Model reduction of linear DAE systems from measurements

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Outline

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Outline

Model reduction: problem Setting

Consider

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{aligned}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^p$. The reduced system is:

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{f}_r(\mathbf{x}_r(t), \mathbf{u}(t)) \\ \mathbf{y}_r(t) &= \mathbf{h}_r(\mathbf{x}_r(t), \mathbf{u}(t)) \end{aligned}$$

where $\mathbf{x}_r \in \mathbb{R}^r$. The number of inputs and outputs, m, p, remain the same, while the internal state-space satisfy: $r \ll n$.

Model reduction: problem Setting

consider

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

where $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$ and $\mathbf{D} \in \mathbb{R}^{p \times m}$; $\mathbf{x}(t) \in \mathbb{R}^{n}$, $\mathbf{u}(t) \in \mathbb{R}^{m}$ and $\mathbf{y}(t) \in \mathbb{R}^{p}$. The reduced system is described by:

$$\begin{aligned} \mathbf{E}_r\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r(t) + \mathbf{D}_r\mathbf{u}(t) \end{aligned}$$

where $\mathbf{A}_r, \mathbf{E}_r \in \mathbb{R}^{r \times r}$, $\mathbf{B}_r \in \mathbb{R}^{r \times m}$, $\mathbf{C}_r \in \mathbb{R}^{p \times r}$ and $\mathbf{D}_r \in \mathbb{R}^{p \times m}$. The number of inputs and outputs, m, p, remain the same, while the internal state-space satisfy: $r \ll n$.

Goals for Reduced Order Models

1. The reduced input-output map should be uniformly ”close” to the original: for the same $\mathbf{u}(t)$, $\mathbf{y} - \mathbf{y}_r$, should be ”small”.
2. Critical system features and structure should be preserved: stability, passivity, Hamiltonian structure, subsystem interconnectivity, or second-order structure.
3. Strategies for computing the reduced system should lead to robust, numerically stable algorithms and require minimal application-specific tuning.

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Problem 1

Interpolatory reduction given state space data

Given a full-order system **E**, **A**, **B**, **C**, **D**, and given

left interpolation points: $\{\mu_i\}_{i=1}^{q} \subset \mathbb{C}$, *with left tangent directions*: $\{\boldsymbol{\ell}_i\}_{i=1}^{q} \subset \mathbb{C}^p$,

and

right interpolation points: $\{\lambda_i\}_{i=1}^{r} \subset \mathbb{C}$ *with right tangent directions*: $\{\mathbf{r}_i\}_{i=1}^{r} \subset \mathbb{C}^m$.

Find a reduced-order system $\mathbf{E}_r$, $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{C}_r$, $\mathbf{D}_r$, such that the transfer function, $\mathbf{H}_r(s)$ is a *tangential interpolant* to $\mathbf{H}(s)$:

$$\boldsymbol{\ell}_i^* \mathbf{H}_r(\mu_i) = \boldsymbol{\ell}_i^* \mathbf{H}(\mu_i) \quad \text{for } i = 1, \cdots, q, \qquad \text{and} \qquad \mathbf{H}_r(\lambda_j)\mathbf{r}_j = \mathbf{H}(\lambda_j)\mathbf{r}_j, \quad \text{for } j = 1, \cdots, r,$$

Interpolation points and tangent directions are selected to realize the model reduction goals.

Problem 2

Interpolatory reduction given input/output data

Given a set of input-output response measurements specified by

left driving frequencies: $\{\mu_i\}_{i=1}^{q} \subset \mathbb{C}$,
using *left input directions*: $\{\boldsymbol{\ell}_i\}_{i=1}^{q} \subset \mathbb{C}^p$,
producing *left responses*: $\{\mathbf{v}_i\}_{i=1}^{q} \subset \mathbb{C}^m$,

and

right driving frequencies: $\{\lambda_i\}_{i=1}^{r} \subset \mathbb{C}$
using *right input directions*: $\{\mathbf{r}_i\}_{i=1}^{r} \subset \mathbb{C}^m$
producing *right responses*: $\{\mathbf{w}_i\}_{i=1}^{r} \subset \mathbb{C}^p$

Find (low order) system matrices $\mathbf{E}_r$, $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{C}_r$, $\mathbf{D}_r$, such that the transfer function, $\mathbf{H}_r(s)$, is a *tangential interpolant* to the data:

$$\boldsymbol{\ell}_i^* \mathbf{H}_r(\mu_i) = \mathbf{v}_i^* \quad \text{for } i = 1, \cdots, q, \qquad \text{and} \qquad \mathbf{H}_r(\lambda_j)\mathbf{r}_j = \mathbf{w}_j, \quad \text{for } j = 1, \cdots, r,$$

Interpolation points and tangent directions are determined by the data.

Problem 2

Interpolatory reduction given input/output data

Given a set of input-output response measurements specified by

left driving frequencies: $\{\mu_i\}_{i=1}^{q} \subset \mathbb{C}$,	and	*right driving frequencies*: $\{\lambda_i\}_{i=1}^{r} \subset \mathbb{C}$
using *left input directions*: $\{\boldsymbol{\ell}_i\}_{i=1}^{q} \subset \mathbb{C}^p$,		using *right input directions*: $\{\mathbf{r}_i\}_{i=1}^{r} \subset \mathbb{C}^m$
producing *left responses*: $\{\mathbf{v}_i\}_{i=1}^{q} \subset \mathbb{C}^m$,		producing *right responses*: $\{\mathbf{w}_i\}_{i=1}^{r} \subset \mathbb{C}^p$

Find (low order) system matrices $\mathbf{E}_r$, $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{C}_r$, $\mathbf{D}_r$, such that the transfer function, $\mathbf{H}_r(s)$, is a *tangential interpolant* to the data:

$$\boldsymbol{\ell}_i^* \mathbf{H}_r(\mu_i) \cong \mathbf{v}_i^* \quad \text{for } i = 1, \cdots, q, \qquad \text{and} \qquad \mathbf{H}_r(\lambda_j)\mathbf{r}_j \cong \mathbf{w}_j, \quad \text{for } j = 1, \cdots, r,$$

Interpolation points and tangent directions are determined by the data.

Outline

Motivation: S-parameters

- Streamlining of the simulation of entire complex electronic systems (chips, packages, boards) is required.
- In circuit simulation, interconnect models must be valid over a wide bandwidth.

An important tool: **S-parameters**

Given a system in I/O representation: $\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$, the associated **S-paremeter representation** is

$$\bar{\mathbf{y}}(s) = \mathbf{S}(s)\bar{\mathbf{u}}(s) = \underbrace{[\mathbf{H}(s)+\mathbf{I}][\mathbf{H}(s)-\mathbf{I}]^{-1}}_{\mathbf{S}(s)}\,\bar{\mathbf{u}}(s),$$

where: $\bar{\mathbf{y}} = \frac{1}{2}(\mathbf{y}+\mathbf{u})$ are the *transmitted waves* and,

$\bar{\mathbf{u}} = \frac{1}{2}(\mathbf{y}-\mathbf{u})$ are the *reflected waves*.

Measurement of S-parameters

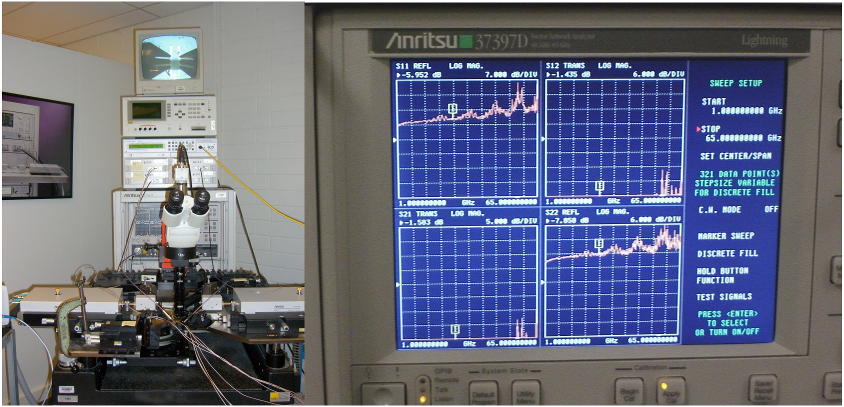

Figure: VNA (Vector Network Analyzer) and VNA screen showing the magnitude of the S-parameters for a 2 port device.

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Outline

Classical realization

Given $\mathbf{h}_t \in \mathbb{R}^{p\times m}$, $t = 1, 2, \cdots$, find $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, such that

$$\mathbf{h}_t = \mathbf{C}\mathbf{A}^{t-1}\mathbf{B},\ \ t > 0$$

Main tool: *Hankel matrix*

$$\mathcal{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 & \cdots \\ \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 & \cdots \\ \mathbf{h}_3 & \mathbf{h}_4 & \mathbf{h}_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \end{bmatrix}}_{\mathcal{O}} \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots \end{bmatrix}}_{\mathcal{R}}$$

Classical realization

Solvability $\Leftrightarrow \operatorname{rank}\mathcal{H} = n < \infty$

Solution: Let $\Delta \in \mathbb{R}^{n\times n}$, be a submatrix of $\mathcal{H}$ such that $\det\Delta \neq 0$; let also $\sigma\Delta \in \mathbb{R}^{n\times n}$ be the matrix with the same rows but columns shifted by m columns; finally, let $\Gamma \in \mathbb{R}^{n\times n}$ have the same rows as Δ but the first m columns only, while $\Lambda \in \mathbb{R}^{m\times n}$ be the submatrix of $\mathcal{H}$ composed of the same columns as Δ, but its first p rows. Then

$$\mathbf{A} = \Delta^{-1}\sigma\Delta, \quad \mathbf{B} = \Delta^{-1}\Gamma, \quad \mathbf{C} = \Lambda.$$

Consequences. If the sequence h_t, $t > 0$, is realizable, it is also summable:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \sum_{t>0} \mathbf{h}_t s^{-t}$$

Notice that in terms of the data:

$$\mathbf{H}(s) = \Lambda(s\Delta - \sigma\Delta)^{-1}\Gamma$$

Remark. $\mathbf{h}_t$ are the *Markov parameters* of the underlying linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$.

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Model Reduction from Measurements

Consider a set of scalar points: (s_i, ϕ_i), $i = 1, 2, \cdots, N$, $s_i \neq s_j$, $i \neq j$.
We seek a rational function $\mathbf{H}(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)}$, such that $\mathbf{H}(s_i) = \phi_i$, $i = 1, 2, \cdots, N$, and $\mathbf{n}, \mathbf{d}$ are coprime polynomials. The data is now divided in disjoint sets: (σ_i, w_i), $i = 1, 2, \cdots, r$, (μ_j, v_j), $j = 1, 2, \cdots, q$, $k + q = N$. Consider:

$$\sum_{i=1}^{r} \gamma_i \frac{\phi(s) - w_i}{s - \sigma_i} = 0.$$

Then as long as $\gamma_i \neq 0$, there holds $\phi(\sigma_i) = w_i$, for $i = 1, \cdots, q$. Making use of the freedom in satisfying the remaining interpolation conditions, we get:

The Loewner matrix

$$\mathbb{L}\mathbf{c} = 0 \quad \text{where} \quad \mathbb{L} = \underbrace{\begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \sigma_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_r}{\mu_1 - \sigma_r} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \sigma_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \sigma_r} \end{bmatrix}}_{\textbf{Loewner matrix}} \in \mathbb{C}^{q \times r}, \quad \mathbf{c} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{bmatrix} \in \mathbb{C}^r.$$

Model construction from data

Main result

The rank of $\mathbb{L}$ encodes the information about the minimal degree interpolants:

$$n = \operatorname{rank} \mathbb{L}.$$

Remark. If $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$, then

$$\mathbb{L} = -\underbrace{\begin{bmatrix} \mathbf{C}(\lambda_1\mathbf{I} - \mathbf{A})^{-1} \\ \mathbf{C}(\lambda_2\mathbf{I} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\lambda_k\mathbf{I} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \underbrace{\begin{bmatrix} (\mu_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\mu_q\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}}$$

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Scalar interpolation – multiple points

Special case. single point with multiplicity: $(s_0; \phi_0, \phi_1, \cdots, \phi_{N-1})$, i.e. the value of the function and that of a number of derivatives is provided. The **Loewner matrix** becomes:

$$\mathbb{L} = \begin{bmatrix} \frac{\phi_1}{1!} & \frac{\phi_2}{2!} & \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots \\ \frac{\phi_2}{2!} & \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots & \\ \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & & & \\ \frac{\phi_4}{4!} & \vdots & & \ddots & \\ \vdots & & & & \end{bmatrix} \quad \Rightarrow \quad \textbf{HANKEL MATRIX}$$

Thus the **Loewner matrix generalizes Hankel matrix** when general interpolation replaces realization.

Outline

General framework – tangential interpolation

Given: • right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$

• left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

We assume for simplicity that all points are distinct.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i \qquad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*$$

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \begin{array}{l} \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \\ \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k} \end{array}$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

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General framework – tangential interpolation

Input-output data. The Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^*\mathbf{r}_1 - \boldsymbol{\ell}_1^*\mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1^*\mathbf{r}_k - \boldsymbol{\ell}_1^*\mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^*\mathbf{r}_1 - \boldsymbol{\ell}_q^*\mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q^*\mathbf{r}_k - \boldsymbol{\ell}_q^*\mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

Recall:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \;\; \boldsymbol{\ell}_j^*\mathbf{H}(\mu_j) = \mathbf{v}_j^*$$

Therefore $\mathbb{L}$ satisfies the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{VR} - \mathbf{LW}$$

General framework – tangential interpolation

State space data. Suppose that $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$.

Let $\mathbf{X}, \mathbf{Y}$ satisfy the following **Sylvester equations**

$$\boxed{\mathbf{E}\mathbf{X}\Lambda - \mathbf{A}\mathbf{X} = \mathbf{B}\mathbf{R}} \quad \text{and} \quad \boxed{\mathbf{M}\mathbf{Y}\mathbf{E} - \mathbf{Y}\mathbf{A} = \mathbf{L}\mathbf{C}}$$

$$\Downarrow$$

$\mathbf{x}_i = (\lambda_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{r}_i \;\Rightarrow \mathbf{X}$: *generalized* reachability matrix

$\mathbf{y}_j^* = \ell_j^*\mathbf{C}(\mu_j\mathbf{E} - \mathbf{A})^{-1} \;\Rightarrow \mathbf{Y}$: *generalized* observability matrix.

$$\Rightarrow \boxed{\mathbb{L} = -\mathbf{Y}\mathbf{E}\mathbf{X}}$$

The shifted Loewner matrix

- The shifted Loewner matrix, $\mathbb{L}_\sigma$, is the Loewner matrix of $s\mathbf{H}(s)$:

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^* \mathbf{r}_1 - \boldsymbol{\ell}_1^* \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1^* \mathbf{r}_k - \boldsymbol{\ell}_1^* \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^* \mathbf{r}_1 - \boldsymbol{\ell}_q^* \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q^* \mathbf{r}_k - \boldsymbol{\ell}_q^* \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

- $\mathbb{L}_\sigma$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}_\sigma \Lambda - M \mathbb{L}_\sigma = M\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\Lambda}$$

- $\mathbb{L}_\sigma$ can be factored as

$$\Rightarrow \boxed{\mathbb{L}_\sigma = -\mathbf{Y}\mathbf{A}\mathbf{X}}$$

- $\mathbb{L}_\sigma - M\mathbb{L} + \mathbf{L}\mathbf{W} = 0$ and $\mathbb{L}_\sigma - \mathbb{L}\Lambda + \mathbf{V}\mathbf{R} = 0.$

Construction of Interpolants (Models)

Theorem: right amount of data

Assume that $k = \ell$, and let

$$\det(x\mathbb{L} - \mathbb{L}_\sigma) \neq 0, \quad x \in \{\lambda_i\} \cup \{\mu_j\}$$

Then

$$\mathbf{E} = -\mathbb{L}, \ \ \mathbf{A} = -\mathbb{L}_\sigma, \ \ \mathbf{B} = \mathbf{V}, \ \ \mathbf{C} = \mathbf{W}$$

is a minimal realization of an interpolant of the data, i.e., the function

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V}$$

interpolates the data.

Proof

Multiplying the first equation by s and subtracting it from the second we get

$$(\mathbb{L}_\sigma - s\mathbb{L})\Lambda - M(\mathbb{L}_\sigma - s\mathbb{L}) = \mathbf{L}\mathbf{W}(\Lambda - s\mathbf{I}) - (M - s\mathbf{I})\mathbf{V}\mathbf{R}.$$

Multiplying this equation by $\mathbf{e}_i$ on the right and setting $s = \lambda_i$, we obtain

$$(\lambda_i\mathbf{I} - M)(\mathbb{L}_\sigma - \lambda_i\mathbb{L})\mathbf{e}_i = (\lambda_i\mathbf{I} - M)\mathbf{V}\mathbf{r}_i \quad \Rightarrow$$

$$(\lambda_i\mathbb{L} - \mathbb{L}_\sigma)\mathbf{e}_i = \mathbf{V}\mathbf{r}_i \quad \Rightarrow \quad \mathbf{W}\mathbf{e}_i = \mathbf{W}(\lambda_i\mathbb{L} - \mathbb{L}_\sigma)^{-1}\mathbf{V}$$

Therefore $\mathbf{w}_i = \mathbf{H}(\lambda_i)\mathbf{r}_i$. This proves right tangential interpolation.

To prove the left tangential interpolation property, we multiply the above equation by $\mathbf{e}_j^*$ on the left and set $s = \mu_j$:

$$\mathbf{e}_j^*(\mathbb{L}_\sigma - \mu_j\mathbb{L})(\Lambda - \mu_j\mathbf{I}) = \mathbf{e}_j^*\mathbf{L}\mathbf{W}(\mu_j\mathbf{I} - \Lambda) \quad \Rightarrow$$

$$\mathbf{e}_j^*(\mathbb{L}_\sigma - \mu_j\mathbb{L}) = \ell_j\mathbf{W} \quad \Rightarrow \quad \mathbf{e}_j^*\mathbf{V} = \ell_j\mathbf{W}(\mathbb{L}_\sigma - \mu_j\mathbb{L})^{-1}\mathbf{V}$$

Therefore $\mathbf{v}_j = \ell_j\mathbf{H}(\mu_j)$.

The case of more data than necessary

Consider the following short SVDs:

$$[\mathbb{L} \quad \mathbb{L}_\sigma] = \mathbf{Y}\Sigma_\ell\tilde{\mathbf{X}}^* \text{ and } \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_\sigma \end{bmatrix} = \tilde{\mathbf{Y}}\Sigma_r\mathbf{X}^*, \text{ where } \Sigma_\ell, \Sigma_r \in \mathbb{R}^{k\times k}, \ \mathbf{Y}, \mathbf{X} \in \mathbb{C}^{N\times k}.$$

Proposition

From the above construction we have:

$$\mathbf{Y}\mathbf{Y}^*\mathbb{L} = \mathbb{L}, \quad \mathbf{Y}\mathbf{Y}^*\mathbb{L}_\sigma = \mathbb{L}_\sigma, \quad \mathbf{Y}\mathbf{Y}^*\mathbf{V} = \mathbf{V},$$
$$\mathbb{L}\mathbf{X}\mathbf{X}^* = \mathbb{L}, \quad \mathbb{L}_\sigma\mathbf{X}\mathbf{X}^* = \mathbb{L}_\sigma, \quad \mathbf{W}\mathbf{X}\mathbf{X}^* = \mathbf{W}.$$

Theorem

A realization $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$, of an (approximate) interpolant is given as follows:

$\mathbf{E} = -\mathbf{Y}^*\mathbb{L}\mathbf{X}$	$\mathbf{B} = \mathbf{Y}^*\mathbf{V}$
$\mathbf{A} = -\mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X}$	$\mathbf{C} = \mathbf{W}\mathbf{X}$

Consequences

If we have more data than necessary, we can consider

$$(\mathbb{L}_\sigma, \ \ \mathbb{L}, \ \ \mathbf{V}, \ \ \mathbf{W}),$$

as a **singular** model of the data.

Corollary 1: Interpolation property

Let $\mathbf{z}_i$ satisfy

$$(\lambda_i \mathbb{L} - \mathbb{L}_\sigma)\mathbf{z}_i = \mathbf{V}\mathbf{r}_i.$$

It follows that

$$\mathbf{W}\mathbf{z}_i = \mathbf{w}_i$$

This follows because $\mathbf{z}_i = \mathbf{e}_i + \mathbf{z}_0$, where $\mathbf{W}\mathbf{z}_0 = 0$.

Consequences

Corollary 2

The original pencil $(\mathbb{L}_\sigma, \mathbb{L})$ and the projected pencil $(\mathbf{A}, \mathbf{E})$, have the same non-trivial eigenvalues.

Proof

Let $(\mathbf{z}, \lambda)$ be a right eigenpair of $(\mathbb{L}_\sigma, \mathbb{L})$.

Then: $\mathbb{L}_\sigma \mathbf{z} = \lambda \mathbb{L} \mathbf{z} \Rightarrow \mathbb{L}_\sigma \mathbf{X}\mathbf{X}^* \mathbf{z} = \lambda \mathbb{L} \mathbf{X}\mathbf{X}^* \mathbf{z} \Rightarrow \underbrace{\mathbf{Y}^* \mathbb{L}_\sigma \mathbf{X}}_{\mathbf{A}} \mathbf{X}^* \mathbf{z} = \lambda \underbrace{\mathbf{Y}^* \mathbb{L} \mathbf{X}}_{\mathbf{E}} \mathbf{X}^* \mathbf{z}$.

Thus $(\mathbf{X}^* \mathbf{z}, \lambda)$ is an eigenpair of $(\mathbf{A}, \mathbf{E})$.

Conversely, if $(\mathbf{z}, \lambda)$ is an eigenpair of $(\mathbf{A}, \mathbf{E})$ then $(\mathbf{X}\mathbf{z}, \lambda)$ is an eigenpair of the original pencil $(\mathbb{L}_\sigma, \mathbb{L})$.

Similarly for left eigenpairs.

Consequences

Corollary 3

Let $\mathbf{\Phi}$ and $\mathbf{\Psi}$ be such that $\mathbf{X}^*\mathbf{\Phi}$ and $\mathbf{\Psi}^*\mathbf{Y}$ are square and non-singular. Then

$$(\mathbf{Y}^*\mathbb{L}\mathbf{X},\ \mathbf{Y}^*\mathbb{L}_\sigma\mathbf{X},\ \mathbf{Y}^*\mathbf{V},\ \mathbf{W}\mathbf{X}) \quad \text{and} \quad (\mathbf{\Phi}^*\mathbb{L}\mathbf{\Psi},\ \mathbf{\Phi}^*\mathbb{L}_\sigma\mathbf{\Psi},\ \mathbf{\Phi}^*\mathbf{V},\ \mathbf{W}\mathbf{\Psi}),$$

are minimal realizations for the same system.

This means that the projection may in essence be chosen arbitrarily.

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Coupled mechanical system

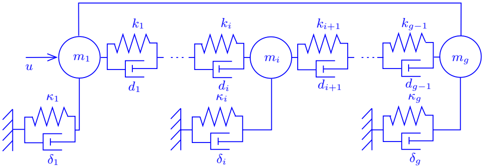


Figure: Constrained mechanical system

The vibration is described by: $\mathbf{E}\dot{\mathbf{x}}(t)$=$\mathbf{A}\mathbf{x}(t)$+$\mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t)$=$\mathbf{C}\mathbf{x}(t)$,
M: mass, **K**: stiffness, **D**: damping, $\mathbf{G} = [1,\ 0, \cdots,\ 0,\ -1]$, constraint:

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{K} & \mathbf{D} & -\mathbf{G}^* \\ \mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathbf{B} = \mathbf{C} = \mathbf{I}.$$

$$\Rightarrow \quad \mathbf{H}(s) = (s\mathbf{E} - \mathbf{A})^{-1}.$$

Example: mechanical system $g = 2$

For $g = 2$ masses, we have

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & 1 & -10 & 5 & -1 \\ 1 & -2 & 5 & -6 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \; \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$\mathbf{H}(s) = \frac{1}{6s^2+6s+3}\mathbf{N}(s)$, where $\mathbf{N}(s) =$

$$\left(\begin{array}{c|c|c|c|c} 5s+5 & s+1 & 1 & 1 & -s^2-s-1 \\ 5s+5 & s+1 & 1 & 1 & 5s^2+5s+2 \\ -s^2-s-3 & s\,(s+1) & s & s & -s\left(s^2+s+1\right) \\ 5s\,(s+1) & -5s^2-5s-3 & s & s & s\left(5s^2+5s+2\right) \\ 5s^3+40s^2+40s+20 & -5s^3-40s^2-37s-17 & s^2+s+1 & -5s^2-5s-2 & 5s^4+40s^3+48s^2+28s+5 \end{array}\right)$$

The pair $(\mathbf{A}, \mathbf{E})$ has 2 finite eigenvalues $-\frac{1}{2} \pm \frac{i}{2}$, and 3 eigenvalues at infinity.

Example: continued

We take 4 measurements at $s = 0$:

$$\mathbf{\Theta}_0 = \begin{pmatrix} \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \frac{20}{3} & -\frac{17}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix}, \quad \mathbf{\Theta}_1 = \begin{pmatrix} -\frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & -1 & -\frac{1}{3} & -\frac{1}{3} & 6 \end{pmatrix},$$

$$\mathbf{\Theta}_2 = \begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{5}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \mathbf{\Theta}_3 = \begin{pmatrix} \frac{10}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{10}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}.$$

We will consider two resulting systems. First, just the right amount of data:

$$\hat{\mathbf{E}} = \mathbf{\Theta}_1,\ \hat{\mathbf{A}} = \mathbf{\Theta}_0,\ \hat{\mathbf{B}} = \hat{\mathbf{C}} = \mathbf{\Theta}_0 \ \Rightarrow\ \mathbf{H}(s) = (s\mathbf{E} - \mathbf{A})^{-1} = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}.$$

Example: continued

The second model uses all the available data:

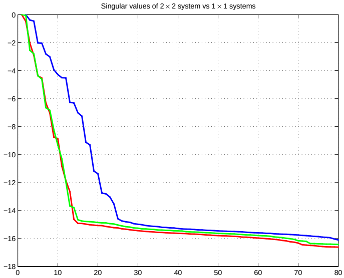
$$\begin{pmatrix} \mathbf{\Theta}_0 & \mathbf{\Theta}_1 \end{pmatrix}, \ \mathbb{L} = \begin{pmatrix} \mathbf{\Theta}_1 & \mathbf{\Theta}_2 \\ \mathbf{\Theta}_2 & \mathbf{\Theta}_3 \end{pmatrix}, \ \mathbb{L}_\sigma = \begin{pmatrix} \mathbf{\Theta}_0 & \mathbf{\Theta}_1 \\ \mathbf{\Theta}_1 & \mathbf{\Theta}_2 \end{pmatrix}, \ \begin{pmatrix} \mathbf{\Theta}_0 \\ \mathbf{\Theta}_1 \end{pmatrix},$$

and it is singular. We want to compute the eigenvalues of the pencil $(\mathbb{L}_\sigma, \mathbb{L})$. The QZ algorithm yields

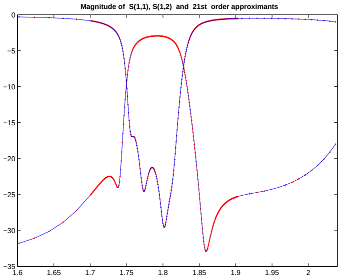
2.6317e − 001 + 1.3878e − 017i	**1.8719e − 013**	↔ infinite eig
$8.5009e-013-3.8885e-018i$	$6.8324e-017$	
−2.2417e − 002 + 2.2417e − 002i	**4.4834e − 002**	↔ finite eig
−6.2394e − 001 − 6.2394e − 001i	**1.2479e + 000**	↔ finite eig
−2.6999e − 004	**0**	↔ infinite eig
5.2379e − 001	**0**	↔ infinite eig
$1.3623e-014$	$1.4393e-015$	
$9.3845e-017$	$1.8285e-016$	
$-1.5898e-016$	$5.1864e-016$	
$-1.1214e-017$	$2.2332e-016$	

Example: Four-pole band-pass filter

- **1000 measurements between 40 and 120 GHz; S-parameters** 2×2, **MIMO (approximate) interpolation** $\Rightarrow \mathbb{L}, \mathbb{L}_\sigma \in \mathbb{R}^{2000\times 2000}$.



The singular values of $\mathbb{L}$, $\mathbb{L}_\sigma$



The $S(1,1)$ and $S(1,2)$ parameter data

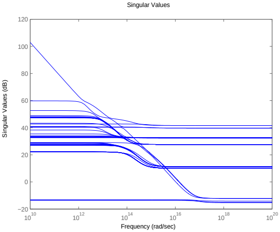
17-th order model

Multi-port example from Qimonda AG

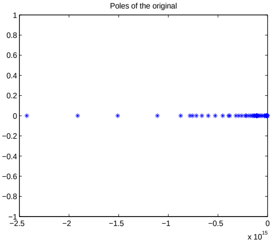
System

$$\mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{G}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{L}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where $m = p = 70$ and $n = 141$:



(a) Frequency response



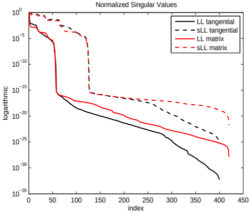
(b) Finite poles

$\Rightarrow$ 84 finite poles and 57 infinite poles.

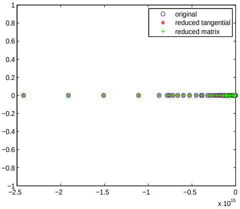
Take 400 measurements between 10^{13} and 10^{15}.

RICE

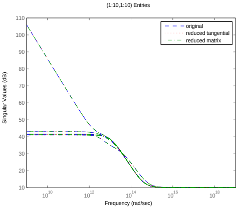
Multi-port example from Qimonda AG



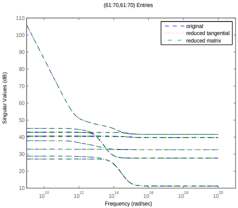
(a) Drop of the singular values of the Loewner matrix pencil for tangential and matrix interpolatio



(b) Poles of the original and reduced systems



(a) Top left 10×10 entries of the transfer function



(b) Bottom right 10×10 entries of the transfer function

Outline

Recursive Loewner-matrix framework

Interpolation data:

$$\mathbf{R} \in \mathbb{C}^{m\times k}, \mathbf{W} \in \mathbb{C}^{p\times k}, \Lambda \in \mathbb{C}^{k\times k} \text{ and } \mathbf{L} \in \mathbb{C}^{\ell\times p}, \mathbf{V} \in \mathbb{C}^{\ell\times m}, \mathbf{M} \in \mathbb{C}^{\ell\times\ell},$$

and Loewner matrices which satisfy:

$$\mathbb{L}\Lambda - \mathbf{M}\mathbb{L} = \mathbf{LW} - \mathbf{VR}, \quad \mathbb{L}_\sigma\Lambda - \mathbf{M}\mathbb{L}_\sigma = \mathbf{LW}\Lambda - \mathbf{MVR}$$

We now define the $(p+m)\times(p+m)$ rational matrix

$$\Theta(s) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{W} \\ -\mathbf{R} \end{bmatrix} (s\mathbb{L} - \mathbb{L}\Lambda)^{-1} \begin{bmatrix} \mathbf{L} & \mathbf{V} \end{bmatrix} = \begin{pmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{pmatrix},$$

and its inverse

$$\bar{\Theta}(s) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} + \begin{bmatrix} -\mathbf{W} \\ \mathbf{R} \end{bmatrix} (s\mathbb{L} - \mathbf{M}\mathbb{L})^{-1} \begin{bmatrix} \mathbf{L} & \mathbf{V} \end{bmatrix} = \begin{pmatrix} \bar{\Theta}_{11}(s) & \bar{\Theta}_{12}(s) \\ \bar{\Theta}_{21}(s) & \bar{\Theta}_{22}(s) \end{pmatrix}.$$

Recursive interpolation

Lemma

$$[\mathbf{L}_j \quad \mathbf{V}_j]\mathbf{\Theta}(\mu_j) = \mathbf{0}_{\ell\times(p+m)}, \ \forall\, j, \ \text{and} \ \bar{\mathbf{\Theta}}(\lambda_k)\begin{pmatrix} -\mathbf{W}_k \\ \mathbf{R}_k \end{pmatrix} = \mathbf{0}_{(p+m)\times k}, \ \forall\, k.$$

All interpolants can be obtained as matrix fractions involving $\mathbf{\Theta}$ and $\bar{\mathbf{\Theta}}$.

Theorem

Ψ is an interpolant iff $\exists\ \Gamma(s)$:

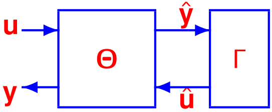
$$\Psi(s) = [\mathbf{\Theta}_{11}(s)\Gamma(s) + \mathbf{\Theta}_{12}(s)][\mathbf{\Theta}_{21}(s)\Gamma(s) + \mathbf{\Theta}_{22}(s)]^{-1}.$$

Similarly, Ψ can also be written as

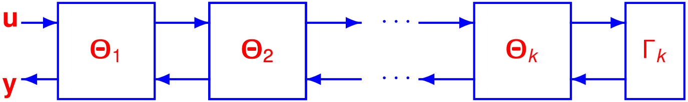
$$\Psi(s) = [\bar{\mathbf{\Theta}}_{11}(s) - \Gamma(s)\bar{\mathbf{\Theta}}_{21}(s)]^{-1}[\bar{\mathbf{\Theta}}_{12}(s) - \Gamma(s)\bar{\mathbf{\Theta}}_{22}(s)].$$

Cascade representation of recursive interpolation

Feedback interpretation of the parametrization of all solutions of the rational interpolation problem



Cascade representation of the recursive interpolation problem.



Recursive Loewner and shifted Loewner matrices

For the recursive procedure, the **error quantities** at each step are the **key**, and are computed as follows:

$$[\mathbf{L}_{k,e} \quad \mathbf{V}_{k,e}] = [\mathbf{L}_k \quad \mathbf{V}_k]\boldsymbol{\Theta}_{k-1}(\mu_k) \text{ and } \begin{pmatrix} -\mathbf{W}_{k,e} \\ \mathbf{R}_{k,e} \end{pmatrix} = \hat{\boldsymbol{\Theta}}_{k-1}(\lambda_k)\begin{pmatrix} -\mathbf{W}_k \\ \mathbf{R}_k \end{pmatrix}.$$

The resulting generating system is

$$\boldsymbol{\Theta}_e(s) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} + \begin{bmatrix} \mathbf{W}_e \\ -\mathbf{R}_e \end{bmatrix}(s\mathbb{L}_e - \mathbb{L}_{\sigma\, e} + \mathbf{V}_e\mathbf{R}_e)^{-1}\begin{bmatrix} \mathbf{L}_e & \mathbf{V}_e \end{bmatrix}$$

Thus the recursive quantities for 3 stages are:

$$\mathbf{W}_e = [\mathbf{W}_{e1} \ \ \mathbf{W}_{e2} \ \ \mathbf{W}_{e3}],\ \mathbb{L}_e = \begin{bmatrix} \mathbb{L}_{1e} & & \\ & \mathbb{L}_{2e} & \\ & & \mathbb{L}_{3e} \end{bmatrix},\ \mathbb{L}_{\sigma\, e} = \begin{bmatrix} \mathbb{L}_{\sigma\, 1e} & \mathbf{L}_{1e}\mathbf{W}_{2e} & \mathbf{L}_{1e}\mathbf{W}_{3e} \\ \mathbf{V}_{2e}\mathbf{R}_{1e} & \mathbb{L}_{\sigma\, 2e} & \mathbf{L}_{2e}\mathbf{W}_{3e} \\ \mathbf{V}_{3e}\mathbf{R}_{1e} & \mathbf{V}_{3e}\mathbf{R}_{2e} & \mathbb{L}_{\sigma\, 3e} \end{bmatrix},\ \mathbf{V}_e = \begin{bmatrix} \mathbf{V}_{1e} \\ \mathbf{V}_{2e} \\ \mathbf{V}_{3e} \end{bmatrix}.$$

The above procedure recursively constructs an L-D-U factorization of the Loewner matrix.

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Summary: recursive interpolation procedure

Given interpolation data: $\mathbf{L}, \mathbf{V}, \mathbf{R}, \mathbf{W}, \Lambda, \mathbf{M}$.

1. Partition the data: $\mathbf{L}_i,\ \mathbf{V}_i,\ \mathbf{R}_i,\ \mathbf{W}_i,\ \Lambda_i,\ \mathbf{M}_i,\quad i = 1, \cdots, n.$
2. Set $\boldsymbol{\Theta}_0(s) = \bar{\boldsymbol{\Theta}}_0(s) = \mathbf{I}_{p+m}$.
3. At the k^{th} step, $k = 1, \cdots, n$, the quantities $\mathbf{L}_k,\ \mathbf{V}_k,\ \mathbf{R}_k,\ \mathbf{W}_k,\ \Lambda_k,\ \mathbf{M}_k$,

$$\begin{aligned} &\boldsymbol{\Theta}_{k-1}(s), \quad \boldsymbol{\Theta}_{1,k-1}(s) = \boldsymbol{\Theta}_0(s)\boldsymbol{\Theta}_1(s)\cdots\boldsymbol{\Theta}_{k-1}(s), \\ &\bar{\boldsymbol{\Theta}}_{k-1}(s), \quad \bar{\boldsymbol{\Theta}}_{1,k-1}(s) = \bar{\boldsymbol{\Theta}}_{k-1}(s)\cdots\bar{\boldsymbol{\Theta}}_1(s)\bar{\boldsymbol{\Theta}}_0(s), \end{aligned}$$

   are available. Compute the k^{th} error quantities:

$$[\mathbf{L}_{k,e}\quad \mathbf{V}_{k,e}] = [\mathbf{L}_k\quad \mathbf{V}_k]\boldsymbol{\Theta}_{1,k-1}(\mu_k), \begin{pmatrix} -\mathbf{W}_{k,e} \\ \mathbf{R}_{k,e} \end{pmatrix} = \hat{\boldsymbol{\Theta}}_{1,k-1}(\lambda_k)\begin{pmatrix} -\mathbf{W}_k \\ \mathbf{R}_k \end{pmatrix}.$$

4. Compute $\mathbb{L}_k, \mathbb{L}_{\sigma k}$, associated with the error data

$$\mathbf{L}_{k,e},\ \mathbf{V}_{k,e},\ \mathbf{R}_{k,e},\ \mathbf{W}_{k,e},\ \Lambda_k,\ \mathbf{M}_k.$$

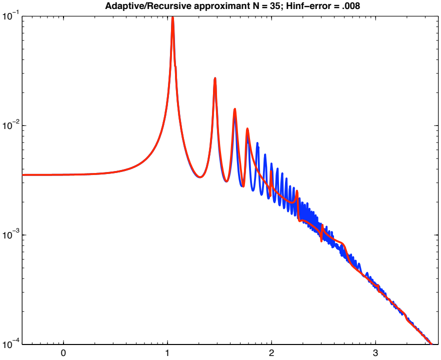
   $\Rightarrow$ construct $\boldsymbol{\Theta}_{k+1}(s)$, $\bar{\boldsymbol{\Theta}}_{k+1}(s)$.

Delay system

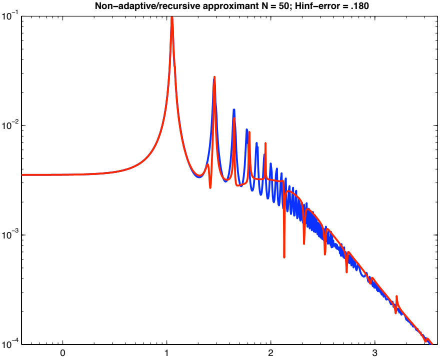
$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau) + \mathbf{B}u(t), \ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

where $\mathbf{E}$, $\mathbf{A}_0$, $\mathbf{A}_1$ are 500×500 and $\mathbf{B}$, $\mathbf{C}^*$ are 500-vectors.

Procedure: compute 1000 frequency response samples. Then apply **recursive/adaptive** Loewner-framework procedure. (Blue: original, red: approximants.)



35-th order recursively constructed model;
H_∞ norm of error: 0.008.



50-th order non-recursively constructed model;
H_∞ norm of error: 0.180.

Outline

Summary and conclusions

Reduction from data (e.g. S-parameters)

- Given input/output data, we can construct with no computation, a singular high order model in generalized state space form.
- Key tool: Loewner matrix pencil and tangential interpolation.
- Since $(\mathbb{L}_\sigma, \mathbb{L})$ is a singular pencil:
  $\Rightarrow$ reduction of $\mathbb{L}$, $\mathbb{L}_\sigma$, required,
  $\Rightarrow$ Recursive procedure.
- Natural way to construct full and reduced models:
  $\Rightarrow$ does not **force** inversion of **E**,
  $\Rightarrow$ does not require persistence of excitation,
  $\Rightarrow$ can deal with many input/output ports,
  $\Rightarrow$ SVD of $[\mathbb{L},\ \mathbb{L}_\sigma]$ or $[\mathbb{L}^*,\ {\mathbb{L}_\sigma}^*]^*$, provides trade-off between accuracy and complexity.

Key references: Model reduction from data

- A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).
- Lefteriu, Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, vol. 29, pages 14-27 (2010).
- General reference: Antoulas SIAM 2005