HOW OFTEN SHOULD ONE COOPERATE?

PARETO-INEFFICIENCY OF PURE NASH EQUILIBRIUM IN SOME FINITE RANDOM GAMES

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Banff 2010 Workshop on Evolutionary Games

Prisoner's Dilemma

	C	D
C	b-c	-c
D	b	0

Penalty Kick Game

How to take penalties: Freakonomics explains, S. J. Dubner and S. D. Levitt, *Times Online, June 12, 2010*

Penalty Kick Game

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	L	R
L	-0.5,0.3	0.3,0.1
R	0.5,-0.8	-0.2,-0.7

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Finite random games

Cooperation and self-interest: Pareto-inefficiency of Nash equilibria in finite random games, J. E. Cohen, *Proc. Natl. Acad. Sci. USA 1998*

Finite random games

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- Cooperation arises: when NE is Pareto dominated by another strategic profile in which every player fares at least as well, and some fares better.

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- 1951 - John Nash: every game has an equilibrium in mixed strategies. The proof relies on Brouwer's fixed point theorem, highly non-constructive.
- Finding NE is NP hard…,
- but finding PNE is easy.

Finite random games

* Probability distribution of k, the number of PNEs:

$$P(k, m_1, m_2) = \sum_{j=0} (-1)^j \binom{k+j}{k} \binom{m_1}{k+j} (m_1 m_2)^{-(k+j)} \binom{m_2}{k+j} (k+j)!$$

* Probability that a PNE is PPO is given by

$$\int_{x\in[0,1]^n} (1 - \prod_{p=1}^{n} (1 - x_p))^{\prod_{p=1}^{n} m_p - \sum_{p=1}^{n} m_p + n - 1} \prod_{p=1}^{n} (m_p x_p^{m_p - 1} dx_p)$$

* If all m_p=m, the probability that a PNE is PPO is not monotonic in n, the number of players. However, the probability that a PNE is PPO decreases as m_p increases.

* For fixed n, the probability that a PNE is PPO is bounded from below by 1/e when all m_p tends to infinity.

* If all players have the same number of strategies, as n tends to infinity, a PNE is always PPO.

The Probability of an Equilibrium Point, K. Goldberg, A. Goldman, M. Newman, J. Res. Nat. Bur. Stand. U.S.A. 72, 93-101 1968.

Cooperation and self-interest: Pareto-inefficiency of Nash equilibria in finite random games, J. E. Cohen, *Proc. Natl. Acad. Sci. USA 1998*

2-person m-strategy random games

(A,B): a random two-person m-strategy game

A,B are $m \times m$ payoff matrices, one for each player. The m^2 payoff entries a_{ij} and b_{ij} are i.i.d. (real-valued, independent, identically distributed continuous random variables), we shall assume them to be U(0,1) for this talk.

The pure strategy pair (i^*, j^*) is a PNE if $a_{i^*,j^*} = \max_i a_{i,j^*}, b_{i^*,j^*} = \max_j b_{i^*,j}$

In symmetric random games, $a_{ij} = b_{ji}$

In zero-sum games $a_{ij} = -b_{ij}$

In common payoffs games, $a_{ij} = b_{ij}$

2-person 2-strategy two-role games

- Trust Game: PNE (3,3) is Pareto-Dominated by (1,1), (4,4), (1,2), (2,4), (1,4)

	$\mathbf{f_1}$	$\mathbf{f_2}$
$\mathbf{e_1}$	$(\beta - c, rc - \beta)$	$(-c, rc)$
$\mathbf{e_2}$	$(0,0)$	$(0,0)$

$$c < \beta < rc$$

	G_1	G_2	G_3	G_4
G_1	$(r-1)c$	$\beta - c$	$-c$	$(r-1)c - \beta$
G_2	$rc - \beta$	0	0	$rc - \beta$
G_3	rc	0	0	rc
G_4	$(r-1)c + \beta$	$\beta - c$	$-c$	$(r-1)c$

$$G_1 = \mathbf{e_1 f_1}, \qquad \mathbf{G_2 = e_2 f_1}, \qquad \mathbf{G_3 = e_2 f_2}, \qquad \mathbf{G_4 = e_1 f_2}$$

- Ultimatum Game: PNEs (1,1) and (3,3) are PPO

	$\mathbf{f_1}$	$\mathbf{f_2}$
$\mathbf{e_1}$	$(1-h, h)$	$(1-h, h)$
$\mathbf{e_2}$	$(0,0)$	$(1-l, l)$

$$0 < l < h < 1$$

	G_1	G_2	G_3	G_4
G_1	1	$1-h$	$1-h$	1
G_2	h	0	$1-l$	$1+h-l$
G_3	h	l	1	$1+h-l$
G_4	1	$1-h+l$	$1-h+l$	1

Public good games with incentives: the role of reputation. H. De Silva and K. Sigmund, in *Games, Groups, and the Global good,* S. A. Levin (ed.), Springer Series in Game Theory, 2009

2-person m-strategy two-role games

Consider a game with two roles I and II and m strategies for each role. Let a_{ij} and b_{ij} be the respective payoffs to role I and II players when the role I player uses strategy i and the role II player uses strategy j. A and B are mxm payoff matrices whose entries are independent U(0,1) distribution. A coin toss decides which role to assign to each player. The resulting game is a 2-person m^2-strategy symmetric game whose m^2xm^2 payoff matrix C has entries given by $c_{ij,kl} = a_{il} + b_{kj}$

The strategic profile (i^*j^*, k^*l^*) is PNE if

$$a_{i^*l^*} + b_{k^*j^*} = max_{(i,j)} a_{il^*} + b_{k^*j}, \qquad a_{i^*l^*} + b_{k^*j^*} = max_{(k,l)} a_{i^*l} + b_{kj^*}$$

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* How often is there a PNE?   What is the probability distribution of the number of PNEs?

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- How often is there a PNE? What is the probability distribution of the number of PNEs?
- How often is a PNE not PPO? i.e. how often can cooperation lead to improvement for all players involved.

Probability distribution of the number of PNES

* random game

$$P(k,m) = \nu(k,m)\sum_{i=0}^{m-k}(-1)^i \frac{1}{m^{2i+2k}}\nu(i,m-k), \qquad \nu(k,m) = \begin{pmatrix} m \\ k \end{pmatrix} \frac{m!}{(m-k)!}$$

* symmetric random game

$$P(k,m) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{m!}{(k-2j)!2^j j! m^k} \sum_{i=0}^{\lfloor \frac{m-k}{2} \rfloor} (-1)^i \frac{1}{i! 2^i m^{2i}} \sum_{l=0}^{m-k-2i} (-1)^l \frac{1}{l!(m-k-2i-l)! m^l}$$

* zero-sum game

$$P(0,m) = 1 - \frac{(m!)^2}{(2m-1)!}, \qquad P(1,m) = \frac{(m!)^2}{(2m-1)!}$$

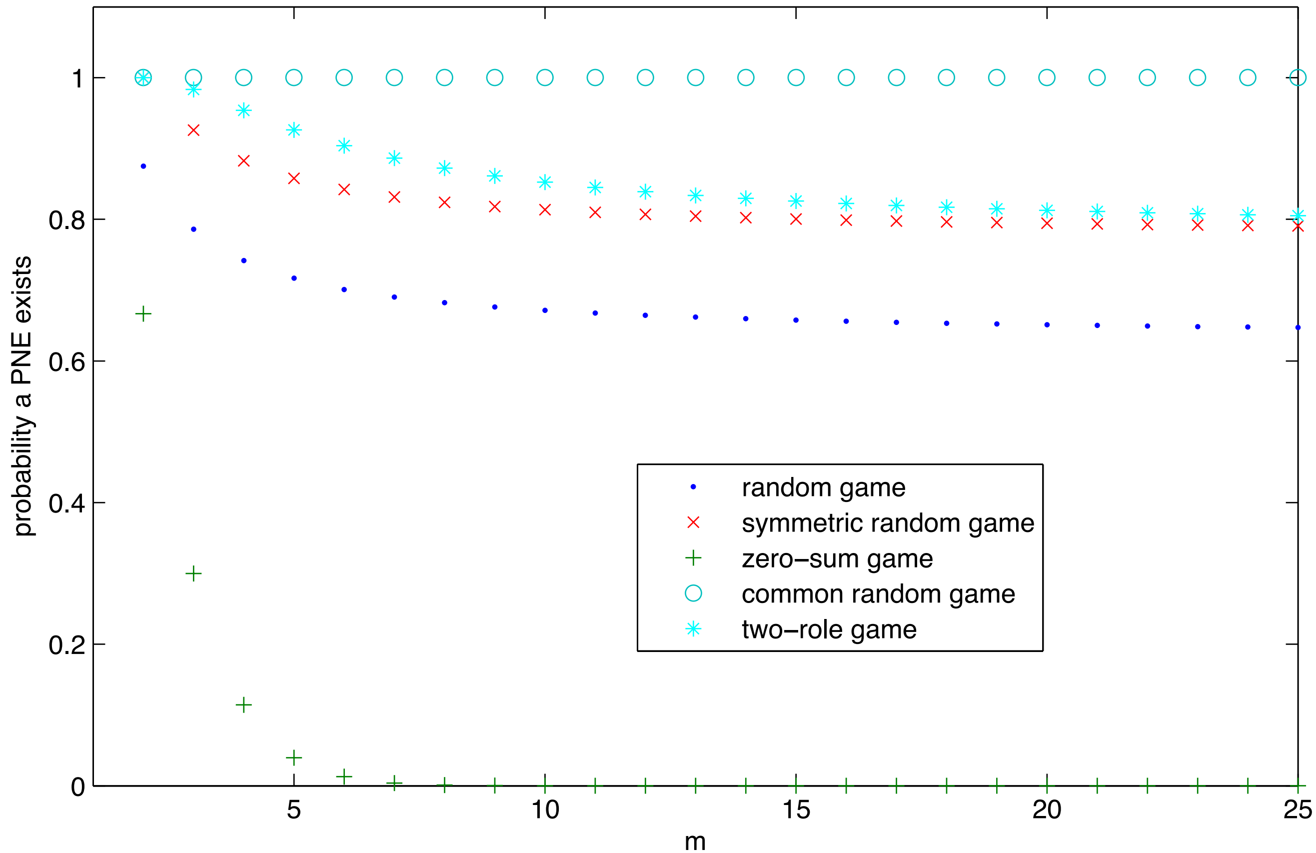
* common payoffs game

$$P(k,m) = \frac{(m!)^2}{((m-k)!)^2 k!} \sum_{j=0}^{m-k} (-1)^j \nu(j, m-k) \frac{(2m-1-k-j)!}{(2m-1)!}$$

* two-role game

$$P(k,m^2) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(m!)^2}{(k-2j)! j! 2^j} \sum_{i=0}^{\lfloor \frac{m-k}{2} \rfloor} \frac{(-1)^i}{i! 2^i} \sum_{l=0}^{m-k-2i} \frac{(-1)^l}{m^{2k+4i+2l} l! ((m-k-2i-l)!)^2}$$

How often does PNE exist?



Asymptotic behavior of number of PNEs for large m

* random game:

$$P(k,m) \to \frac{e^{-1}}{k!}$$

* symmetric random game:

$$P(k,m) \to e^{-1.5} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{j!2^j(k-2j)!}$$

* zero-sum game:

$$P(0,m) \to 1, \qquad P(1,m) \to 0$$

* common payoffs game:

$$P(k,m) \to \frac{m^k}{k!2^k e^{m/2}}$$

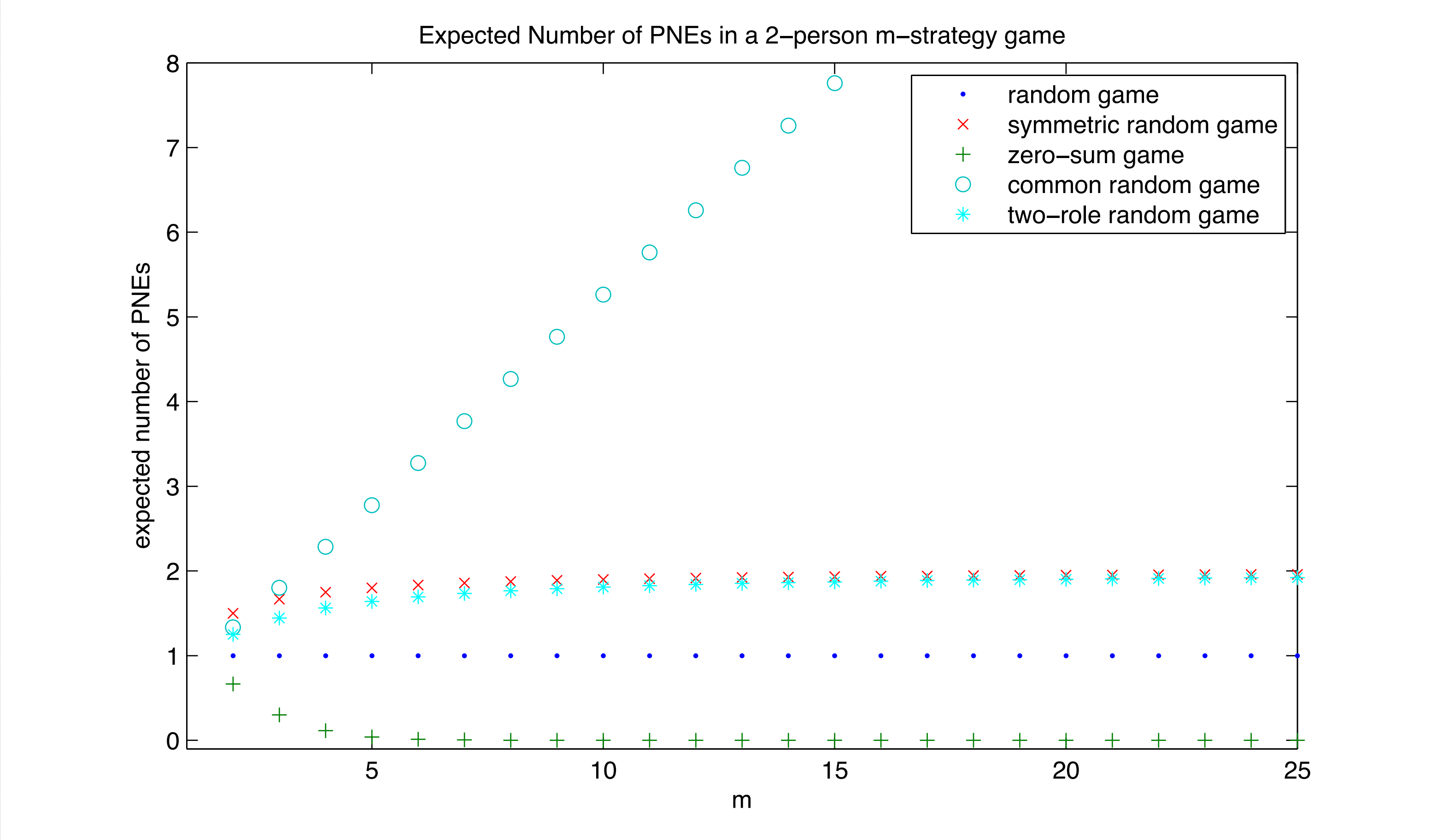
* two-role game

$$P(k,m^2) \to e^{-1.5} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{j!2^j(k-2j)!}$$

Expected the number of PNES

* random game: 1
* symmetric random game: $1\frac{m-1}{m}$
* zero-sum game: $\frac{(m!)^2}{(2m-1)!}$
* common random game: $\frac{m^2}{2m-1}$
* two-role game $1\frac{(m-1)^2}{m^2}$

Expected the number of PNES



How often is a PNE PPO?

- random game

$$\Pi = \int_0^1 \int_0^1 m^2 x^{m-1} y^{m-1} (1-(1-x)(1-y))^{(m-1)^2} dy dx$$

- symmetric random game

$$\Pi = \frac{m}{2m-1}(J_m + \frac{m-1}{m} K_m)$$

$$J_m = \int_0^1 m x^{2(m-1)} (1-(1-x)^2)^{(m-2)(m-1)/2} dx$$

$$K_m = 2 \int_0^1 \int_0^x m^2 x^{2m-3} y^{m-1} (x^2 + 2(1-x)y)^{(m-2)(m-3)/2} dy dx$$

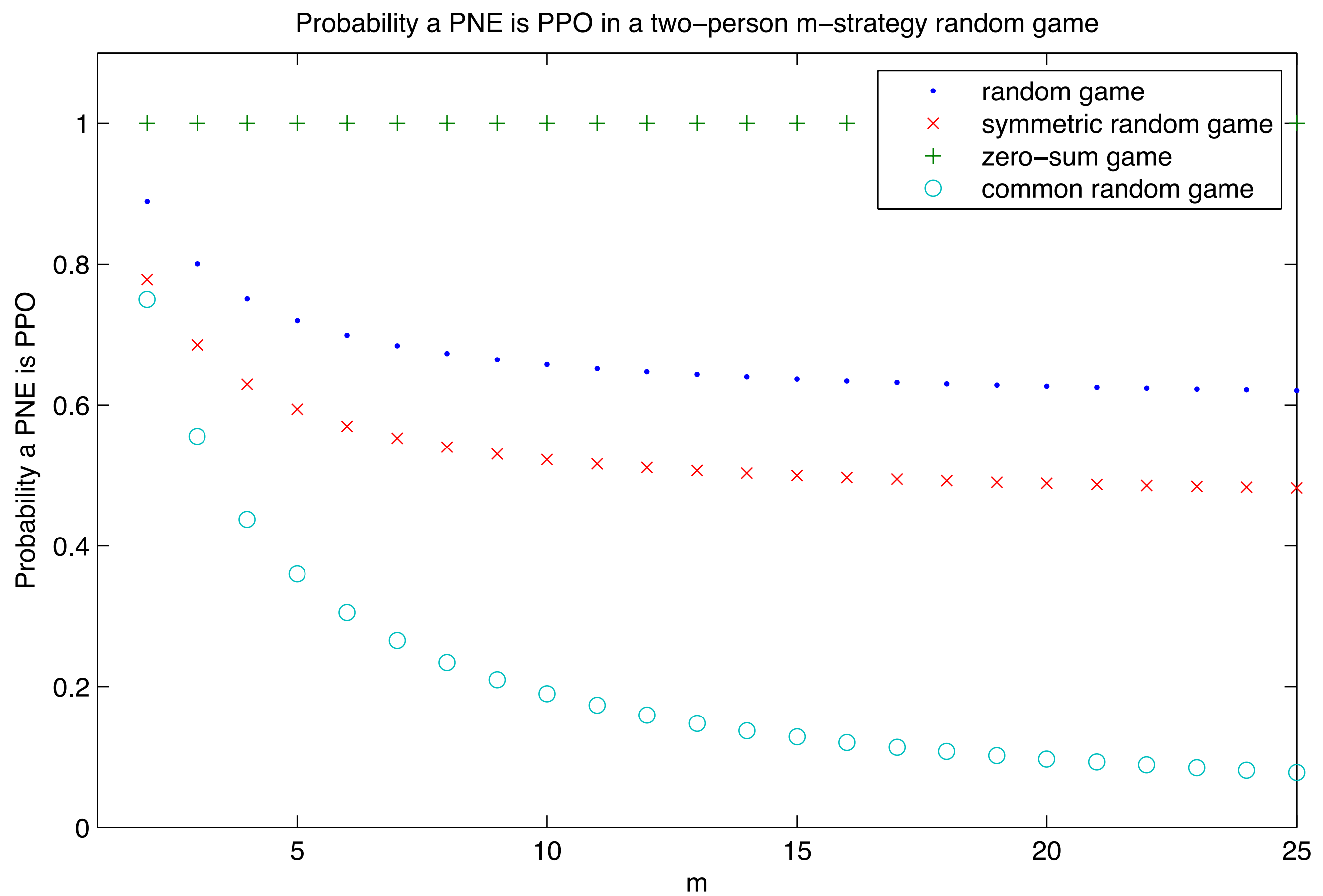
- zero-sum game

$$\Pi = 1$$

- common random game

$$\Pi = \frac{2m-1}{m^2}$$

How often is a PNE PPO?



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- Probability that a PNE is PPO is independent of distribution.
- As m, the number of strategies increases, cooperation becomes more favorable.
- As the correlation between payoffs increases, cooperation becomes more desirable.

n-person 2-strategy symmetric random games

- 2 strategies A and B, with payoff values

$$\vec{\alpha} = (\alpha_1, \cdots, \alpha_n), \qquad \vec{\beta} = (\beta_0, \beta_1, \cdots, \beta_{n-1})$$

- i* is PNE if

$$\alpha_i > \beta_{i-1}, \qquad \beta_i > \alpha_{i+1}$$

- Probability distribution of k, the number of PNEs.

$$P(k,n) = \frac{1}{2^n}\binom{n+1}{2k-1}, \qquad E(X) = \frac{n+3}{4}$$

n-person 2-strategy symmetric random games

- PNE 0* or n* are PPO with probability

$$\sum_{i=0}^{n-2}(-1)^i \binom{n-2}{i} 2^{n-2-i} \frac{2}{n+1+i}$$

- PNE 1* or (n-1)* are PPO with probability

$$\sum_{i=0}^{n-3}(-1)^i \binom{n-3}{i} \sum_{j=0}^{n-3-i} \binom{n-3-i}{j} \frac{4}{(n-1-j)(2+i+j)} \left(1 - \frac{1}{n+2+i}\right)$$

- PNE 2*, 3*,…, (n-2)* are PPO with probability

$$\sum_{i=0}^{n-4}(-1)^i \binom{n-4}{i} \sum_{j=0}^{n-4-i} \binom{n-4-i}{j} \frac{4}{(n-2-j)(2+i+j)} \left(1 - \frac{2}{n+1+i} + \frac{2}{(n+2+i)(n+1+i)}\right)$$

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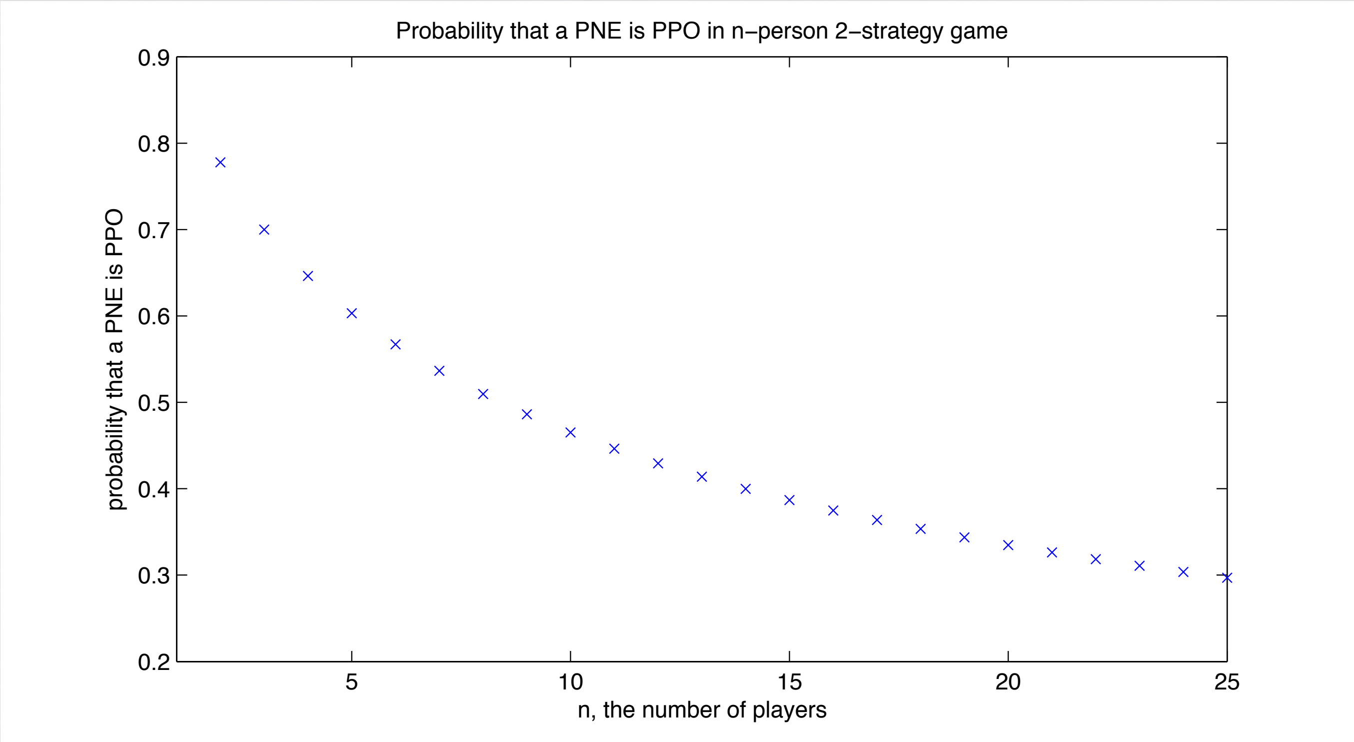

asymmetric case: Π(2,2)> Π(2,2,2)< Π(2,2,2,2)< Π(2,2,2,2,2)< Π(2,2,2,2,2,2)

To be continued…

* Expected gain from cooperation.
* Evolution of cooperation in repeated finite random games.