Large-Scale Differential-Algebraic Equations Arising in VLSI Interconnect Analysis

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Outline

- Numerical simulation of electronic circuits
- DAEs arising in VLSI interconnect analysis
- Structure-preserving model order reduction
- Thick restarts and multiple expansion points
- Open problems

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VLSI circuit simulation

- Circuit simulation uses computational methods to simulate and analyze the behavior of electronic circuits

- A circuit can be viewed as a **network** of electronic **devices**: transistors, resistors, capacitors, inductors, . . .

- Today's VLSI circuits can have $\mathcal{O}(10^9)$ transistors

Are we really just solving DAEs?

- Network topology is described by a graph:
  - Kirchhoff's current laws
  - Kirchhoff's voltage laws

- Equations that characterize the circuit devices:

$$f(i, v) = 0, \quad g\Big(i, \frac{d}{dt}v\Big) = 0, \quad \ldots$$

- All these equations can be summarized as a system of DAEs:

$$\mathbf{F}\Big(\mathbf{x}, \frac{d}{dt}\mathbf{x}, t\Big) = \mathbf{0}$$

The catch

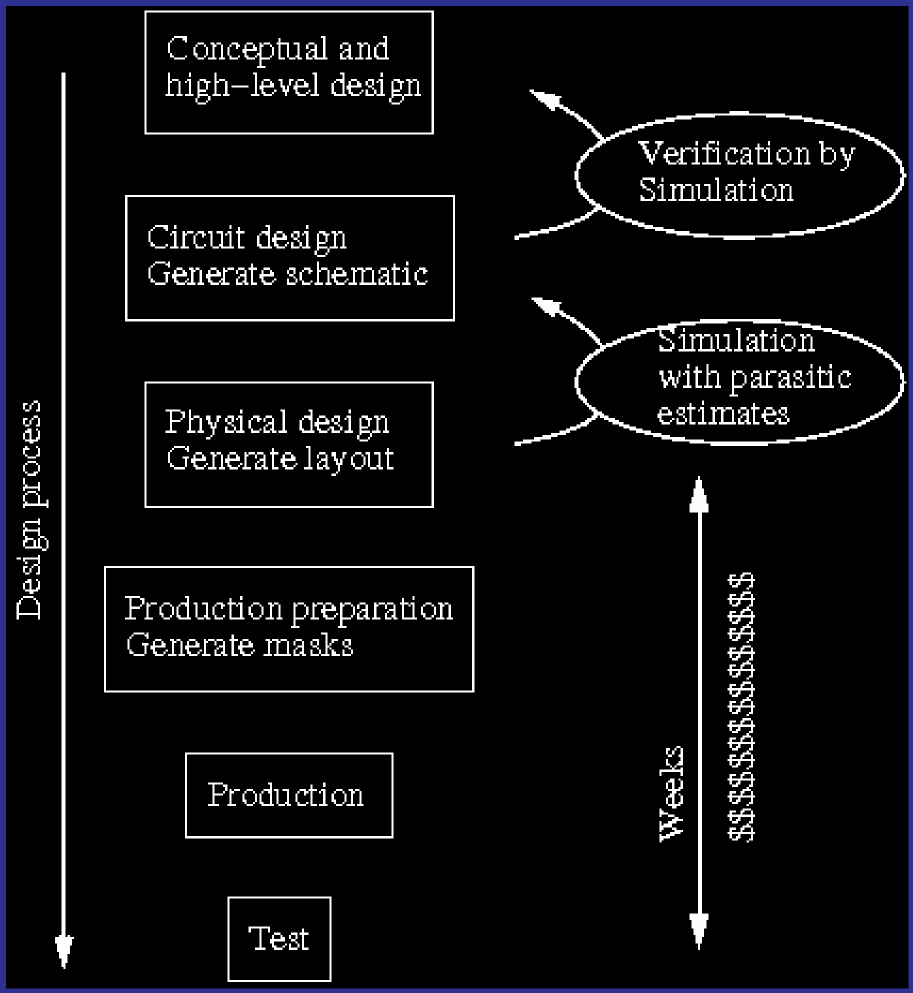
- $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}^N$ where N is of the order of the number of devices in the circuit

- For a state-of-the-art circuit: $N = \mathcal{O}(10^9)$

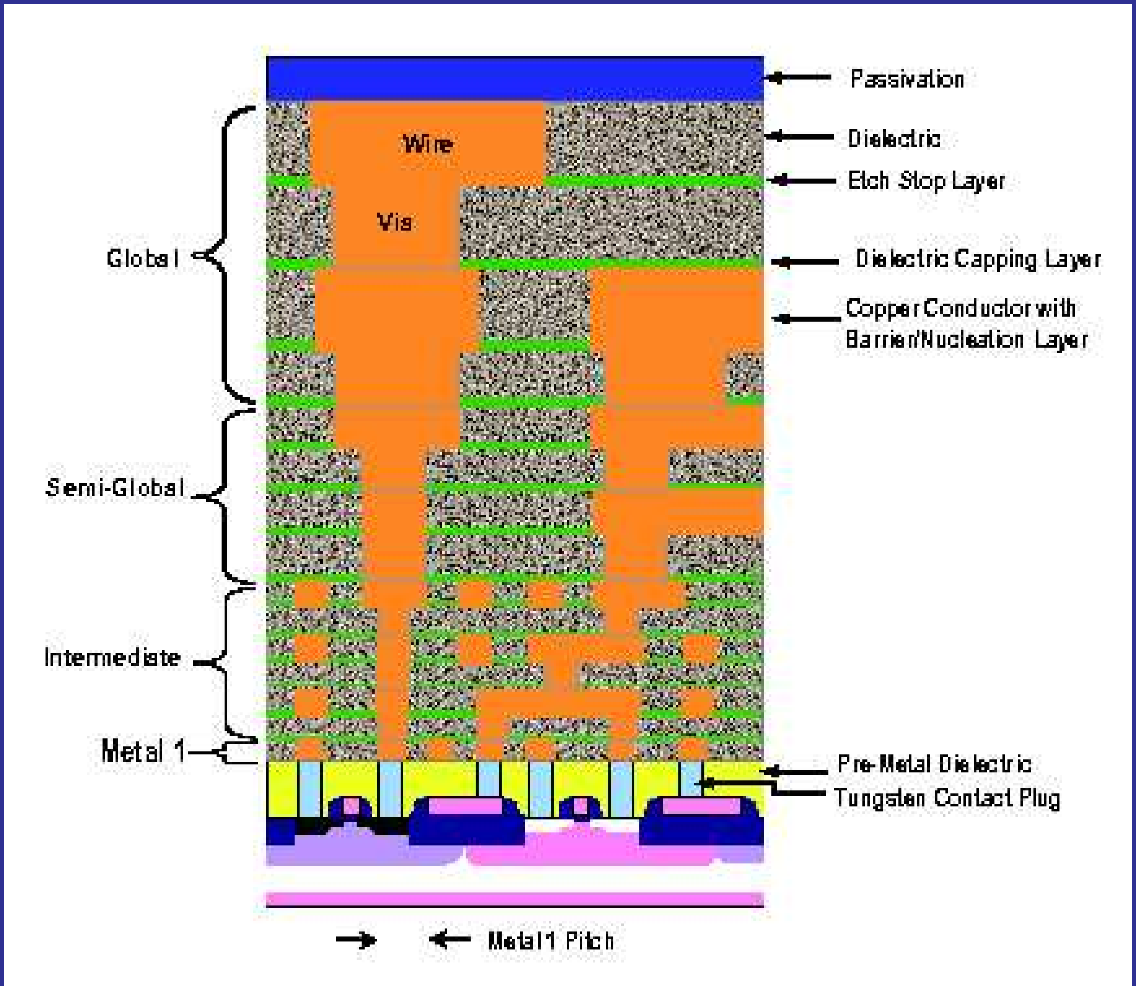
- No way!

- We are always using today's computers to design tomorrow's largest and more complicated machines

The VLSI circuit design process



A small piece of a chip cross section



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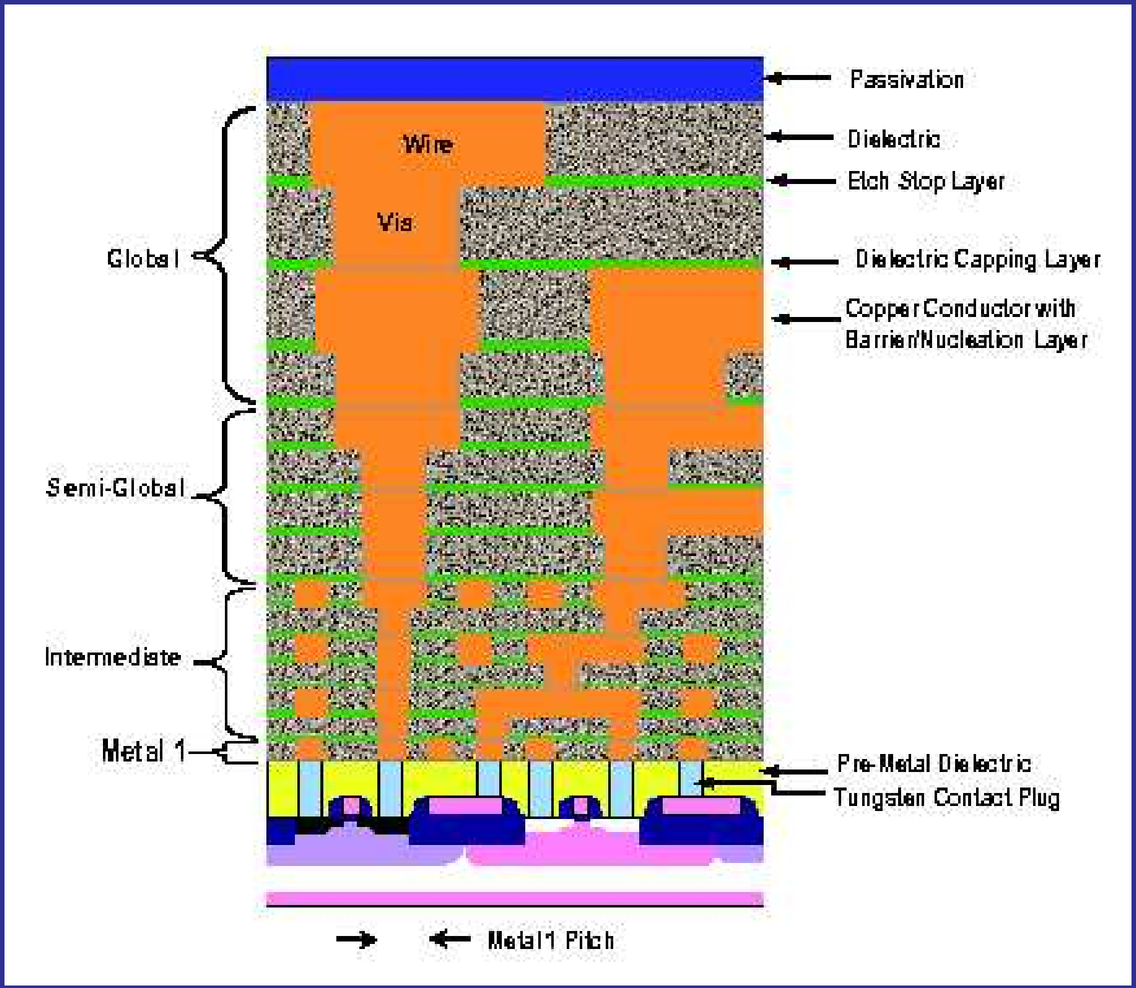
State-of-the-art VLSI circuits

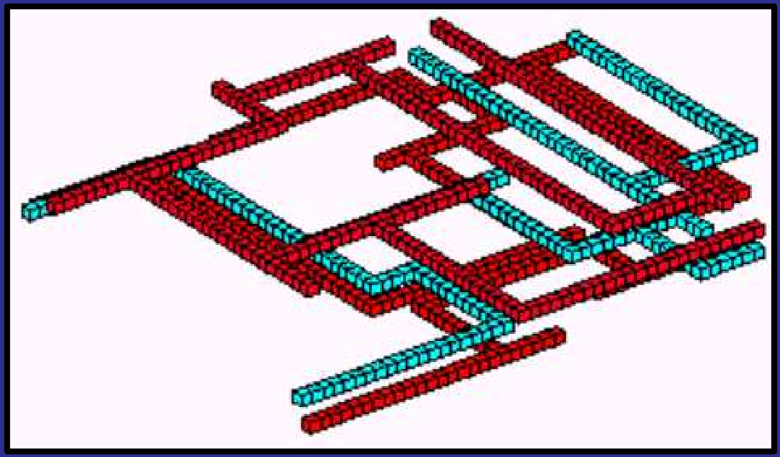
- 45 nm feature size
- $\mathcal{O}(10^9)$ transistors
- $\mathcal{O}(10)$ km of 'wires' (the *interconnect*)
- Up to 15 layers



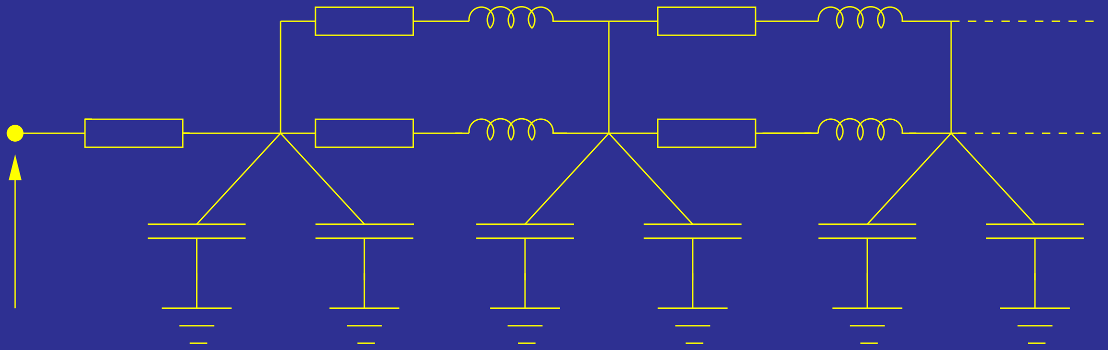
VLSI interconnect parasitics

- Wires are not ideal:
    - Resistance
    - Capacitance
    - Inductance
- Consequences:
    - Timing behavior
    - Noise
    - Energy consumption
    - Power distribution

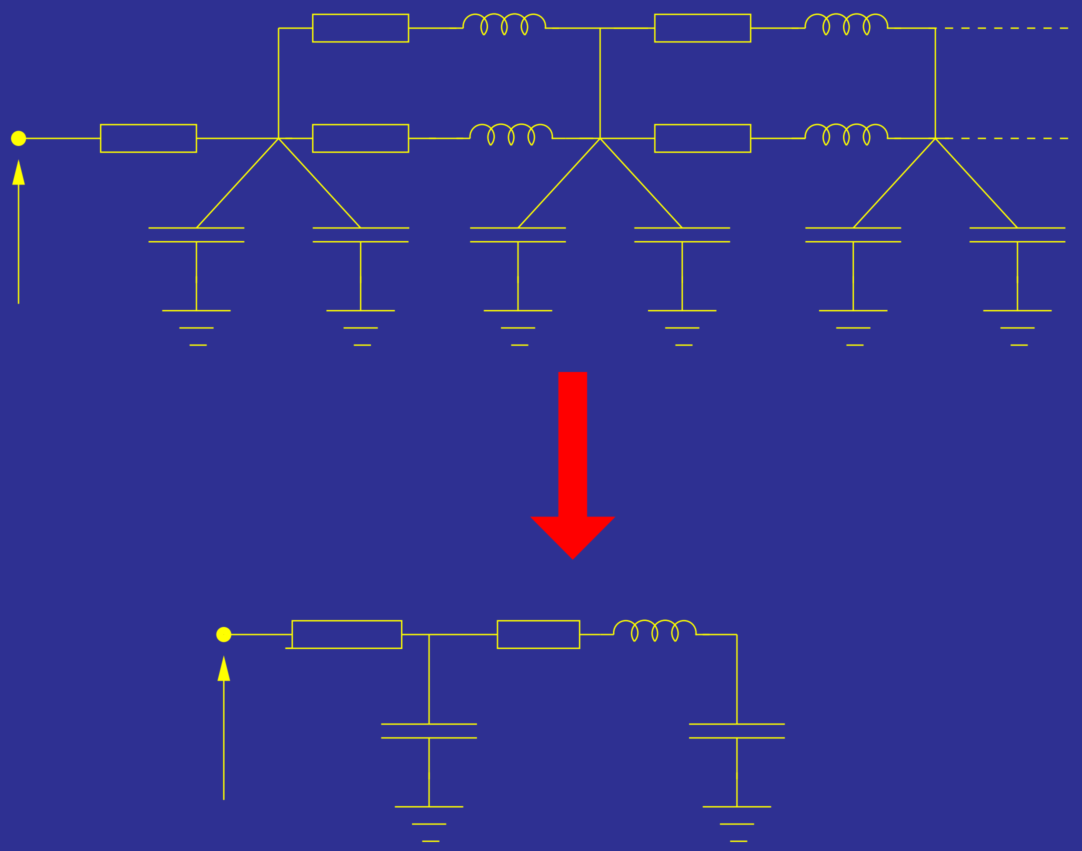
Interconnect parasitics extraction



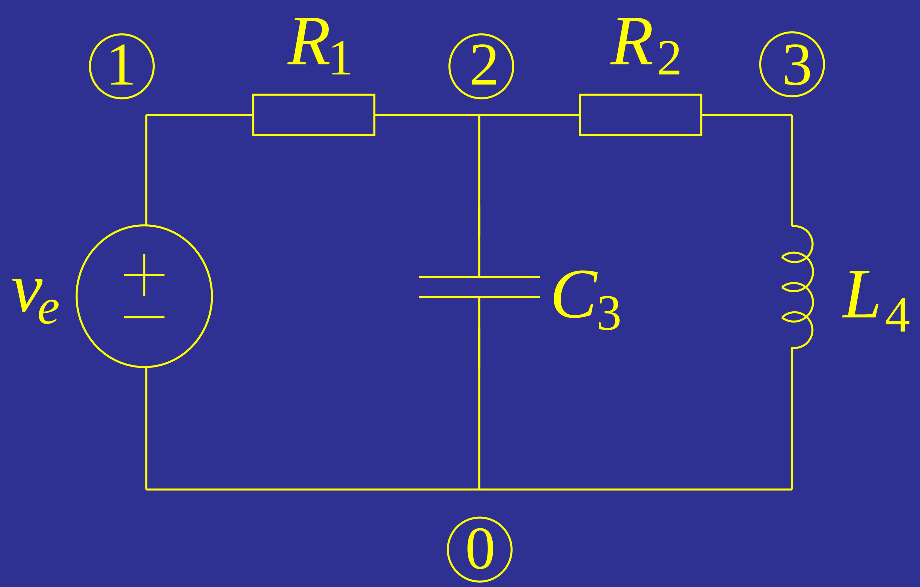
- Replace 'pieces' of the interconnect by RCL networks:



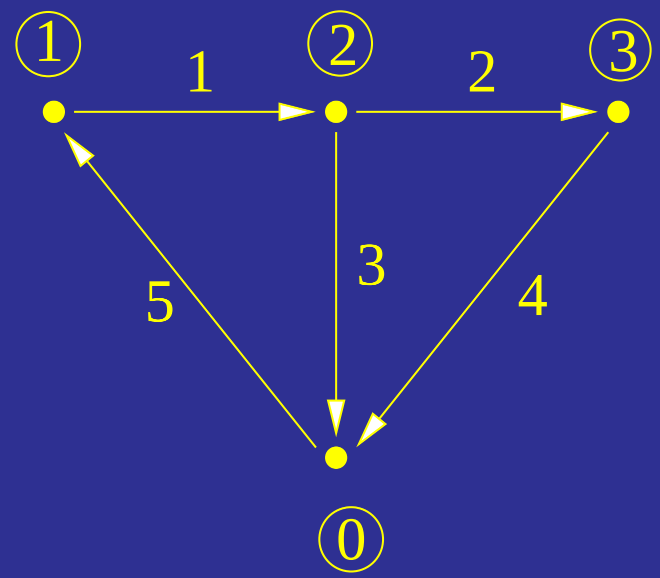
Need for model order reduction



RCL networks as directed graphs



RCL network

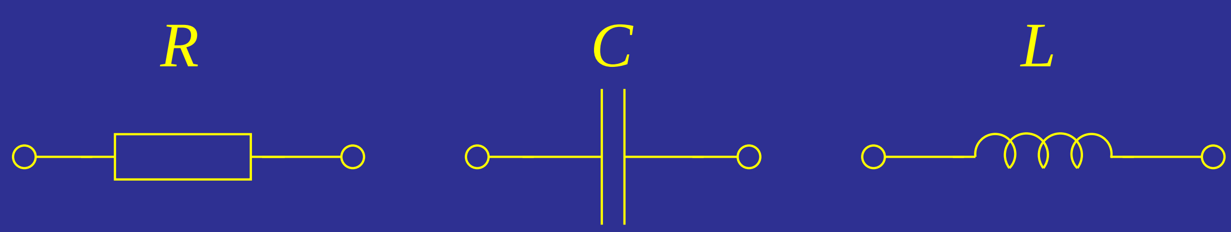


directed graph

- Network topology $\Longleftrightarrow$ Graph incidence matrix $\mathcal{A}$

RCL network equations

- Kirchhoff's current laws: $\mathcal{A}\, i_{\mathcal{E}} = 0$
- Kirchhoff's voltage laws: $\mathcal{A}^{\mathsf{T}} v = v_{\mathcal{E}}$
- Equations for R's, C's, and L's:



$$v_R = R\, i_R \qquad i_C = C\,\frac{d}{d\,t}\,v_C \qquad v_L = L\,\frac{d}{d\,t}\,i_L$$

RCL networks as descriptor systems

- System of linear time-invariant DAEs of the form

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\mathsf{T}}\mathbf{x}(t)$$

  where $\mathbf{C},\ \mathbf{G} \in \mathbb{R}^{N \times N}$ and $\mathbf{B} \in \mathbb{R}^{N \times m}$

- $\mathbf{x}(t) \in \mathbb{R}^{N}$ is the unknown vector of state variables

- Large state-space dimension N

- m inputs, m outputs

Reduced-order models

- System of DAEs of the same form:

$$\mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n\,\mathbf{z}(t) = \mathbf{B}_n\,\mathbf{u}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{B}_n^{\top}\,\mathbf{z}(t)$$

- But now:

$$\mathbf{C}_n,\ \mathbf{G}_n \in \mathbb{R}^{n\times n} \quad \text{and} \quad \mathbf{B}_n \in \mathbb{R}^{n\times m}$$

where $n \ll N$

Transfer functions

- Original descriptor system:

$$\mathbf{H}(s) = \mathbf{B}^{\top} (s\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

- Reduced-order model:

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\top} (s\,\mathbf{C}_n + \mathbf{G}_n)^{-1}\,\mathbf{B}_n$$

- 'Good' reduced-order model

$\Longleftrightarrow$ 'Good' approximation $\mathbf{H}_n \approx \mathbf{H}$

- Original dimension $N \approx 10^{4-6}$

$$\mathbf{H}(s) = \mathbf{B}^{\top} \left(s\,\mathbf{C} + \mathbf{G} \right)^{-1} \mathbf{B}$$

- Reduced dimension $n \ll N$ $(n \approx 10^{0-2})$

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\top} \left(s\,\mathbf{C}_n + \mathbf{G}_n \right)^{-1} \mathbf{B}_n$$

Moment matching

- Choose a suitable expansion point $s_0 \in \mathbb{C}$ and expand $\mathbf{H}(s)$ about that point

$$\mathbf{H}(s) = \mathrm{M}_0 + \mathrm{M}_1 (s - s_0) + \cdots + \mathrm{M}_i (s - s_0)^i + \cdots$$

- Determine reduced-order transfer function $\mathbf{H}_n(s)$ such that

$$\begin{aligned}\mathbf{H}_n(s) &= \mathrm{M}_0 + \mathrm{M}_1 (s - s_0) + \cdots + \mathrm{M}_{q-1} (s - s_0)^{q-1} \\ &\qquad + \widetilde{\mathbf{M}}_{\mathbf{q}} (\mathbf{s} - \mathbf{s}_0)^{\mathbf{q}} + \widetilde{\mathbf{M}}_{\mathbf{q+1}} (\mathbf{s} - \mathbf{s}_0)^{\mathbf{q+1}} + \cdots \\ &= \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^q\right)\end{aligned}$$

for some $q = q(n)$

Padé and Padé-type approximation

- **Padé approximation**: $\mathbf{C}_n,\ \mathbf{G}_n \in \mathbb{R}^{n\times n},\ \mathbf{B}_n \in \mathbb{R}^{n\times m}$ such that

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s-s_0)^{q(n)}\right)$$

  and $q(n)$ is maximal

- $q(n) \geq 2\left\lfloor \frac{n}{m} \right\rfloor$ with equality in the 'generic' case

- **Padé-type approximation**:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s-s_0)^{\tilde{q}(n)}\right)$$

  where $\tilde{q}(n)$ is no longer maximal, e.g., $\tilde{q}(n) = \left\lfloor \frac{n}{m} \right\rfloor$

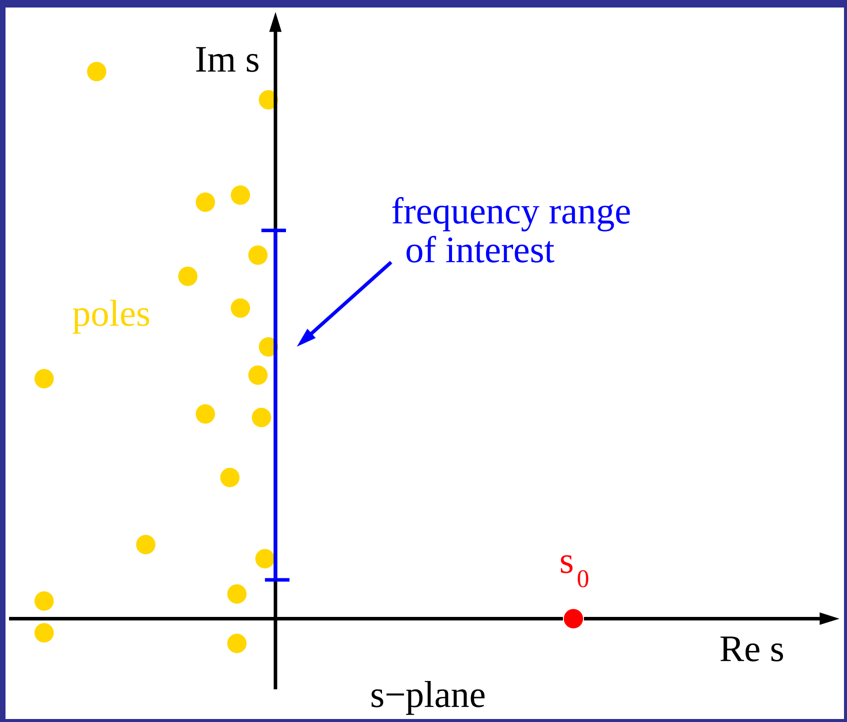
Some history

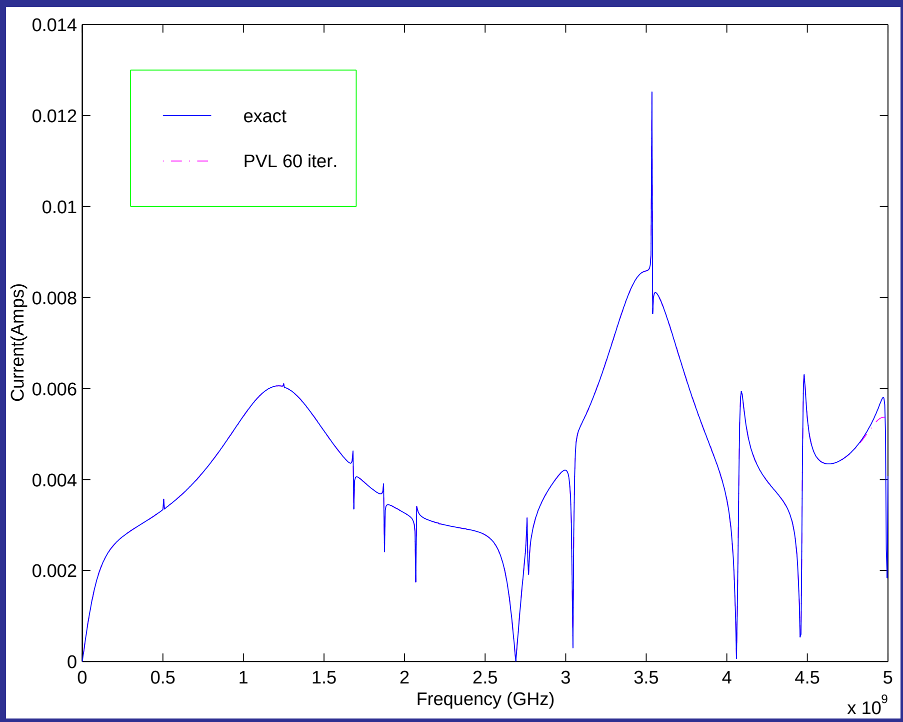
- **AWE** (Pillage and Rohrer, ‘90):
  Explicit computation and matching of moments

- **PVL, MPVL** (Feldmann and F., ‘94 and ‘95):
  Avoids numerical issues of AWE by computing Padé reduced-order models via the Lanczos process

- **Arnoldi-based reduction** (Silveira et al, ‘96):
  Padé-type reduced-order models via the Arnoldi process

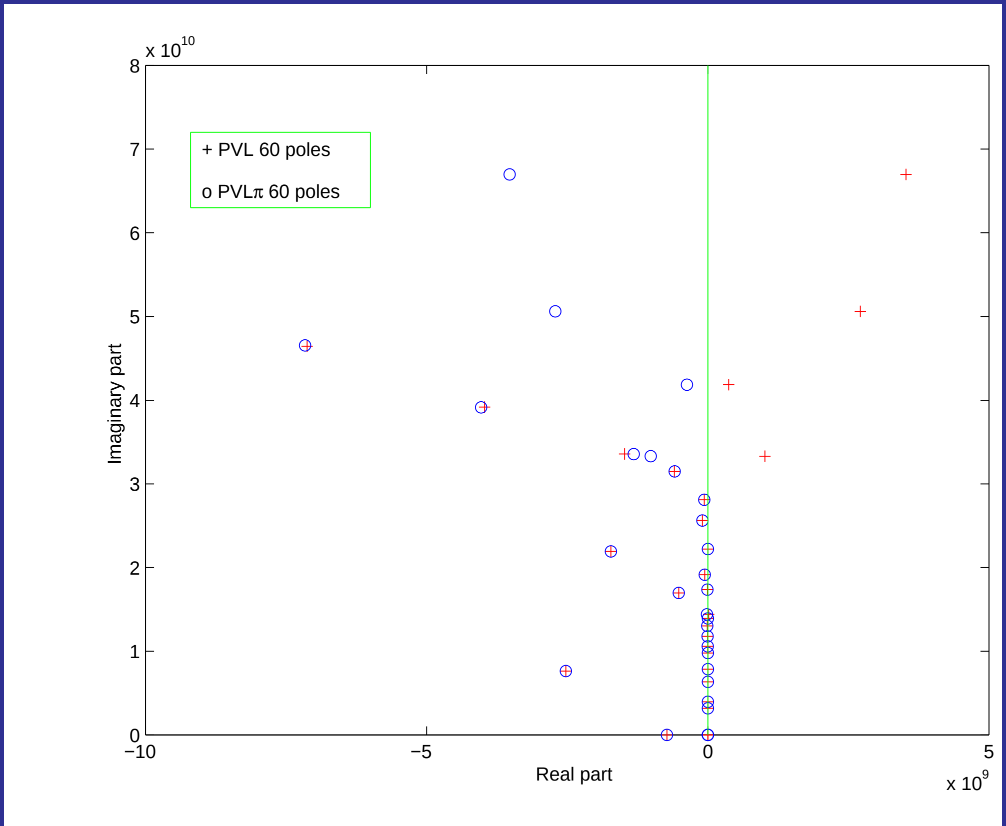
An RCL network





Exact and Padé model

Padé may produce unstable poles



Some more history

- **PRIMA** (Odabasioglu, Celik, and Pileggi, '97):
  Passive reduced-order models via explicit projection onto Krylov subspaces

- **SPRIM** (F., '04, '09, and '11)
  Structure-Preserving Reduced Interconnect Macromodeling

Projection-based order reduction

- Choose an $N \times n$ matrix

$\mathbf{V}_n =$  with $\text{Rank}\,\mathbf{V}_n = n$

and explicitly project the data matrices of

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\top}\mathbf{x}(t)$$

onto the subspace spanned by the columns of $\mathbf{V}_n$

Projection-based order reduction

- Resulting reduced-order model:

$$\mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n \mathbf{z}(t) = \mathbf{B}_n \mathbf{u}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{B}_n^\top \mathbf{z}(t)$$

where

$$\mathbf{C}_n := \mathbf{V}_n^\top \mathbf{C} \mathbf{V}_n, \quad \mathbf{G}_n := \mathbf{V}_n^\top \mathbf{G} \mathbf{V}_n, \quad \mathbf{B}_n := \mathbf{V}_n^\top \mathbf{B}$$

- Preserves **passivity**:

$$\mathbf{C} \succeq \mathbf{0}, \quad \mathbf{G} + \mathbf{G}^\top \succeq \mathbf{0} \quad \Rightarrow \quad \mathbf{C}_n \succeq \mathbf{0}, \quad \mathbf{G}_n + \mathbf{G}_n^\top \succeq \mathbf{0}$$

Choice of projection matrix

- Choose expansion point $s_0 \in \mathbb{C}$ for transfer function and rewrite:

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{T}} (s\,\mathbf{C} + \mathbf{G})^{-1} \mathbf{B} = \mathbf{B}^{\mathsf{T}} \Big(\mathbf{I} - (s - s_0)\,\mathbf{A} \Big)^{-1} \mathbf{R}$$

  where

$$\mathbf{A} := -\,(s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C} \quad \text{and} \quad \mathbf{R} := (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

- $\widehat{n}$-th **block Krylov subspace**:

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) := \text{colspan}_{\widehat{n}} \Big[\, \mathbf{R} \quad \mathbf{A}\mathbf{R} \quad \mathbf{A}^2\mathbf{R} \quad \cdots \,\Big]$$

Krylov + Projection = Padé-type

- $\widehat{n}$-th block Krylov subspace:

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) := \mathsf{colspan}_{\widehat{n}} \left[\mathbf{R} \quad \mathbf{AR} \quad \mathbf{A}^2\mathbf{R} \quad \cdots \right]$$

- Choose the projection matrix $\mathbf{V}_n$ such that

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) \subseteq \mathsf{Range}\, \mathbf{V}_n$$

- Krylov subspace + Projection = **Padé-type approximant**:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq \lfloor \widehat{n}/m \rfloor$$

- PRIMA and SPRIM are methods of this type

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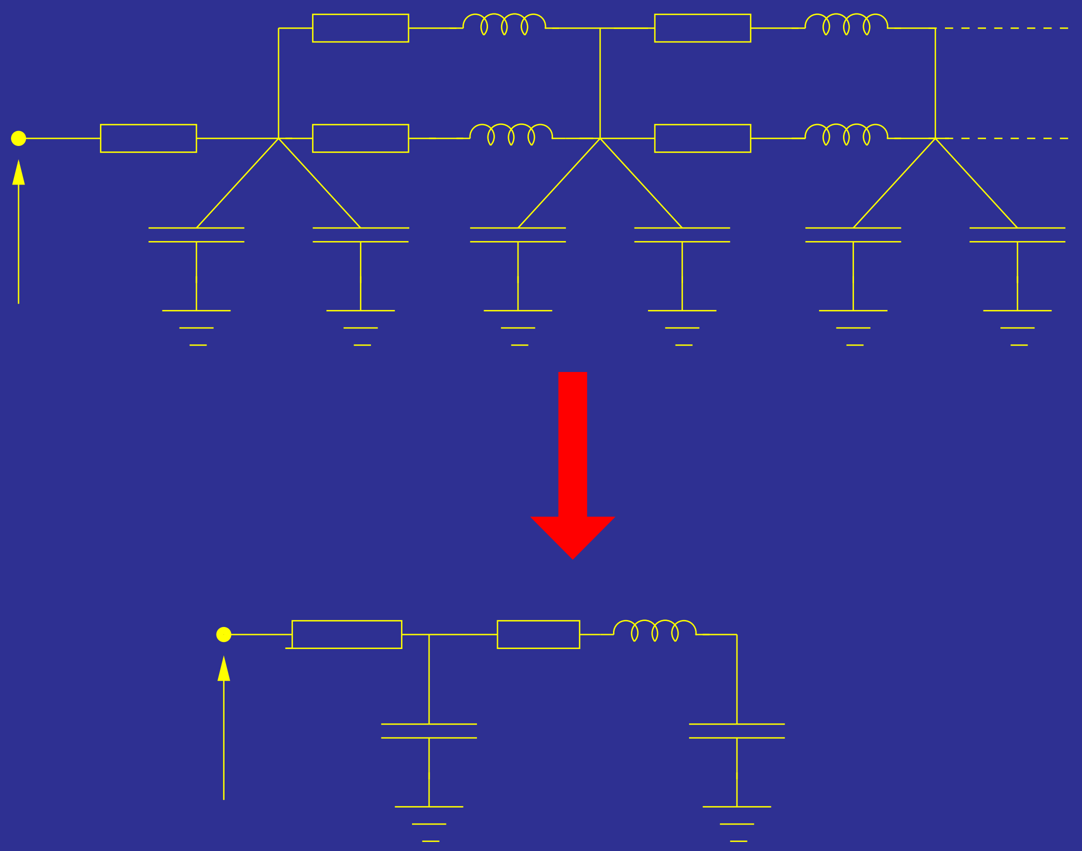
Preservation of RCL structure



General RCL network equations

- System of linear time-invariant DAEs of the form

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\top}\mathbf{x}(t)$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \\ -\mathbf{G}_2^{\top} & \mathbf{0} & \mathbf{0} \\ -\mathbf{G}_3^{\top} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$$

- Passivity:

$$\mathbf{C} \succeq \mathbf{0} \quad \text{and} \quad \mathbf{G} + \mathbf{G}^{\top} \succeq \mathbf{0}$$

PRIMA does not preserve structure

- PRIMA = projection onto n-th block Krylov subspace:

$$\text{Range}\,\mathbf{V}_n = \mathcal{K}_n(\mathbf{A}, \mathbf{R})$$

- Block structure of the data matrices:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & 0 & 0 \\ 0 & \mathbf{C}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \\ -\mathbf{G}_2^\mathsf{T} & 0 & 0 \\ -\mathbf{G}_3^\mathsf{T} & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & 0 \\ 0 & 0 \\ 0 & \mathbf{B}_2 \end{bmatrix}$$

- PRIMA reduced-order matrices:

$$\mathbf{C}_n = \begin{bmatrix} \blacksquare \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \blacksquare \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \blacksquare \end{bmatrix}$$

SPRIM does preserve block structure

- Structure of SPRIM reduced-order matrices:

$$\mathbf{C}_n = \begin{bmatrix} \tilde{\mathbf{C}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{C}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \tilde{\mathbf{G}}_1 & \tilde{\mathbf{G}}_2 & \tilde{\mathbf{G}}_3 \\ -\tilde{\mathbf{G}}_2^T & \mathbf{0} & \mathbf{0} \\ -\tilde{\mathbf{G}}_3^T & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \tilde{\mathbf{B}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{B}}_2 \end{bmatrix}$$

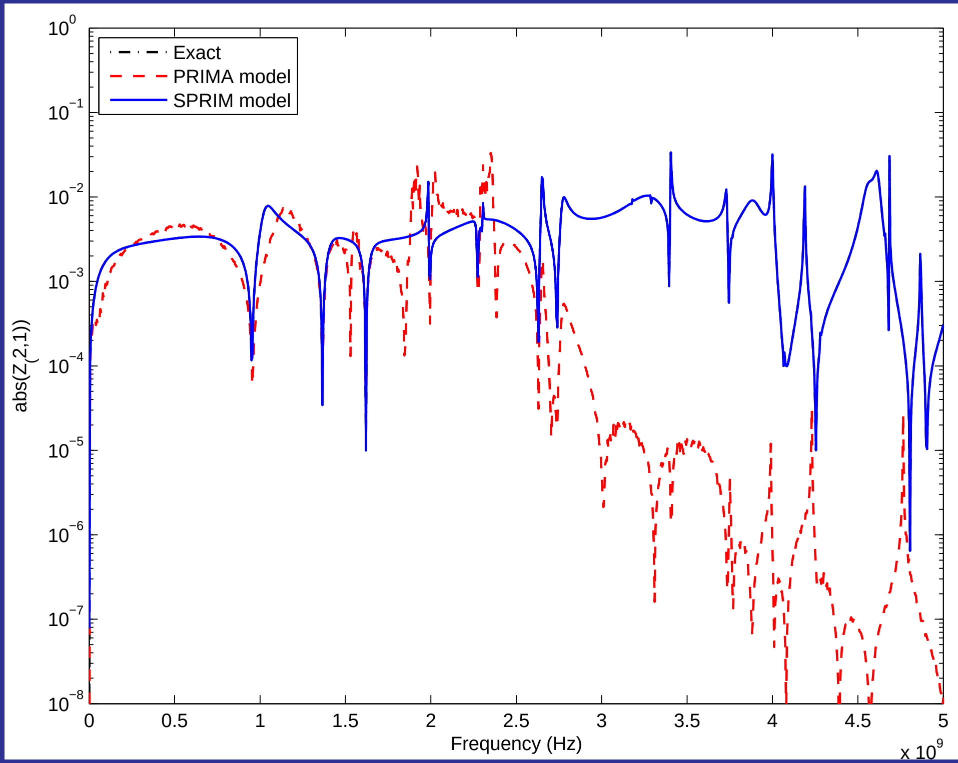
- Projection onto Krylov subspaces guarantees a Padé-type property:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right)$$

  with $\tilde{q}$ the same integer as for PRIMA

- For SPRIM, we actually observe higher accuracy

An RCL network with mostly C's and L's



Exact and models corresponding to
block Krylov subspace of dimension $\widehat{n} = 120$

SPRIM

- Let $\widehat{\mathbf{V}}_{\widehat{n}}$ be any matrix such that

$$\text{Range}\,\widehat{\mathbf{V}}_{\widehat{n}} = \mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R})$$

- Recall:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 & \mathbf{G}_3 \\ -\mathbf{G}_2^\top & \mathbf{0} & \mathbf{0} \\ -\mathbf{G}_3^\top & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$$

SPRIM, continued

- Partition $\hat{\mathbf{V}}_{\widehat{n}}$ accordingly:

$$\hat{\mathbf{V}}_{\widehat{n}} = \begin{bmatrix} \mathbf{V}_{\widehat{n}}^{(1)} \\ \mathbf{V}_{\widehat{n}}^{(2)} \\ \mathbf{V}_{\widehat{n}}^{(3)} \end{bmatrix}$$

- For $l = 1, 2, 3$:
  If $\text{Rank}\, \mathbf{V}_{\widehat{n}}^{(i)} < \widehat{n}$, replace $\mathbf{V}_{\widehat{n}}^{(i)}$ by matrix of full column rank

SPRIM, continued

- Set

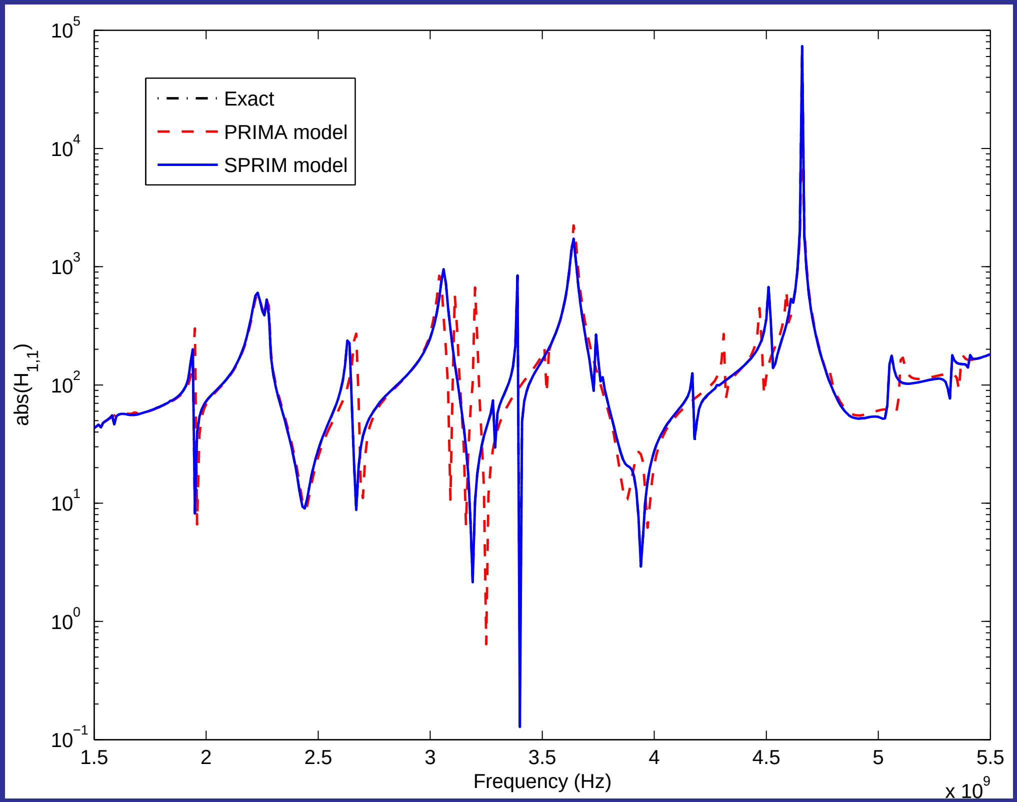
$$\mathbf{V}_n = \begin{bmatrix} \mathbf{V}_n^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_n^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_n^{(3)} \end{bmatrix}$$

- Block structure is preserved:

$$\mathbf{C}_n = \begin{bmatrix} \tilde{\mathbf{C}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{C}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \tilde{\mathbf{G}}_1 & \tilde{\mathbf{G}}_2 & \tilde{\mathbf{G}}_3 \\ -\tilde{\mathbf{G}}_2^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ -\tilde{\mathbf{G}}_3^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \tilde{\mathbf{B}}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{B}}_2 \end{bmatrix}$$

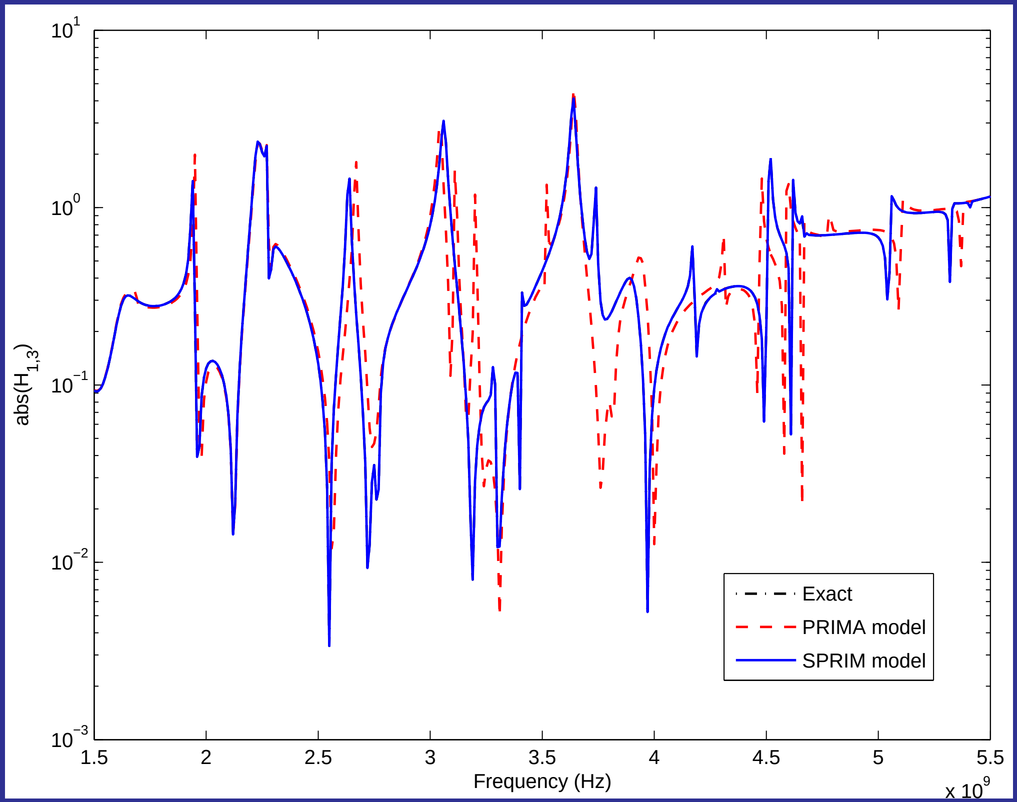
- $\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) = \text{Range}\,\mathbf{V}_{\widehat{n}} \subseteq \text{Range}\,\mathbf{V}_n$ $\Rightarrow$ Padé-type property!

An RCL network with mostly C's and L's



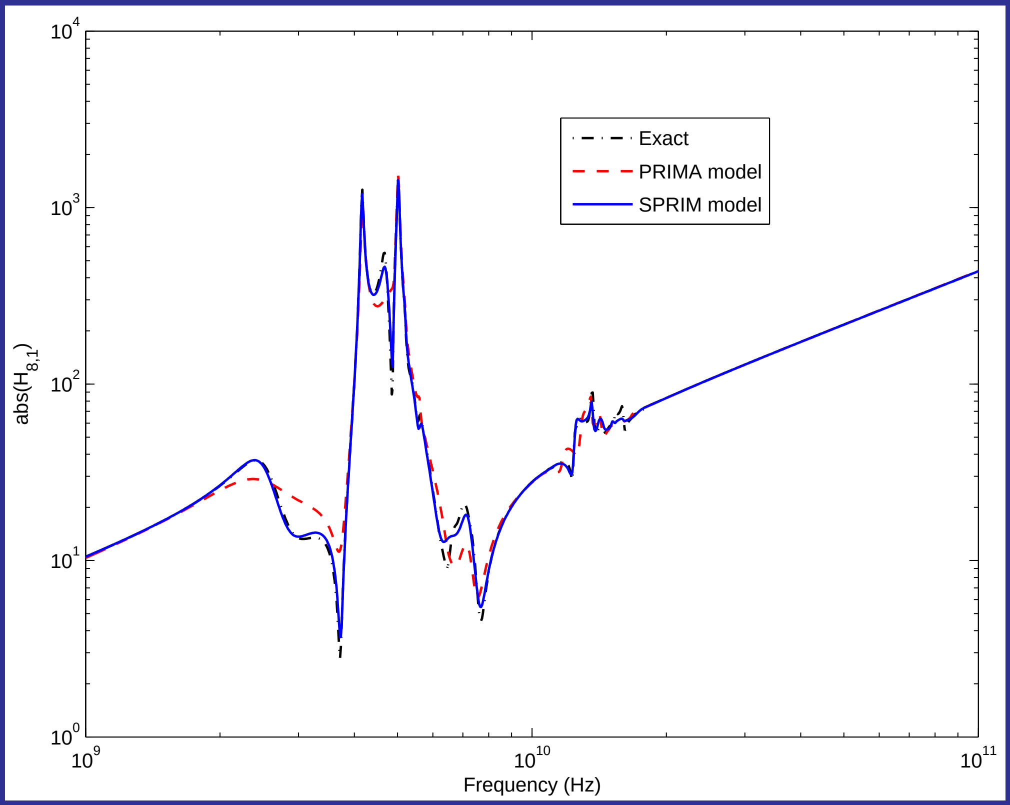
Exact and models corresponding to $\hat{n} = 90$

An RCL network with mostly C's and L's



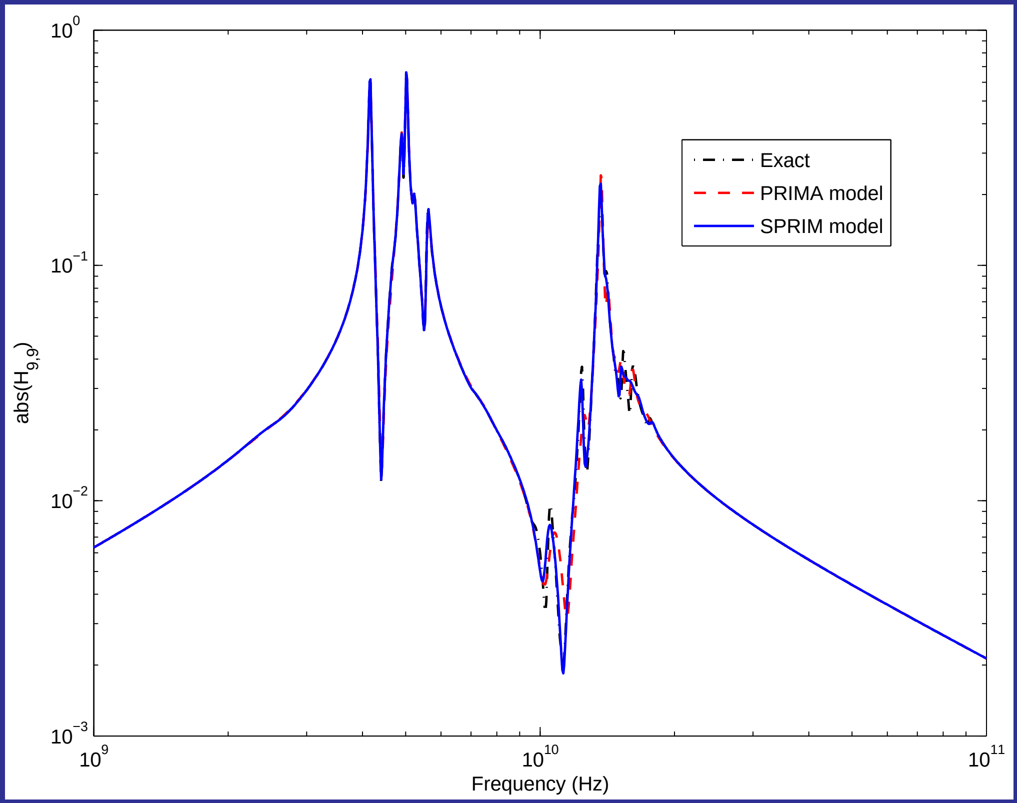
Exact and models corresponding to $\hat{n} = 90$

A package example



Exact and models corresponding to $\widehat{n} = 128$

A package example



Exact and models corresponding to $\hat{n} = 128$

Padé-type property of SPRIM

- General theory of projection onto block Krylov subspaces: PRIMA and SPRIM produce Padé-type models with

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq \lfloor \hat{n}/m \rfloor$$

- **Theorem** (F., '08)

  The n-th SPRIM model satisfies

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq 2\lfloor \hat{n}/m \rfloor$$

- Twice as accurate as PRIMA!

- This is a consequence of structure preservation!

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- Structure-preserving model order reduction
- *Thick restarts and multiple expansion points*
  (with Efrem B. Rensi)
- Open problems

Need for restarts

- To obtain a Padé-type property, projection matrix $\mathbf{V}_n$ with

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) \subseteq \text{Range}\, \mathbf{V}_n$$

- Need to first generate $\widehat{\mathbf{V}}_{\widehat{n}}$ such that

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) = \text{Range}\, \widehat{\mathbf{V}}_{\widehat{n}}$$

- Use suitable (band) variant of the Arnoldi process
- But: prohibitive for large $\widehat{n}$
- Remedy: (thick) restarts

Using restarts

- Motivated by recent work by Eiermann et al.
- Restart after each cycle of r Arnoldi steps
- Extract ‘good’ eigenvector information Y from the last batch of r Arnoldi vectors
- Use the columns of Y as the first vectors in the next cycle
- Repeat
- Project with

$$\mathbf{V}_n = \left[\mathbf{V}^{(1)} \quad \mathbf{V}^{(2)} \quad \ldots \quad \mathbf{V}^{(l)} \right]$$

‘Good’ eigenvector information

- Recall:

$$\mathbf{H}(s) = \mathbf{B}^{\top}\Big(\mathbf{I} - (s - s_0)\,\mathbf{A}\Big)^{-1}\mathbf{R}$$

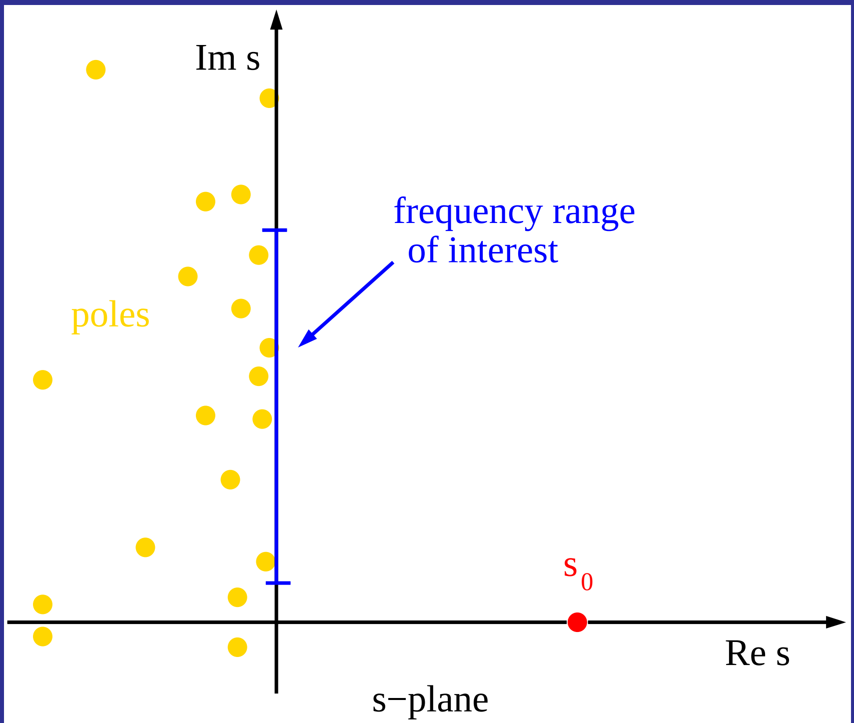
- Poles of $\mathbf{H}$ are of the form

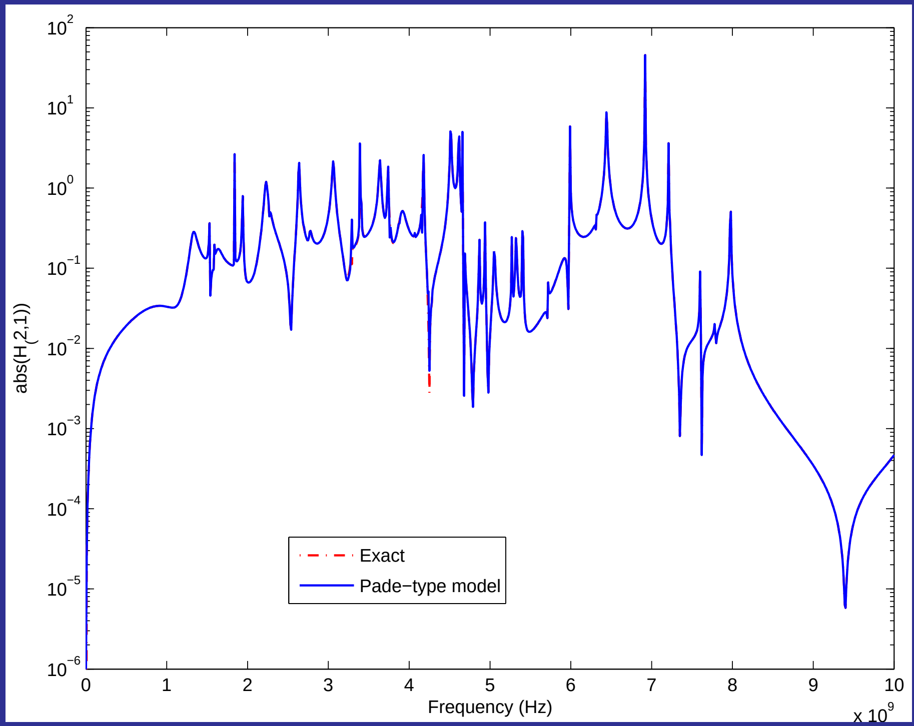
$$s = s_0 + \frac{1}{\lambda}, \quad \lambda \in \sigma(\mathbf{A})$$

- ‘Good’ eigenvector information:

  Good approximate eigenvectors corresponding to poles close to the frequency range of interest

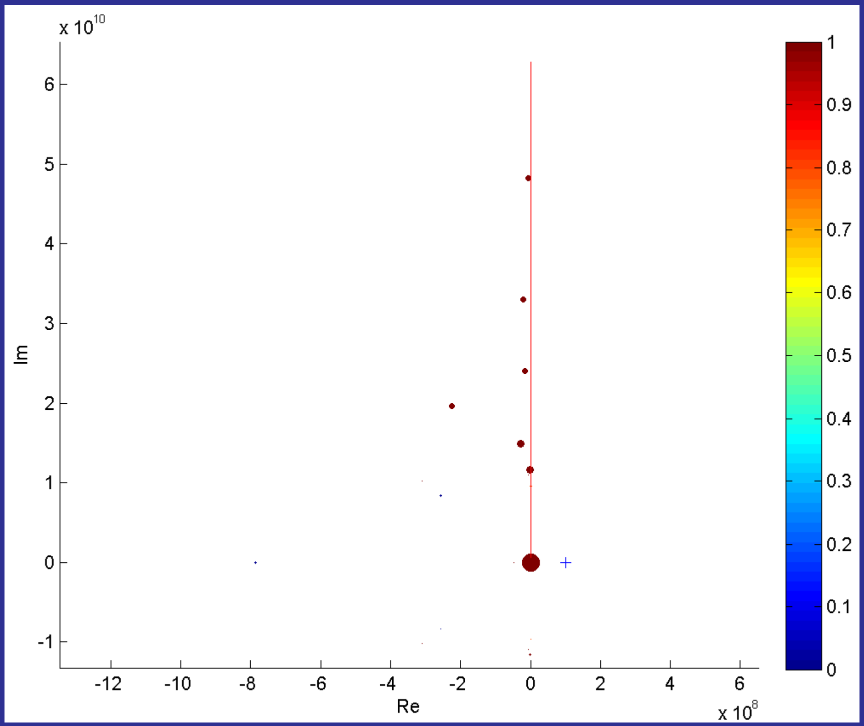
Without restarts

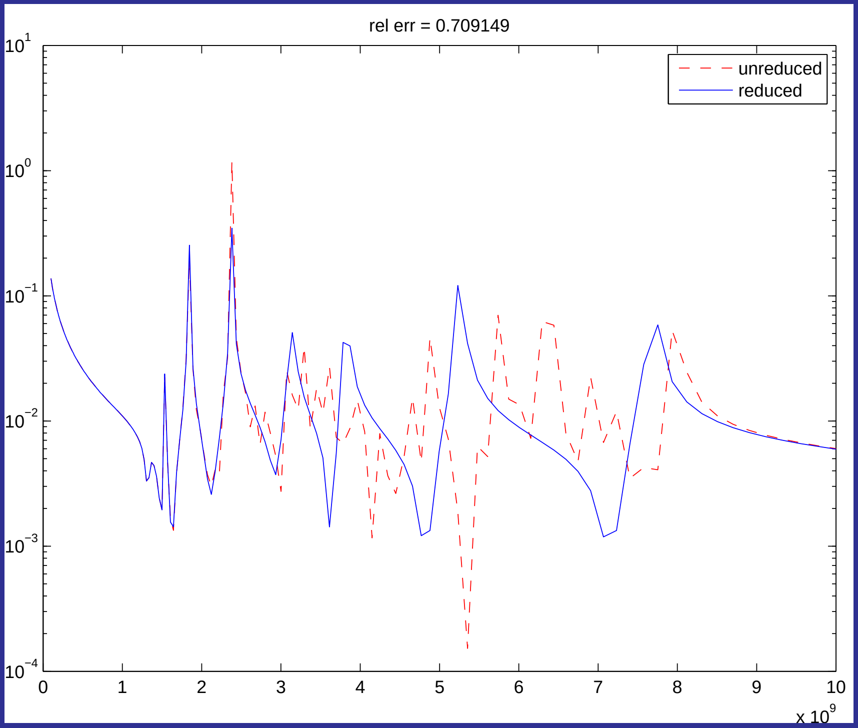




Uniform convergence throughout frequency range

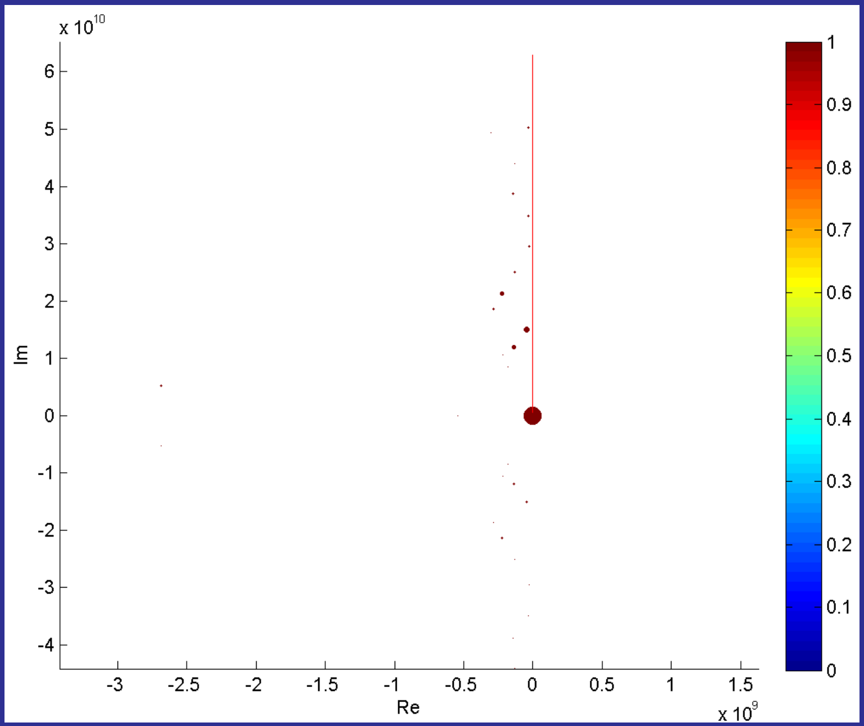
Obtaining ‘good’ eigenvector information

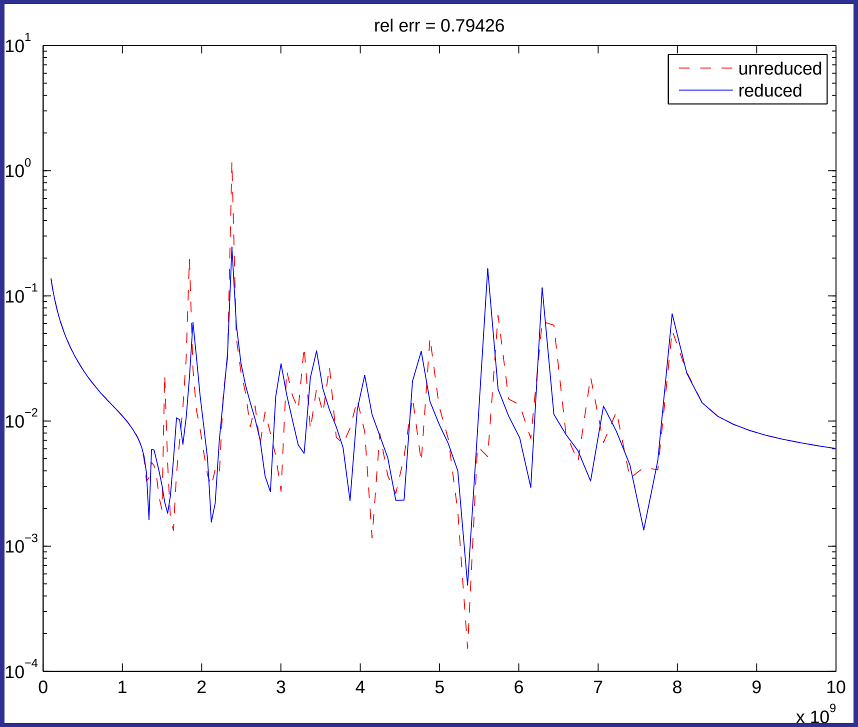




Single point $s_0 = 1 \times 10^8$

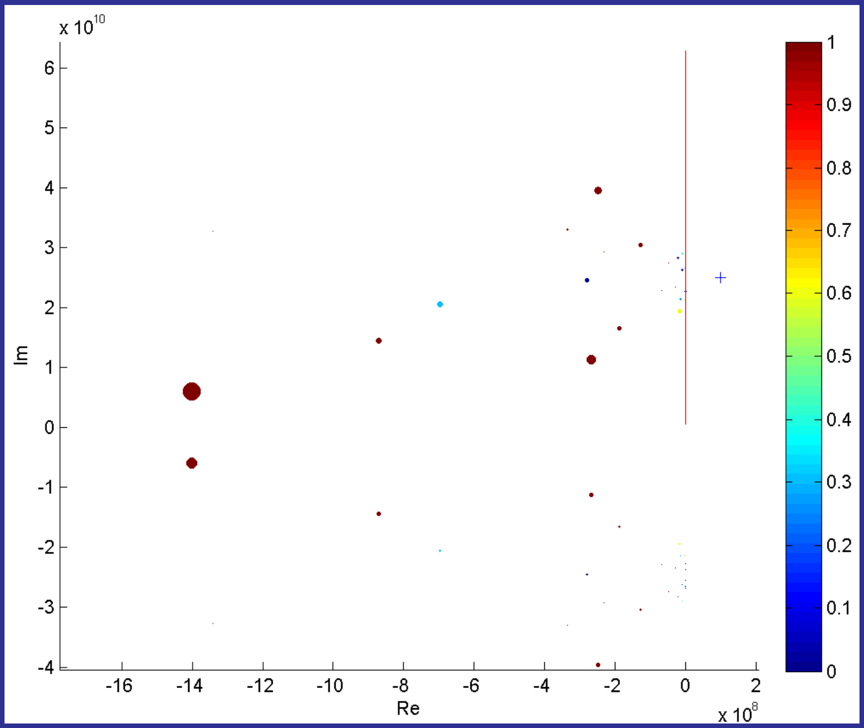
Obtaining 'good' eigenvector information

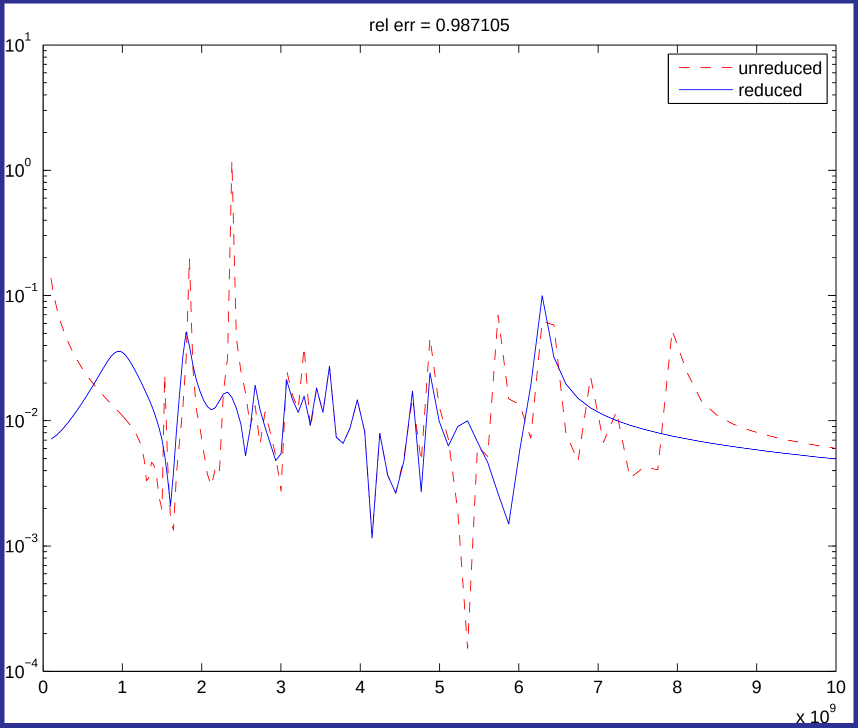




Single point $s_0 = 250 \times 10^8$

Obtaining 'good' eigenvector information





Single point $s_0 = (1 + 250\,\mathrm{i}) \times 10^8$

Changing expansion points

- Extract 'good' eigenvector information Y from the last batch of r Arnoldi vectors

- At each restart allow for changing expansion point:

$$\mathbf{A}(s_0) = -\left(s_0\,\mathbf{C} + \mathbf{G}\right)^{-1}\mathbf{C} \quad \Longrightarrow \quad \mathbf{A}(\tilde{s}_0) = -\left(\tilde{s}_0\,\mathbf{C} + \mathbf{G}\right)^{-1}\mathbf{C}$$

- 'Converged' eigenvectors $\mathbf{v}$ do not change:

$$\mathbf{A}(s_0)\,\mathbf{v} = \lambda\,\mathbf{v} \quad \Longleftrightarrow \quad \mathbf{A}(\tilde{s}_0)\,\mathbf{v} = \tilde{\lambda}\,\mathbf{v}$$

where

$$\frac{1}{\lambda} - \frac{1}{\tilde{\lambda}} = \tilde{s}_0 - s_0$$

Multiple expansion points

- Due to changing expansion points

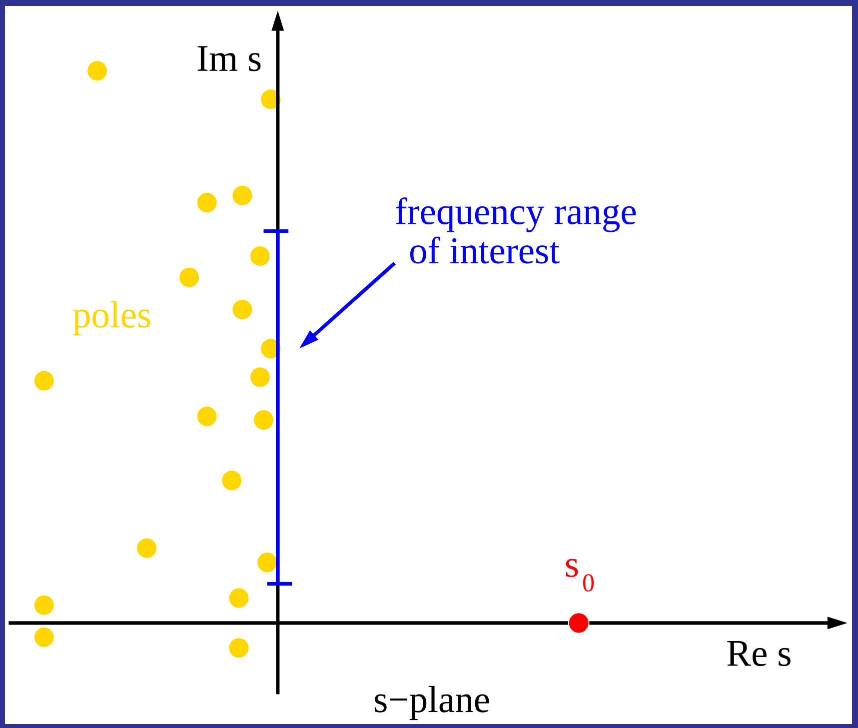
$$s_0^{(1)},\ s_0^{(2)},\ \ldots,\ s_0^{(l)},$$

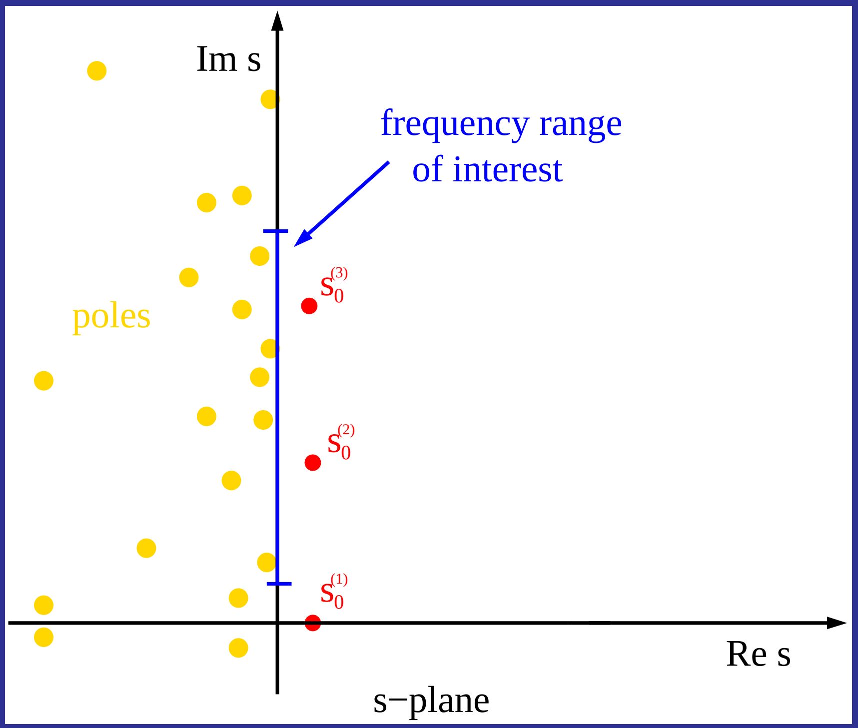
  the resulting reduced-order model is characterized by a **multi-point Padé-type property**:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left(\left(s - s_0^{(j)}\right)^{q_j}\right), \quad j = 1, 2, \ldots, l$$

- Except for $s_0^{(1)}$, the other expansion points are complex

Single vs. multiple expansion points

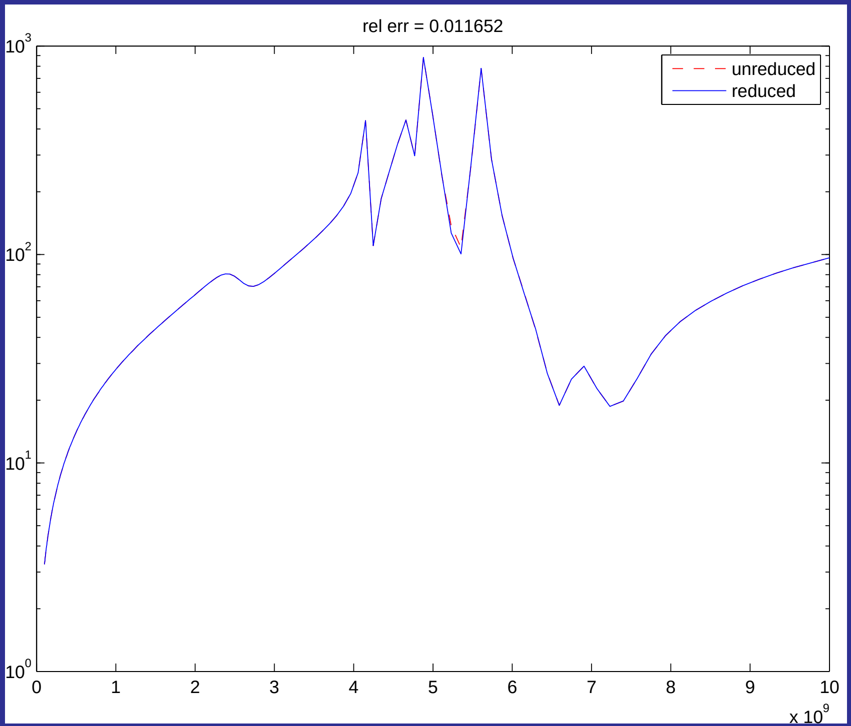


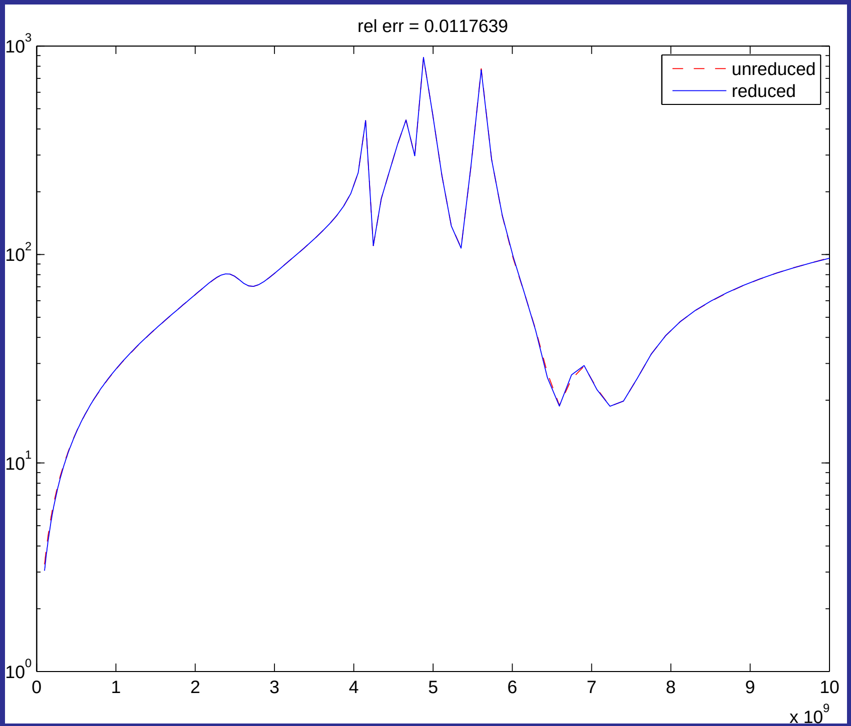


Single point — no restarts

Three points — thick restarts

Single vs. multiple expansion points





Single point — no restarts

$n = 80$

Three points — thick restarts

$n = 42$

Outline

- Numerical simulation of electronic circuits
- DAEs arising in VLSI interconnect analysis
- Structure-preserving model order reduction
- Thick restarts and multiple expansion points
- *Open problems*

Open problems

- SPRIM preserves the block structures of RCL networks
- Preservation of the fine structure of the blocks?
- Optimal structure-preserving Padé-type reduction?
- Automated selection of changing expansion points to make thick restarts practical?
- We still cannot handle RCL descriptor systems as large as we would need to
- Meaningful reduced-order models for very inaccurate system data?