε of the form

$$J_\varepsilon(u,x) = \frac{1}{2}\left\langle x(T)-\xi_{N+1}, E'G_{N+1}E\left(x(T)-\xi_{N+1}\right)\right\rangle +$$

$$+\frac{\varepsilon}{2}\sum_{j=1}^{N}\left\langle x(t_j)-\xi_j, E'G_jE\left(x(t_j)-\xi_j\right)\right\rangle +$$

$$+\frac{1}{2}\int_0^T \left\langle \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} W(t) & S(t) \\ S(t)' & R(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\rangle dt \to \min_u, \qquad (26)$$

$$\frac{d(Ex(t))}{dt} = A(t)x(t)+B(t)u(t)+f(t), \quad Ex(0)=x^0. \qquad (27)$$

Here $\varepsilon \geq 0$ is a small parameter.

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- admissible controls $u(\cdot)$ in the perturbed problem are piecewise continuous functions ensuring the solvability of a state equation with a given condition,
  trajectories $x(\cdot)$ of a state equation are piecewise continuous functions satisfying the state equation almost everywhere such that $Ex(\cdot)$ are continuous,
- the operator

$$QA(t)P : KerE \to KerE' \tag{28}$$

is invertible and

$$\begin{pmatrix} PW(t)P & PS(t) \\ S(t)'P & R(t) \end{pmatrix} > 0, \quad t \in [0, T]. \tag{29}$$

Formalism of Asymptotic Expansions Construction

We will seek a solution of the perturbed problem (26), (27) in the series form

$$u(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j u_j(t), \quad x(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j x_j(t), \tag{30}$$

We substitute the relations (30) into the problem condition and expand the right-hand sides of equalities in series in powers of ε. Then the functional to be minimized may be written in the form

$$J_\varepsilon(u,x) = \sum_{j\geq 0} \varepsilon^j J_j. \tag{31}$$

Equating the coefficients of like powers of ε in (27), we obtain the equations for the terms of the decompositions (30).

We will determine a series of optimal control problems in order to find the coefficients in (30).

Formalism of Asymptotic Expansions Construction

When $\varepsilon = 0$ we obtain from (26), (27) the degenerate problem without intermediate points

$$P_0 : J_0 = J_0(u_0) = \frac{1}{2}\langle x_0(T) - \xi_{N+1}, F_{N+1}(x_0(T) - \xi_{N+1})\rangle +$$

$$+ \int_0^T \left(\frac{1}{2}\langle x_0, Wx_0\rangle + \langle x_0, Su_0\rangle + \frac{1}{2}\langle u_0, Ru_0\rangle\right)dt \to \min_{u_0}, \qquad (32)$$

$$\frac{d(Ex_0)}{dt} = Ax_0 + Bu_0 + f, \quad Ex_0(0) = x^0. \qquad (33)$$

Here and further, we denote the operator $E^* G_j E$ by $F_j, j = 1, \ldots, N+1$.
The optimal control for the problem P_0 is a continuous function.

Formalism of Asymptotic Expansions Construction

Further, in order to determine the pair of the functions (u_k, x_k) for $k \geq 1$, we define the following problem

$$P_k : \widetilde{J}_k(u_k, x_k) = \frac{1}{2}\langle x_k(T), F_{N+1}x_k(T)\rangle + \sum_{j=1}^{N}\Big\langle x_k(t_j), F_j\big(x_{k-1}(t_j) - \xi_{j,k-1}\big)\Big\rangle +$$

$$+\frac{1}{2}\int_0^T \left\langle \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \begin{pmatrix} W & S \\ S' & R \end{pmatrix}\begin{pmatrix} x_k \\ u_k \end{pmatrix} \right\rangle dt \to \min_{u_k}, \qquad (34)$$

$$\frac{d(Ex_k)}{dt} = Ax_k + Bu_k, \quad Ex_k(0) = 0, \qquad (35)$$

where

$$\xi_{j,k-1} = \begin{cases} \xi_j, & k = 1, \\ 0, & k > 1. \end{cases}$$

Formalism of Asymptotic Expansions Construction

The solution of the problem P_k can be found from the following relations

$$\frac{d(Ex_k)}{dt} = Ax_k + Bu_k, \quad Ex_k(0) = 0, \tag{13}$$

$$E'\frac{d\psi_k}{dt} = Wx_k - A'\psi_k + Su_k, \quad t \neq t_j, \tag{36}$$

$$E'(\psi_k(t_j - 0) - \psi_k(t_j + 0)) = -F_j(x_{k-1}(t_j) - \xi_{j,k-1}), \quad j = 1, \ldots, N, \tag{37}$$

$$E'\psi_k(T) = -F_{N+1}x_k(T), \tag{38}$$

$$0 = -S'x_k + B'\psi_k - Ru_k. \tag{39}$$

In general, the optimal control for the problem P_k is a discontinuous function when $k \geq 1$.

The following theorem shows the structure of the coefficients in the decomposition of the minimized functional: $J_\varepsilon = \sum\limits_{j\geq 0} \varepsilon^j J_j$.

Theorem 3. The coefficient J_{2k-1} is known after problem P_{k-1} has been solved. The performance index for the problem P_k is the transformed expression for the coefficient J_{2k} when $k \geq 1$.

Theorem 3 is valid without the assumption on the invertibility of the operator (28).

Estimates of Approximate Solution

Let us assume that the solutions (u_j, x_j) have been found for the problems P_j, $j = 0, \ldots, n$.
We shall estimate the approximate solution of the perturbed problem P_ε:

$$\widetilde{u}_n(t) = \sum_{j=0}^{n} \varepsilon^j u_j(t), \quad \widetilde{x}_n(t) = \sum_{j=0}^{n} \varepsilon^j x_j(t). \tag{40}$$

It is not difficult to see that the function $\widetilde{x}_n(t)$ is a solution of the problem (27) when $u(t) = \widetilde{u}_n(t)$.

Estimates of Approximate Solution

We will denote the solution of the problem P_ε by (u_*, x_*).

Theorem 4. The following estimates

$$\begin{gathered}\|u_*(t) - \widetilde{u}_n(t)\| \le c\varepsilon^{n+1}, \|x_*(t) - \widetilde{x}_n(t)\| \le c\varepsilon^{n+1}, \\ J_\varepsilon(\widetilde{u}_n, \widetilde{x}_n) - J_\varepsilon(u_*, x_*) \le c\varepsilon^{2(n+1)},\end{gathered} \tag{41}$$

are true for all $t \in [0, T]$ and sufficiently small $\varepsilon > 0$.

A constant c does not depend on t and ε.

It follows from this theorem that $\{\widetilde{u}_n(\cdot)\}$ is a minimizing sequence for the considered functional.

Estimates of Approximate Solution

It has been proved that the sequence

$$\{J_\varepsilon(\widetilde{u}_i, \widetilde{x}_i)\} \tag{42}$$

is decreasing for fixed ε.

Theorem 5. For sufficiently small $\varepsilon > 0$, we have

$$J_\varepsilon(\widetilde{u}_i, \widetilde{x}_i) \leq J_\varepsilon(\widetilde{u}_{i-1}, \widetilde{x}_{i-1}), \quad i = 1, \ldots, n. \tag{43}$$

If $u_i \neq 0$ then (43) is a strict inequality.

Illustrative Example

As the obtained results are new for problems with a state equation resolved with respect to the derivative, we consider the problem P_ε of minimizing the functional

$$J_\varepsilon(u,x) = J_\varepsilon(u) = \frac{1}{2}((x(4)-100)^2 + (y(4)-100)^2) + \frac{\varepsilon}{2}((x(1)+5000)^2 +$$

$$+(y(1)-5000)^2 + (x(2)-4000)^2 + (y(2)+3000)^2 +$$

$$+(x(3)+1000)^2 + (y(3)-100)^2) + \int_0^4 u^2 dt \qquad (44)$$

on trajectories of the system

$$\begin{aligned} \frac{dx}{dt} &= y + e^t, \quad x(0) = 1, \\ \frac{dy}{dt} &= u, \quad y(0) = 100, \end{aligned} \qquad (45)$$

when $\varepsilon = 0.1$.

Illustrative Example

Fig.1. $x = x(t)$

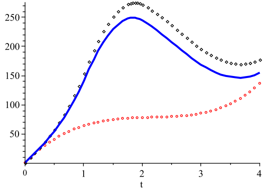


Fig.2. $y = y(t)$

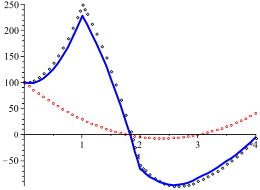
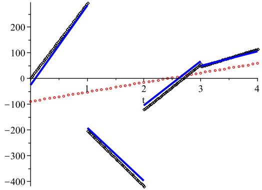


Fig.3. $u = u(t)$



The exact solution is denoted by blue continuous line, the zero approximation for the solution is denoted by red circles, and the first approximation is denoted by diamonds.

Singularly Perturbed Problems for Discontinuous Systems

The formalism of asymptotics constructing for solutions of singularly perturbed linear-quadratic optimal control problems with discontinuous coefficients is based on immediate substituting a postulated asymptotic expansion of boundary layer type for a solution into the problem condition and on defining four types of optimal control problems for finding asymptotics terms.

The unique solvability of the problems, the solutions of which form the asymptotic solution, is proved.

We note that the asymptotic analysis of singularly perturbed linear-quadratic optimal control problems was previously made only for the case of continuous coefficients.

We consider linear-quadratic optimal control problem P_ε of the form

$$J_\varepsilon(u) = \frac{1}{2}\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\left\{\left\langle \overset{(j)}{z}(t,\varepsilon), \overset{(j)}{\mathbb{W}}(t,\varepsilon)\overset{(j)}{z}(t,\varepsilon)\right\rangle + \right.$$
$$\left. + \left\langle \overset{(j)}{u}(t,\varepsilon), \overset{(j)}{\mathbb{R}}(t,\varepsilon)\overset{(j)}{u}(t,\varepsilon)\right\rangle\right\} dt \to \min_u, \tag{46}$$

$$\mathbb{E}(\varepsilon)\dot{\overset{(j)}{z}}(t,\varepsilon) = \overset{(j)}{\mathbb{A}}(t,\varepsilon)\overset{(j)}{z}(t,\varepsilon) + \overset{(j)}{\mathbb{B}}(t,\varepsilon)\overset{(j)}{u}(t,\varepsilon), \ t \in [t_{j-1}, t_j], \ \ j = 1, 2,$$
$$\overset{(1)}{z}(0,\varepsilon) = z^0, \quad \overset{(1)}{z}(t_1,\varepsilon) = \overset{(2)}{z}(t_1,\varepsilon). \tag{47}$$

Problem Formulation

Here $0 = t_0 < t_1 < t_2 = T$, the values $t_j (j = 0, 1, 2)$ are fixed;

$$\mathbb{E}(\varepsilon) = diag(I, \varepsilon I), \tag{48}$$

$z = (x', y')'$, $\overset{(j)}{x} = \overset{(j)}{x}(t, \varepsilon) \in R^n$, $\overset{(j)}{y} = \overset{(j)}{y}(t, \varepsilon) \in R^m$, $\overset{(j)}{u} = \overset{(j)}{u}(t, \varepsilon) \in R^r$;
$\varepsilon \geq 0$ is a small parameter, matrices $\overset{(j)}{\mathbb{W}}(t, \varepsilon)$, $\overset{(j)}{\mathbb{R}}(t, \varepsilon)$, $\overset{(j)}{\mathbb{A}}(t, \varepsilon)$,
$\overset{(j)}{\mathbb{B}}(t, \varepsilon)$ are sufficiently smooth for all $t \in [t_{j-1}, t_j]$ and $\varepsilon \geq 0$;
matrices $\overset{(j)}{\mathbb{W}}(t, \varepsilon)$ and $\overset{(j)}{\mathbb{R}}(t, \varepsilon)$ are symmetric, $\overset{(j)}{\mathbb{W}}(t, 0) > 0$,
$\overset{(j)}{\mathbb{R}}(t, 0) > 0$ for all $t \in [t_{j-1}, t_j]$, $j = 1, 2$.

If $\varepsilon = 0$, then, in general case, the solution of the state equation does not satisfy given conditions and the fast variable may be discontinuous function although trajectories of the perturbed problem are continuous.

Formalism of Asymptotic Expansions Construction

We will seek a solution of the perturbed problem in the series form

$$\overset{(j)}{v}(t,\varepsilon) = \sum_{i\geq 0} \varepsilon^i \overset{(j)}{v_i}(t,\varepsilon) = \sum_{i\geq 0} \varepsilon^i \big(\overset{(j)}{\bar{v}_i}(t) + \overset{(j)}{\Pi}_i v(\tau_{j-1}) + \overset{(j)}{Q}_i v(\tau_j)\big), \tag{49}$$

where $v = (u', z')'$, symbols $\overset{(j)}{\Pi}$, $j \in \{1,2\}$, denote boundary value functions of the exponential type in a neighborhood of left hand sides of the segments $[0, t_1]$ and $[t_1, T]$, and $\overset{(j)}{Q}$, $j \in \{1,2\}$, denote boundary value functions of the exponential type in a neighborhood of right hand sides of the same segments,

$$\tau_0 = \frac{t}{\varepsilon}, \tau_1 = \frac{t - t_1}{\varepsilon}, \tau_2 = \frac{t - T}{\varepsilon}. \tag{50}$$

Formalism of Asymptotic Expansions Construction

We substitute the last expansions into problem conditions and write the right hand side of the state equation and integrand function in the asymptotic series form with respect to powers of ε with coefficients depending on $t, \tau_0, \tau_1, \tau_2$.

Separately equating the coefficients of like powers of ε depending on $t, \tau_0, \tau_1, \tau_2$ in the state equation, we obtain the equations for the terms of the decompositions.

Then the functional to be minimized may be written in the form

$$J_\varepsilon(u,x) = \sum_{j \geq 0} \varepsilon^j J_j. \tag{51}$$

We will determine a series of optimal control problems in order to find the coefficients of the expansions.

Formalism of Asymptotic Expansions Construction

We will use the following notations for the expansion of any function $h = h(\varepsilon)$ with respect to powers of ε:

$$h(\varepsilon) = \sum_{i\geq 0} \varepsilon^i h_i =$$

$$= \{h\}_{n-1} + \varepsilon^n [h]_n + \alpha(\varepsilon^{n+1}), [h]_n = h_n, \{h\}_{n-1} = \sum_{i=0}^{n-1} \varepsilon^i h_i. \tag{52}$$

We will also use the notation

$$\overset{(j)}{\widetilde{v}}_n (t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \overset{(j)}{v}_i (t), \overset{(j)}{\widetilde{\Pi}}_n v(\tau_{j-1},\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \overset{(j)}{\Pi}_i v(\tau_{j-1}),$$

$$\begin{gathered} \overset{(j)}{\widetilde{Q}}_n v(\tau_j,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \overset{(j)}{Q}_i v(\tau_j), \overset{(j)}{f} (\overset{(j)}{z}, \overset{(j)}{u}, t, \varepsilon) = \overset{(j)}{\mathbb{A}}(t,\varepsilon) \overset{(j)}{z} + \overset{(j)}{\mathbb{B}}(t,\varepsilon) \overset{(j)}{u}, \\ \overset{(j)}{d} (\overset{(j)}{z}, \overset{(j)}{\omega}, t, \varepsilon) = \overset{(j)}{\mathbb{W}}(t,\varepsilon) \overset{(j)}{z} - \overset{(j)}{\mathbb{A}}(t,\varepsilon)' \overset{(j)}{\omega}, \\ \overset{(j)}{g} (\overset{(j)}{u}, \overset{(j)}{\omega}, t, \varepsilon) = \overset{(j)}{R}(t,\varepsilon) \overset{(j)}{u} - \overset{(j)}{\mathbb{B}}(t,\varepsilon)' \overset{(j)}{\omega}. \end{gathered} \tag{53}$$

So, we have the following systems for coefficients in (49).

$$[\mathbb{E}(\varepsilon)\dot{\overset{(j)}{\overline{z}}}(t,\varepsilon)]_i = \overset{(j)}{\mathbb{A}}_0(t)\overset{(j)}{\overline{z}_i} + \overset{(j)}{\mathbb{B}}_0(t)\overset{(j)}{\overline{u}_i} + [\overset{(j)}{\widehat{f}}_{i-1}]_i,\ t_{j-1} \le t \le t_j,\ j = 1,2, \tag{54}$$

$$\frac{d\overset{(j)}{\Pi}_i z}{d\tau_{j-1}} = E_2(\overset{(j)}{\mathbb{A}}_0(t_{j-1})\overset{(j)}{\Pi}_i z + \overset{(j)}{\mathbb{B}}_0(t_{j-1})\overset{(j)}{\Pi}_i u) + E_1[\overset{(j)}{\widehat{\Pi}}_{i-1} f]_i + E_2[\overset{(j)}{\widehat{\Pi}}_{i-1} f]_{i-1}, \tag{55}$$

$$\frac{d\overset{(j)}{Q}_i z}{d\tau_j} = E_2(\overset{(j)}{\mathbb{A}}_0(t_j)\overset{(j)}{Q}_i z + \overset{(j)}{\mathbb{B}}_0(t_j)\overset{(j)}{Q}_i u) + E_1[\overset{(j)}{\widehat{Q}}_{i-1} f]_i + E_2[\overset{(j)}{\widehat{Q}}_{i-1} f]_{i-1}, \tag{56}$$

where

$$E_1 = diag(I,0), \quad E_2 = diag(0,I). \tag{57}$$

Functions $\overset{(j)}{\overline{u}_0}$, $\overset{(j)}{\overline{z}_0}$ may be found from the problem

$$\overline{P}_0 \colon \overline{J}_0(\overline{u}_0) = \frac{1}{2}\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\Bigg(\Big\langle \overset{(j)}{\overline{z}_0}, \overset{(j)}{\mathbb{W}_0}(t)\overset{(j)}{\overline{z}_0}\Big\rangle + \\ + \Big\langle \overset{(j)}{\overline{u}_0}, \overset{(j)}{\mathbb{R}_0}(t)\overset{(j)}{\overline{u}_0}\Big\rangle\Bigg)dt \to \min_{(\overset{(1)}{\overline{u}_0}, \overset{(2)}{\overline{u}_0})}, \tag{58}$$

$$E_1 \dot{\overset{(j)}{\overline{z}_0}} = \overset{(j)}{\mathbb{A}_0}(t)\overset{(j)}{\overline{z}_0} + \overset{(j)}{\mathbb{B}_0}(t)\overset{(j)}{\overline{u}_0},\ t \in [t_{j-1}, t_j],\ \ j = 1, 2,$$

$$E_1 \overset{(1)}{\overline{z}_0}(0) = E_1 z^0,\ E_1 \overset{(2)}{\overline{z}_0}(t_1) = E_1 \overset{(1)}{\overline{z}_0}(t_1).$$

It is proved that this problem has a unique solution. Its adjoint variable is denoted by $\overset{(j)}{\overline{\omega}_0}$.

Transforming the expression for J_1 and reject addends which are known after solving problem $\overline{P}_0$ we obtain the sum

$$\Pi 1 J_0 + \Pi 2 J_0 + \Pi 3 J_0, \tag{59}$$

where

$$\Pi 1 J_0 = \Pi 1 J_0(\overset{(1)}{\Pi}_0 u) = \frac{1}{2}\int_0^{+\infty}\left\{\left\langle \overset{(1)}{\Pi}_0 z, \overset{(1)}{\mathbb{W}}_0(0)\overset{(1)}{\Pi}_0 z\right\rangle + \left\langle \overset{(1)}{\Pi}_0 u, \overset{(1)}{\mathbb{R}}_0(0)\overset{(1)}{\Pi}_0 u\right\rangle\right\} d\tau_0,$$

$$\Pi 2 J_0 = \Pi 2 J_0(\overset{(1)}{Q}_0 u, \overset{(2)}{\Pi}_0 u) = \left\langle \overset{(1)}{Q}_0 z(0), E_2 \overset{(1)}{\overline{\omega}}_0(t_1)\right\rangle - \left\langle \overset{(2)}{\Pi}_0 z(0), E_2 \overset{(2)}{\overline{\omega}}_0(t_1)\right\rangle +$$

$$+\frac{1}{2}\int_{-\infty}^0 \left\{\left\langle \overset{(1)}{Q}_0 z, \overset{(1)}{\mathbb{W}}_0(t_1)\overset{(1)}{Q}_0 z\right\rangle + \left\langle \overset{(1)}{Q}_0 u, \overset{(1)}{\mathbb{R}}_0(t_1)\overset{(1)}{Q}_0 u\right\rangle\right\} d\tau_1 +$$

$$+\frac{1}{2}\int_0^{+\infty} \left\{\left\langle \overset{(2)}{\Pi}_0 z, \overset{(2)}{\mathbb{W}}_0(t_1)\overset{(2)}{\Pi}_0 z\right\rangle + \left\langle \overset{(2)}{\Pi}_0 u, \overset{(2)}{\mathbb{R}}_0(t_1)\overset{(2)}{\Pi}_0 u\right\rangle\right\} d\tau_1,$$

$$\Pi 3 J_0 = \Pi 3 J_0(\overset{(2)}{Q}_0 u) = \left\langle \overset{(2)}{Q}_0 z(0), E_2 \overset{(2)}{\overline{\omega}}_0(T)\right\rangle +$$

$$+\frac{1}{2}\int_{-\infty}^0 \left\{\left\langle \overset{(2)}{Q}_0 z, \overset{(2)}{\mathbb{W}}_0(T)\overset{(2)}{Q}_0 z\right\rangle + \left\langle \overset{(2)}{Q}_0 u, \overset{(2)}{\mathbb{R}}_0(T)\overset{(2)}{Q}_0 u\right\rangle\right\} d\tau_2. \tag{60}$$

As the equations for finding boundary layer coefficients depend only on one type from four considered types of boundary layer functions therefore coefficients of boundary layers series in zero approximation may be found from following three problems:

$$\Pi 1 P_0 : \ \Pi 1 J_0(\overset{(1)}{\Pi}_0 u) \to \min_{\overset{(1)}{\Pi}_0 u},$$

$$\frac{d\overset{(1)}{\Pi}_0 z}{d\tau_0} = E_2(\overset{(1)}{\mathbb{A}}_0(0)\overset{(1)}{\Pi}_0 z + \overset{(1)}{\mathbb{B}}_0(0)\overset{(1)}{\Pi}_0 u), \quad \tau_0 \geq 0, \tag{61}$$

$$E_1\overset{(1)}{\Pi}_0 z(+\infty) = 0, \ \ E_2\overset{(1)}{\Pi}_0 z(0) = E_2(z^0 - \overset{(1)}{\bar{z}_0}(0)),$$

Formalism of Asymptotic Expansions Construction

$$
\begin{gathered}
\Pi 2P_0:\ \Pi 2J_0(\overset{(1)}{Q}_0u, \overset{(2)}{\Pi}_0u) \to \min_{(\overset{(1)}{Q}_0u, \overset{(2)}{\Pi}_0u)}, \\
\frac{d\overset{(1)}{Q}_0z}{d\tau_1} = E_2(\overset{(1)}{\mathbb{A}}_0(t_1)\overset{(1)}{Q}_0z + \overset{(1)}{\mathbb{B}}_0(t_1)\overset{(1)}{Q}_0u), \quad \tau_1 \le 0, \\
\frac{d\overset{(2)}{\Pi}_0z}{d\tau_1} = E_2(\overset{(2)}{\mathbb{A}}_0(t_1)\overset{(2)}{\Pi}_0z + \overset{(2)}{\mathbb{B}}_0(t_1)\overset{(2)}{\Pi}_0u), \quad \tau_1 \ge 0, \\
\overset{(1)}{Q}_0z(-\infty) = 0,\ E_1\overset{(2)}{\Pi}_0z(+\infty) = 0, \\
E_2\overset{(2)}{\Pi}_0z(0) = E_2(\overset{(1)}{\bar{z}}_0(t_1) + \overset{(1)}{Q}_0z(0) - \overset{(2)}{\bar{z}}_0(t_1)),
\end{gathered}
\tag{62}
$$

Formalism of Asymptotic Expansions Construction

$$\Pi 3P_0 : \Pi 3J_0(\overset{(2)}{Q}_0 u) \to \min_{\overset{(2)}{Q}_0 u},$$

$$\frac{d\overset{(2)}{Q}_0 z}{d\tau_2} = E_2(\overset{(2)}{\mathbb{A}}_0(T)\overset{(2)}{Q}_0 z + \overset{(2)}{\mathbb{B}}_0(T)\overset{(2)}{Q}_0 u), \quad \tau_2 \le 0, \tag{63}$$

$$\overset{(2)}{Q}_0 z(-\infty) = 0.$$

These problems have unique solutions. Control optimality conditions for these problems have be obtained.

Let us introduce the recurrent formulas for finding expansions terms with positive indexes. Let the problems $\overline{P}_i$, $\Pi 1 P_i$, $\Pi 2 P_i$, $\Pi 3 P_i$, $0 \le i \le n-1$, have been solved. We denote the adjoint variables for these problems respectively by $\overset{(j)}{\overline{\omega}}_i(t),\ j = 1,2$, $\overset{(1)}{\Pi}_i \omega(\tau_0)$, $(\overset{(1)}{Q}_i \omega(\tau_1), \overset{(2)}{\Pi}_i \omega(\tau_1))$, $\overset{(2)}{Q}_i \omega(\tau_2)$.

The functions $\bar{v}_n(t)$, $t \in [0,T]$, are found from the problem $\overline{P}_n$ of minimizing the functional

$$\bar{J}_n(\bar{u}_n) = \langle \overset{(1)}{\bar{z}}_n(t_1), E_1 \overset{(1)}{Q}_{n-1}\omega(0)\rangle - \langle \overset{(2)}{\bar{z}}_n(t_1), E_1 \overset{(2)}{\Pi}_{n-1}\omega(0)\rangle +$$
$$+\langle \overset{(2)}{\bar{z}}_n(T), E_1 \overset{(2)}{Q}_{n-1}\omega(0)\rangle + \sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\left(\left\langle \overset{(j)}{\bar{z}}_n(t), \frac{1}{2}\overset{(j)}{\mathbb{W}}_0(t)\overset{(j)}{\bar{z}}_n(t) + \right.\right.$$
$$\left.\left.+[\overset{(j)}{\widehat{q}}_{n-1}]_n - E_2'\dot{\overset{(j)}{\overline{\omega}}}_{n-1}(t)\right\rangle + \left\langle \overset{(j)}{\bar{u}}_n(t), \frac{1}{2}\overset{(j)}{\mathbb{R}}_0(t)\overset{(j)}{\bar{u}}_n(t) + [\overset{(j)}{\widehat{g}}_{n-1}]_n\right\rangle\right)dt \qquad (64)$$

on trajectories of the system (54) when $i = n$ under the conditions

$$\begin{aligned} E_1 \overset{(1)}{\bar{z}_n}(0) &= -E_1 \overset{(1)}{\Pi}_n z(0), \\ E_1(\overset{(2)}{\bar{z}_n}(t_1) - \overset{(1)}{\bar{z}_n}(t_1)) &= E_1(\overset{(1)}{Q}_n z(0) - \overset{(2)}{\Pi}_n z(0)). \end{aligned} \tag{65}$$

Here the symbols $\overset{(j)}{\widehat{q}}_{n-1}$, $\overset{(j)}{\widehat{g}}_{n-1}$ mean the values of the functions $\overset{(j)}{q}$, $\overset{(j)}{g}$ when $\overset{(j)}{v} = \overset{(j)}{\widetilde{\bar{v}}}_{n-1}$, $\overset{(j)}{\omega} = \overset{(j)}{\widetilde{\bar{\omega}}}_{n-1}$.

The functions $\overset{(1)}{\Pi}_n v(\tau_0)$, $\tau_0 \in [0, +\infty)$, are found from the problem $\Pi 1 P_n$ of minimizing the functional

$$\Pi 1 J_n(\overset{(1)}{\Pi}_n u) = \int_0^{+\infty} \left(\left\langle \overset{(1)}{\Pi}_n z(\tau_0), \frac{1}{2} \overset{(1)}{\mathbb{W}}_0(0) \overset{(1)}{\Pi}_n z(\tau_0) + [\overset{(1)}{\widehat{\Pi}}_{n-1} q]_n \right\rangle + \right.$$
$$\left. + \left\langle \overset{(1)}{\Pi}_n u(\tau_0), \frac{1}{2} \overset{(1)}{\mathbb{R}}_0(0) \overset{(1)}{\Pi}_n u(\tau_0) + [\overset{(1)}{\widehat{\Pi}}_{n-1} g]_n \right\rangle \right) d\tau_0 \tag{66}$$

on trajectories of the system (55) when $j = 1$, $i = n$ under the conditions

$$E_1 \overset{(1)}{\Pi}_n z(+\infty) = 0, \;\; E_2 \overset{(1)}{\Pi}_n z(0) = -E_2 \overset{(1)}{\bar{z}_n} (0). \tag{67}$$

Formalism of Asymptotic Expansions Construction

The functions $\overset{(1)}{Q}_n v(\tau_1)$, $\tau_1 \in (-\infty, 0]$ and $\overset{(2)}{\Pi}_n v(\tau_1)$, $\tau_1 \in [0, +\infty)$ are found from the problem $\Pi 2P_n$ of minimizing the functional

$$
\begin{aligned}
\Pi 2J_n(\overset{(1)}{Q}_n u, \overset{(2)}{\Pi}_n u) &= \langle \overset{(1)}{Q}_n z(0), E_2 \overset{(1)}{\overline{\omega}}_n(t_1)\rangle - \langle \overset{(2)}{\Pi}_n z(0), E_2 \overset{(2)}{\overline{\omega}}_n(t_1)\rangle + \\
&+ \int_{-\infty}^{0} \left(\left\langle \overset{(1)}{Q}_n z(\tau_1), \frac{1}{2} \overset{(1)}{\mathbb{W}}_0(t_1) \overset{(1)}{Q}_n z(\tau_1) + [\overset{(1)}{\widehat{Q}}_{n-1} q]_n \right\rangle + \right. \\
&+ \left. \left\langle \overset{(1)}{Q}_n u(\tau_1), \frac{1}{2} \overset{(1)}{\mathbb{R}}_0(t_1) \overset{(1)}{Q}_n u(\tau_1) + [\overset{(1)}{\widehat{Q}}_{n-1} g]_n \right\rangle \right) d\tau_1 + \\
&+ \int_{0}^{+\infty} \left(\left\langle \overset{(2)}{\Pi}_n z(\tau_1), \frac{1}{2} \overset{(2)}{\mathbb{W}}_0(t_1) \overset{(2)}{\Pi}_n z(\tau_1) + [\overset{(2)}{\widehat{\Pi}}_{n-1} q]_n \right\rangle + \right. \\
&+ \left. \left\langle \overset{(2)}{\Pi}_n u(\tau_1), \frac{1}{2} \overset{(2)}{\mathbb{R}}_0(t_1) \overset{(2)}{\Pi}_n u(\tau_1) + [\overset{(2)}{\widehat{\Pi}}_{n-1} g]_n \right\rangle \right) d\tau_1
\end{aligned}
\qquad (68)
$$

on trajectories of the system (55) when $j = 2$, $i = n$ and (56) when $j = 1$, $i = n$ under conditions

$$\overset{(1)}{Q}_n z(-\infty) = 0, \quad E_1 \overset{(2)}{\Pi}_n z(+\infty) = 0,$$
$$E_2(\overset{(2)}{\Pi}_n z(0) - \overset{(1)}{Q}_n z(0)) = E_2(\overset{(1)}{\bar{z}_n}(t_1) - \overset{(2)}{\bar{z}_n}(t_1)). \tag{69}$$

The functions $\overset{(2)}{Q}_n v(\tau_2)$, $\tau_2 \in (-\infty, 0]$, are found from the problem $\Pi 3 P_n$ of minimizing the functional

$$\Pi 3 J_n(\overset{(2)}{Q}_n u) = \langle \overset{(2)}{Q}_n z(0), E_2 \overset{(2)}{\overline{\omega}}_n(T) \rangle + \\ + \int_{-\infty}^{0} \left(\left\langle \overset{(2)}{Q}_n z(\tau_2), \frac{1}{2} \overset{(2)}{\mathbb{W}}_0(T) \overset{(2)}{Q}_n z(\tau_2) + [\overset{(2)}{\widehat{Q}}_{n-1} q]_n \right\rangle + \\ + \left\langle \overset{(2)}{Q}_n u(\tau_2), \frac{1}{2} \overset{(2)}{\mathbb{R}}_0(T) \overset{(2)}{Q}_n u(\tau_2) + [\overset{(2)}{\widehat{Q}}_{n-1} g]_n \right\rangle \right) d\tau_2 \tag{70}$$

on trajectories of the system (56) when $j = 2$, $i = n$, $-\infty < \tau_2 \leq 0$, under the condition

$$\overset{(2)}{Q}_n z(-\infty) = 0. \tag{71}$$

Formalism of Asymptotic Expansions Construction

In problems $\Pi 1 P_n$, $\Pi 2 P_n$, $\Pi 3 P_n$, the symbols

$$\overset{(j)}{\widehat{\Pi}}_{n-1} q, \quad \overset{(j)}{\widehat{\Pi}}_{n-1} g, \quad \overset{(j)}{\widehat{Q}}_{n-1} q, \quad \overset{(j)}{\widehat{Q}}_{n-1} g \tag{72}$$

mean the values of the functions $\overset{(j)}{q}$, $\overset{(j)}{g}$ with arguments depending on asymptotics terms of lower order than n.

It has been proved that the problems $\overline{P}_n$, $\Pi 1 P_n$, $\Pi 2 P_n$, $\Pi 3 P_n$ $(n \geq 0)$ have unique solutions. These solutions may be found from the obtained control optimality conditions.

The following two theorem explain the form of the problems for finding asymptotics terms.

We denote the coefficients in an asymptotic expansion of the adjoint variable for the perturbed problem:
$\overset{(j)}{\zeta}(t,\varepsilon) = (\overset{(j)}{\xi}(t,\varepsilon', \overset{(j)}{\eta}(t,\varepsilon)')'$ by

$$\overset{(j)}{\bar{\zeta}}_i(t), \quad \overset{(j)}{\Pi}_i \zeta(\tau_{j-1}), \quad \overset{(j)}{Q}_i \zeta(\tau_j). \tag{73}$$

Theorem 6. Problems obtained from the control optimality conditions for the problems

$$\overline{P}_n, \quad \Pi 1 P_n, \quad \Pi 2 P_n, \quad \Pi 3 P_n \tag{74}$$

coincide with the problems for the functions

$$(\overset{(j)}{\overline{v}}_n, \overset{(j)}{\overline{\xi}}_n, \overset{(j)}{\overline{\eta}}_n), \quad j = 1,2, \quad (\overset{(1)}{\Pi}_n v, \overset{(1)}{\Pi}_{n+1} \xi, \overset{(1)}{\Pi}_n \eta),$$
$$(\overset{(1)}{Q}_n v, \overset{(1)}{Q}_{n+1} \xi, \overset{(1)}{Q}_n \eta, \overset{(2)}{\Pi}_n v, \overset{(2)}{\Pi}_{n+1} \xi, \overset{(2)}{\Pi}_n \eta), \quad (\overset{(2)}{Q}_n v, \overset{(2)}{Q}_{n+1} \xi, \overset{(2)}{Q}_n \eta) \tag{75}$$

from the asymptotic expansion of the solution of the problem following from the control optimality conditions for the perturbed problem P_ε.

Theorem 7. The performance index $\overline{J}_n$ is a result of a transformaton of the coefficient J_{2n} in the expansion (51), and the sum of the performance indexes $\Pi 1 J_n + \Pi 2 J_n + \Pi 3 J_n$ is a result of a transformation of the coefficient J_{2n+1} in the expansion (51).

Estimates of Approximate Solution

Let $u_*(.,\varepsilon)$, $z_*(.,\varepsilon)$ be a solution of the perturbed problem P_ε and $\widetilde{u}_n(t,\varepsilon)$, $\widetilde{z}_n(t,\varepsilon)$ be an approximate asymptotic solution.

Theorem 8. The following estimates

$$\begin{gathered}\|\widetilde{u}_n(t,\varepsilon)-u_*(t,\varepsilon)\|\leq c\varepsilon^{n+1},\ \ \|\widetilde{z}_n(t,\varepsilon)-z_*(t,\varepsilon)\|\leq c\varepsilon^{n+1},\\ J_\varepsilon(\widetilde{u}_n)-J_\varepsilon(u_*)\leq c\varepsilon^{2(n+2)},\end{gathered} \tag{76}$$

are true for all $t\in[0,T]$ and sufficiently small $\varepsilon>0$.

A constant c does not depend on t and ε.

It follows from this theorem that $\{\widetilde{u}_n(\cdot,\varepsilon)\}$ is a minimizing sequence for the considered functional.

Estimates of Approximate Solution

It is proved that the sequence

$$\{J_\varepsilon(\widetilde{u}_i\} \tag{77}$$

is decreasing for fixed ε.

Theorem 9. For sufficiently small $\varepsilon > 0$, we have

$$J_\varepsilon(\widetilde{u}_i) \leq J_\varepsilon(\widetilde{u}_{i-1}), \quad i = 1, \ldots, n. \tag{78}$$

Illustrative Example

Let us consider the problem

$$P_\varepsilon: \ J_\varepsilon(u) = \frac{1}{2}\int_0^1 \left((\overset{(1)}{x})^2 + 2\overset{(1)}{x}\overset{(1)}{y} + 3(\overset{(1)}{y})^2 + (\overset{(1)}{u})^2 \right) dt + \\ + \frac{1}{2}\int_1^2 \left((\overset{(2)}{x})^2 + (\overset{(2)}{y})^2 + \frac{1}{3}(\overset{(2)}{u})^2 \right) dt,$$

$$\dot{\overset{(1)}{x}} = (1+\varepsilon)\overset{(1)}{x}, \quad \varepsilon\dot{\overset{(1)}{y}} = -\overset{(1)}{y} + \overset{(1)}{u}, \tag{79}$$

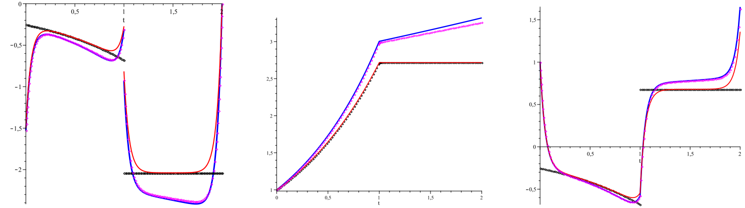
$$\dot{\overset{(2)}{x}} = \varepsilon\overset{(2)}{x}, \quad \varepsilon\dot{\overset{(2)}{y}} = \overset{(2)}{x} - \overset{(2)}{y} + \overset{(2)}{u},$$

$$\overset{(1)}{x}(0,\varepsilon) = 1, \ \overset{(2)}{x}(1,\varepsilon) = \overset{(1)}{x}(1,\varepsilon), \ \overset{(1)}{y}(0,\varepsilon) = 1, \ \overset{(2)}{y}(1,\varepsilon) = \overset{(1)}{y}(1,\varepsilon).$$

The results of evaluations, when $\varepsilon = 0.1$, are presented for the solutions of the perturbed and degenerate problems and for the approximations of zero and first orders in Fig. 4-6 for the functions $u(\cdot)$, $x(\cdot)$, $y(\cdot)$ respectively.

Illustrative Example

Fig.4. $u(\cdot)$ Fig.5. $x(\cdot)$ Fig.6. $y(\cdot)$



The solution of the perturbed problem is denoted by blue line, the solution of the degenerate problem is denoted by red line, the zero order approximation is denoted by black line and the first order approximation is denoted by violet line.

Feedback control

The feedback form for optimal control and asymptotics of a solution of a matrix differential Riccati equation are used for constructing asymptotic solution of singularly perturbed linear-quadratic optimal control problems. Applying this method, we avoid solving boundary value problems.

Problem statement

Let us consider the problem P_ε :

$$J_\varepsilon(u) = \frac{1}{2}\left\langle \overset{(2)}{z}(T,\varepsilon), \mathbb{F}(\varepsilon)\overset{(2)}{z}(T,\varepsilon)\right\rangle + \\ +\frac{1}{2}\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}\left\{\left\langle \overset{(j)}{z}(t,\varepsilon), \overset{(j)}{\mathbb{W}}(t,\varepsilon)\overset{(j)}{z}(t,\varepsilon)\right\rangle + \\ +\left\langle \overset{(j)}{u}(t,\varepsilon), \overset{(j)}{\mathbb{R}}(t,\varepsilon)\overset{(j)}{u}(t,\varepsilon)\right\rangle\right\} dt \to \min_u, \tag{80}$$

$$E(\varepsilon)\dot{\overset{(j)}{z}}(t,\varepsilon) = \overset{(j)}{\mathbb{A}}(t,\varepsilon)\overset{(j)}{z}(t,\varepsilon) + \overset{(j)}{\mathbb{B}}(t,\varepsilon)\overset{(j)}{u}(t,\varepsilon), \;\; t\in[t_{j-1},t_j], \;\; j=1,2, \tag{81}$$

$$\overset{(1)}{z}(0,\varepsilon) = z^0, \quad \overset{(1)}{z}(t_1,\varepsilon) = \overset{(2)}{z}(t_1,\varepsilon), \tag{82}$$

Optimal feedback control

We formulate the theorem, which gives the optimal control in a feedback form.

Optimal feedback control

Theorem 10. Let the operator–function $\mathbb{K}(\cdot)$, consisting of $\overset{(j)}{\mathbb{K}}$, be a solution of the problem

$$\begin{gathered}
E(\varepsilon)'\dot{\overset{(j)}{\mathbb{K}}} = -\overset{(j)}{\mathbb{K}}{}'\overset{(j)}{\mathbb{A}} - \overset{(j)}{\mathbb{A}}{}'\overset{(j)}{\mathbb{K}} + \overset{(j)}{\mathbb{K}}{}'\overset{(j)}{\mathbb{S}}\overset{(j)}{\mathbb{K}} - \overset{(j)}{\mathbb{W}}, \quad \overset{(j)}{\mathbb{S}} = \overset{(j)}{\mathbb{B}}\overset{(j)}{\mathbb{R}}{}^{-1}\overset{(j)}{\mathbb{B}}{}', \\
t \in [t_{j-1}, t_j], \;\; j = 1, 2, \\
E(\varepsilon)'\overset{(2)}{\mathbb{K}}(T, \varepsilon) = \mathbb{F}(\varepsilon), \;\; \overset{(1)}{\mathbb{K}}(t_1, \varepsilon) = \overset{(2)}{\mathbb{K}}(t_1, \varepsilon).
\end{gathered} \tag{83}$$

Let also $z_*(\cdot)$, consisting of $\overset{(j)}{z}_*(\cdot, \varepsilon)$, be a solution of the problem

$$\begin{gathered}
E(\varepsilon)\dot{\overset{(j)}{z}}_* = \left(\overset{(j)}{\mathbb{A}} - \overset{(j)}{\mathbb{S}}\overset{(j)}{\mathbb{K}}\right)\overset{(j)}{z}_*, \; t \in [t_{j-1}, t_j], \;\; j = 1, 2, \\
\overset{(1)}{z}_*(0, \varepsilon) = z^0, \;\; \overset{(2)}{z}_*(t_1, \varepsilon) = \overset{(1)}{z}_*(t_1, \varepsilon).
\end{gathered} \tag{84}$$

Optimal feedback control

Then the function $u_*(\cdot,\varepsilon)$, consisting of

$$\overset{(j)}{u}_* = -\overset{(j)}{\mathbb{R}}{}^{-1}\overset{(j)}{\mathbb{B}}{}'\overset{(j)}{\mathbb{K}}\overset{(j)}{z}_*, \quad t \in [t_{j-1}, t_j], \quad j = 1, 2, \tag{85}$$

is an optimal control for the considered problem and the minimal value of the performance index is

$$J_\varepsilon(u_*) = \frac{1}{2}\left\langle z^0, E(\varepsilon)'\overset{(1)}{\mathbb{K}}(0,\varepsilon)z^0 \right\rangle.$$

Optimal feedback control

In this section, it is assumed that one of two following conditions is valid.

1. The pairs $(\overset{(j)}{A}_4(t,0), \overset{(j)}{B}_2(t,0))$, $t \in [t_{j-1}, t_j],\ j = 1, 2$, are controllable.

2. $\overset{(j)}{B}_2(t,0) = 0$ and the operators $\overset{(j)}{A}_4(t,0)$ are stable, $t \in [t_{j-1}, t_j],\ j = 1, 2$.

We will seek asymptotic solutions of the above problems in the series form

$$\begin{aligned} \overset{(j)}{\mathbb{K}}(t,\varepsilon) &= \sum_{k\geq 0} \varepsilon^k \left(\overset{(j)}{\overline{\mathbb{K}}}_k(t) + \overset{(j)}{Q}_k \mathbb{K}(\tau_j) \right), \\ \overset{(j)}{z}(t,\varepsilon) &= \sum_{k\geq 0} \varepsilon^k \left(\overset{(j)}{\overline{z}}_k(t) + \overset{(j)}{\Pi}_k z(\tau_{j-1}) + \overset{(j)}{Q}_k z(\tau_j) \right). \end{aligned} \tag{86}$$

Then the optimal control may be written in the form

$$\overset{(j)}{u}_*(t,\varepsilon) = \sum_{k\geq 0} \varepsilon^k \left(\overset{(j)}{\overline{u}}_k(t) + \overset{(j)}{\Pi}_k u(\tau_{j-1}) + \overset{(j)}{Q}_k u(\tau_j) \right). \tag{87}$$

Estimates of Approximate Solution

Let us assume that $n+1$ terms of the expansions (86)- (87) have been found. Introduce the notations

$$
\begin{aligned}
\overset{(j)}{\widetilde{\mathbb{K}}}_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \left(\overset{(j)}{\overline{\mathbb{K}}}_k(t) + \overset{(j)}{Q}_k \mathbb{K}(\tau_j) \right), \\
\overset{(j)}{\widetilde{z}}_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \left(\overset{(j)}{\overline{z}}_k(t) + \overset{(j)}{\Pi}_k z(\tau_{j-1}) + \overset{(j)}{Q}_k z(\tau_j) \right), \\
\overset{(j)}{\widetilde{u}}_n(t,\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \left(\overset{(j)}{\overline{u}}_k(t) + \overset{(j)}{\Pi}_k u(\tau_{j-1}) + \overset{(j)}{Q}_k u(\tau_j) \right).
\end{aligned} \tag{88}
$$

The following theorem has been proved.

Theorem 11. The order of the values

$$
\overset{(j)}{\mathbb{K}}(t,\varepsilon) - \overset{(j)}{\widetilde{\mathbb{K}}}_n(t,\varepsilon), \quad \overset{(j)}{z}_*(t,\varepsilon) - \overset{(j)}{\widetilde{z}}_n(t,\varepsilon), \quad \overset{(j)}{u}_*(t,\varepsilon) - \overset{(j)}{\widetilde{u}}_n(t,\varepsilon), \tag{89}
$$

is ε^{n+1}.

Estimates of Approximate Solution

Let us denote the solution of the problem

$$\mathbb{E}(\varepsilon)\dot{\overset{(j)}{\widehat{z}_n}} = \left(\overset{(j)}{\mathbb{A}} - \overset{(j)}{\mathbb{S}}\overset{(j)}{\widetilde{\mathbb{K}}_n} \right) \overset{(j)}{\widehat{z}_n}, \qquad (90)$$

$$\overset{(1)}{\widehat{z}_n}(0,\varepsilon) = z^0, \quad \overset{(2)}{\widehat{z}_n}(t_1,\varepsilon) = \overset{(1)}{\widehat{z}_n}(t_1,\varepsilon).$$

by $\overset{(j)}{\widehat{z}_n}$ and introduce the notation

$$\overset{(j)}{\widehat{u}_n} = -\overset{(j)}{\mathbb{R}}{}^{-1}\overset{(j)}{\mathbb{B}}{}'\overset{(j)}{\widetilde{\mathbb{K}}_n}\overset{(j)}{\widehat{z}_n}\ . \qquad (91)$$

Theorem 12. The estimates

$$\|\overset{(j)}{u_*}(t,\varepsilon) - \overset{(j)}{\widehat{u}_n}(t,\varepsilon)\| \le c\varepsilon^{n+1}, \|\overset{(j)}{z_*}(t) - \overset{(j)}{\widehat{z}_n}(t)\| \le c\varepsilon^{n+1},\ j=1,2,$$

$$J_\varepsilon(\widehat{u}_n) - J_\varepsilon(u_*) \le c\varepsilon^{2(n+1)}.$$

are true for all $t \in [0,T]$ and sufficiently small $\varepsilon > 0$.

Illustrative Example

Let us consider the problem P_ε of minimizing the functional

$$
\begin{aligned}
J_\varepsilon(u) = {} & \frac{1}{2}\left((\overset{(2)}{x}(2,\varepsilon))^2 + \varepsilon(\overset{(2)}{y}(2,\varepsilon))^2 \right) + \\
& + \frac{1}{2}\int_0^1 \left((\overset{(1)}{x})^2 + 3(\overset{(1)}{y})^2 + (\overset{(1)}{u})^2 \right) dt + \\
& + \frac{1}{2}\int_1^2 \left(4(\overset{(2)}{x})^2 + 8\overset{(2)}{x}\overset{(2)}{y} + 8(\overset{(2)}{y})^2 + (\overset{(2)}{u})^2 \right) dt
\end{aligned}
\qquad (92)
$$

on trajectories of the system

$$\dot{\overset{(1)}{x}} = \overset{(1)}{x}, \quad \varepsilon \dot{\overset{(1)}{y}} = -\overset{(1)}{y} + \overset{(1)}{u}, \; t \in [0,1],$$

$$\dot{\overset{(2)}{x}} = 0, \; \varepsilon \dot{\overset{(2)}{y}} = \overset{(2)}{x} - \overset{(2)}{y} - \overset{(2)}{u}, \; t \in [1,2], \tag{93}$$

$$\overset{(1)}{x}(0,\varepsilon) = 1, \; \overset{(1)}{y}(0,\varepsilon) = 1, \overset{(2)}{x}(1,\varepsilon) = \overset{(1)}{x}(1,\varepsilon), \; \overset{(2)}{y}(1,\varepsilon) = \overset{(1)}{y}(1,\varepsilon),$$

when $\varepsilon = 0,15$.

Illustrative Example

Fig.1. $K_1(t,\varepsilon)$

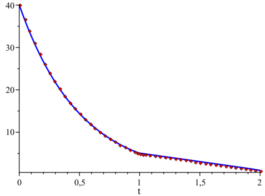


Fig.2. $K_3(t,\varepsilon)$

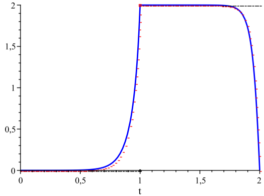


Fig.3. $K_4(t,\varepsilon)$

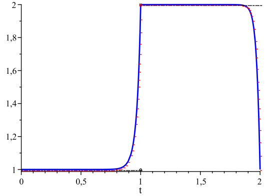
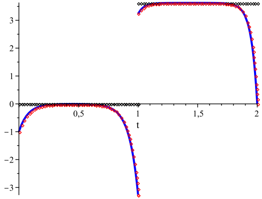


Fig.4. $u(t,\varepsilon)$



The exact solution is denoted by blue continuous line, the zero approximation for the solution is denoted by red circles, and the first approximation is denoted by diamonds.

Illustrative Example

Fig.5. $x(t,\varepsilon)$

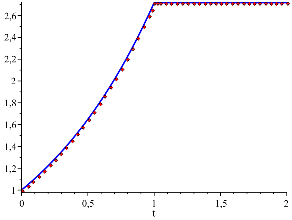
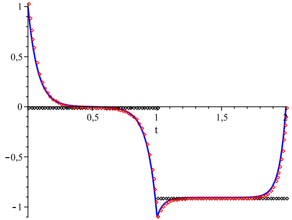


Fig.6. $y(t,\varepsilon)$



The exact solution is denoted by blue continuous line, the zero approximation for the solution is denoted by red circles, and the first approximation is denoted by diamonds.

Inverse problem for variational calculus for systems with deviating argument

The Euler equation for a functional

$$J(y(\cdot)) = \int_a^b F(x, y(x), y(x-\theta), y'(x))dx \tag{94}$$

with conditions

$$y(x) = \varphi(x), \quad x \in [a-\theta, a], \quad y(b) = y_b \tag{95}$$

has the form

$$\begin{aligned} Ay''(x) + B + Cy'(x-\theta) + \widetilde{D} = 0, \quad x \in [a, b-\theta], \\ Ay''(x) + B + Cy'(x-\theta) = 0, \quad x \in (b-\theta, b], \end{aligned} \tag{96}$$

where twice continuously differentiable functions A, B, C, D depend on $x, y(x), y(x-\theta), y'(x)$ and

$$\widetilde{D} = D(x+\theta, y(x+\theta), y(x), y'(x+\theta)). \tag{97}$$

Inverse problem for variational calculus for systems with deviating argument

The inverse problem for variational calculus demands to find a functional of the form (94), the Euler equation for which coincides with the given equation of the form (96) for any twice continuously differentiable function $y(\cdot)$, satisfying (95).

Inverse problem for variational calculus for systems with deviating argument

Theorem 13. The inverse problem for variational calculus has a solution if and only if the functions A, B, C, D from (96) satisfy the relations

$$
\begin{gathered}
B_{y'(x)} - A_x - y'(x)A_{y(x)} = 0, \quad x \in [a,b], \\
C_{y'(x)} - A_{y(x-\theta)} = 0, \quad x \in [a,b], \\
D_{y'(x)} + C = 0, \quad x \in [a+\theta,b], \\
D_{y(x)} + \int\limits_0^{y'(x)} C_{y(x)}\,dy'(x) - G_{y(x)y(x-\theta)} = 0, \quad x \in [a+\theta,b],
\end{gathered} \tag{98}
$$

where

Inverse problem for variational calculus for systems with deviating argument

$$G = G(x, y(x), y(x-\theta)) = \int_0^{y(x)} \Big(B - \int_0^{y'(x)} B_{y'(x)} dy'(x) + E_x\Big) dy(x) + Q, \tag{99}$$

$$Q = \begin{cases} 0, x \in (b-\theta, b], \\ \int_0^{y(x)} \Big(\widetilde{D} + \int_0^{y'(x+\theta)} \widetilde{C} dy'(x+\theta) - \widetilde{G}_{y(x)}\Big) dy(x), x \in [a, b-\theta], \end{cases} \tag{100}$$

$$E = E(x, y(x), y(x-\theta)) = \int_0^{y(x-\theta)} \Big(\int_0^{y'(x)} A_{y(x-\theta)} dy'(x) - C\Big) dy(x-\theta), x \in [a, b], \tag{101}$$

the function Q when $x \in [a, b-\theta]$ is successively found on the intervals $(b-k\theta, b-(k-1)\theta], k = 2, \ldots : b-k\theta \geq a$.

Inverse problem for variational calculus for systems with deviating argument

The function F, defining a solution of an inverse problem for variational calculus, may be determined by the formula

$$F = -\int_{0}^{y'(x)} \left(\int_{0}^{y'(x)} A\,dy'(x) \right) dy'(x) + Ey'(x) + G, \quad x \in [a,b]. \qquad (102)$$

Thank you for your attention!