BIRS workshop 10w5019

Integrable and stochastic Laplacian growth in modern mathematical physics

Final Report

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This workshop took place Oct 31–Nov 5 2010 and consisted of approximately 35 participants representing a broad spectrum of scientific disciplines, from pure and applied mathematicians to physicists. The central topic was Laplacian growth and its various manifestations in different scientific and mathematical areas. The event was a natural sequel to the earlier BIRS worskshop:

July 15th-20th 2007: Workshop on "Quadrature domains and Laplacian growth in modern physics", Banff International Research Station (Pacific Institute for Mathematical Sciences), Banff, Canada.

In the 2010 workshop, several new ideas and theoretical connections which became apparent during the 2007 workshop were explored and developed. The emphasis in the 2010 workshop was in many ways complementary to the earlier workshop; in 2010, more emphasis was placed on the mathematical connection of Laplacian growth and potential theory with stochastic growth problems. This has already led to ongoing collaborations and new ideas.

Among the participants were several PhD students, most of whom were invited to give shorter 30 minute presentations as part of our program.

The following pages include extended abstracts describing the content of the program of talks and presentations that took place during the workshop.

Integrable structures in the relativistic theory of extended objects

Jens Hoppe

A dynamical symmetry, as well as special diffeomorphism algebras generalizing the Witt-Virasoro algebra, related to Poincare-invariance and crucial with regard to quantisation, questions of integrability, and M(atrix) theory, were recently found to exist in the theory of relativistic extended objects of any dimension (and have been presented in my talk).

In a second part of my talk, I discussed the singularity formation for strings moving in a plane. Self-similar string solutions in a graph representation were presented, near the point of singularity formation.

Computation and calculation of some free boundary Hele-Shaw flows

N. Robb McDonald

The time-dependent evolution of source-driven Hele-Shaw free boundary flows in the presence of an obstacle are computed numerically. The Baiocchi transformation is used to convert the Hele-Shaw Laplacian growth problem into a free boundary problem for a streamfunction-like variable $u(x, y, t)$ governed by Poisson's equation (with constant right hand side) with the source becoming a point vortex of strength linearly dependent on time. On the free boundary both u and its normal derivative vanishes, and on the obstacle the normal derivative of u vanishes. Interpreting u as a streamfunction, at a given time the problem becomes that of finding a steady patch of uniform vorticity enclosing a point vortex of given strength such that the velocity vanishes on the free boundary and the tangential velocity vanishes on the obstacle. A combination of contour dynamics and Newton's method is used to compute such equilibria. By varying the strength of the point vortex these equilibria represent a sequence of source-driven growing blobs of fluid in a Hele-Shaw cell.

The practicality and accuracy of the method is demonstrated by computing the evolution of Hele-Shaw flow driven by a source near a plane wall; a case for which there is a known exact solution. Other obstacles for which there are no known exact solutions are also considered, including a source both inside and outside a circular boundary, a source near a finite-length plate and the interaction of an infinite free boundary impinging on a circular disc.

An equation governing the evolution of a Hele-Shaw free boundary flow in the presence of an arbitrary external potential–generalized Hele-Shaw flow–is derived in terms of the Schwarz function $g(z, t)$ of the free boundary. This generalizes the well-known equation $\partial g/\partial t = 2\partial w/\partial z$, where w is the complex potential, which has been successfully employed in constructing many exact solutions in the absence of external potentials.

The new equation is used to re-derive some known explicit solutions for equilibrium and time-dependent free boundary flows in the presence of external potentials including those with singular potential fields, uniform gravity and centrifugal forces.

Some new solutions are also constructed which variously describe equilibrium flows with higher-order hydrodynamic singularities in the presence of electric point sources, unsteady solutions describing bubbles under the combined influence of strain and centrifugal potential.

Random normal matrices by Riemann-Hilbert problem

Seung-Yeop Lee

Consider an ensemble of normal matrices with a certain prescribed probability distribution. We study the induced distribution of the eigenvalues as the size of the matrices gets large. It has been known that the eigenvalues behave like the 2 dimensional Coulomb gas subject to a certain external potential. In this correspondence, the size of the matrix becomes the number of Coulomb particles, and one may consider the scaling limit where the number of particles grows with the strength of the external potential, linearly with a given ratio, say t. In this limit, the limiting density of the particles tends to be supported uniformly on a finite domain $D \in \mathbb{C}$, and D behaves like the non-viscous domain in ideal Hele-Shaw flow when one considers t as the Hele-Shaw time.

The above particle system is known to be a determinantal point process, and the system is closely related to the orthogonal polynomial with respect to an area integral (Weighted Bergman polynomial). Especially, all the probabilistic information is expressed in terms of the reproducing kernel, that can be obtained from the orthogonal polynomial. Therefore, we set our primary goal to obtaining the asymptotic information about the orthogonal polynomial.

A well-known technique for orthogonal polynomial on the real line (or any contour) is the nonlinear steepest descent analysis of the corresponding Riemann-Hilbert problem by Deift-Kricherbauer-McLaughlin-Venakides-Zhou. To apply this technique we redefine the orthogonal polynomial over area integral, in terms of a line integral over some contour. Then the standard technique applies to produce the strong asymptotics for the orthogonal polynomial.

To demonstrate the technique, we consider a specific toy model that has one logarithmic singularity in the external potential. This produces Joukowsky map for the domain D. This toy model, though very simple, contains the critical moment where the topology of the domain D changes. We showed that such criticality is described by a special solution of Painlevé II equation (as expected in literature). Along with the strong asymptotics for the orthogonal polynomial, we have confirmed the theorem by Ameur-Hakan-Makarov that the norm of the orthogonal polynomial is asymptotically related to the harmonic measure on $\mathbb{C} \setminus D$.

Unlike the orthogonal polynomial on the real line, the Christoffel-Darboux identity (that allows one to write the kernel in terms of only "a few" orthogonal polynomials) is more complicated to write down. In our toy model, we have written down the identity, and obtained some preliminary result on the spectral density (i.e. density of the Coulomb particles).

This project is a work in progress with Ferenc Balogh, Marco Bertola and Kenneth McLaughlin.

Asymptotics of the interface of Laplacian growth with multiple point sources

Michiaki Onodera (National Taiwan University)

We study the asymptotic behavior of the interface of a Hele-Shaw ow produced by the injection of fluid from infinitely many points at different speeds. We prove that, as time tends to infinity, the interface approaches the circle centered at the barycenter of the injection points with weights proportional to the injection rates. The distances from the barycenter to the boundary points are estimated both from above and below. The proof is based on a precise estimate of quadrature domains on the measure $\alpha\delta_i + \beta\delta_{-i}$ in the case where $\alpha, \beta > 0$ and $\alpha + \beta$ is sufficiently large. Hence $\delta_{\pm i}$ denote the Dirac measures at ±i respectively. Combining an induction argument with the estimate yields the result.

Large-time behavior of multi-cut solutions to Hele-Shaw flows

Yu-Lin Lin

In the case of zero surface tension Hele-Shaw flows driven by injection, it is found that there is a large set of functions which can give rise to global solutions to the Polubarinova-Galin equation driven by injection. In this talk, we assume solutions are global and hence obtain a most precise way of describing rescaling behavior in terms of some conserved quantities. These conserved quantities are called Richardson's complex moments. In this talk, we focus on the special set of solutions which are called multi-cut solutions and this set of solutions are the combination of rational functions and logarithmic functions.

In this talk, we first look at the polynomial solution case and show how each coefficient of these multi-cut solutions decays in terms of Richardson's complex moments and hence obtain large-time behavior of multi-cut solutions $f(\zeta, t)$ in terms of these conserved quantities. Next, we show that all global multi-cut solutions behave like polynomials and hence obtain similar results to polynomial cases. From this talk, we can see that this method is the most precise way of describing large-time boundary behavior to Hele-Shaw flows. One of the important step of the proof is to formulate ODEs for poles and branch points of $f(\zeta, t)$. Part of this talk is joint-work with B. Gustafsson.

Structure theorems on quadrature domains for subharmonic functions

Makoto Sakai

For a positive Radon measure on the Euclidean space having a compact support, we assign a positive measure called a partial balayage. By using the measure, we obtain the existence and nonexistence theorems on quadrature domains of the given measure for subharmonic functions with finite volume. We describe all quadrature domains with finite volume by using two open sets determined by the measure. As a measure, a quadrature domain with finite volume is uniquely determined.
We also discuss a positive Radon measure whose support may not be compact. In this case, a large number of quadrature domains for subharmonic functions are possible. We

introduce the generalized Newtonian potential of the measure and define the partial balayage of the measure replacing the Newtonian potential of a measure with compact support by the generalized Newtonian potential of the measure. Every quadrature domain can be describe by using the partial measure of a measure which is different from the given measure.

Schwarz symmetry principle and dynamical mother bodies

Tatiana Savin(a)

The talk consists of two separate parts which however have some connections from both mathematical and physical viewpoints. Both considered problems can be thought as applications of the Schwarz function.

First part is devoted to generalizations of the celebrated, point-to-point, Schwarz symmetry principle. This principle serves as a convenient tools in analysis and mathematical physics and its development has been attracting attention of many mathematicians. From the viewpoint of applications it is important to have an explicit reflection formula for every specific problem. One of the open questions is the following: for what partial differential equations, boundary conditions and spatial dimensions such a formula exists and what is the structure of this formula, in other words, whether it is a point to point formula or it has a more complicated structure, for example, a point to a finite set or a point to a continuous set.

It turns out that unfortunately the point-to-point reflection, holding for harmonic functions subject to the Dirichlet or Neumann conditions on a real-analytic curve in the plane, almost always fails for solutions to more general elliptic equations. We will discuss non-local, point-to-compact set, reflection operators for different elliptic equations subject to different boundary conditions.

The second part of the talk is a joint project with Alexander Nepomnyashchy (Technion). We will introduce dynamical mother bodies arising in an attempt to answer the question: what distribution of sinks allows the complete removal of a droplet with an algebraic boundary from a Hele-Shaw cell.

Indeed, it is well-known that in the framework of the internal Hele-Shaw problem the fluid can not be fully removed through a single sink because of the cusp formation before the moving boundary reaches the sink unless the initial domain is a circle with the sink located in its center. We give a definition of a dynamical mother body and use it for developing an algorithm of complete removal of a fluid droplet having algebraic boundary. To illustrate our theory we consider examples, where fluid can be completely removed through the sinks distributed along arcs of curves and/or points.

Originally, the concept of a (static) mother body was introduced by D. Zidarov, who studied gravitational potential of a family of heavy bodies producing the same gravitational field. His starting point was as follows: a sphere uniformly filled by masses generates the same gravitational field as the point mass of the same magnitude placed at the center of the sphere. Thus, this point mass is a minimal element of the infinite family of concentric balls, each member of which generates the same gravitational field outside of the biggest one. Using Poincare sweeping method one may start from an arbitrary body with a smooth boundary and try to construct the family of graviequivalent bodies. The minimal element for the family is called a mother body. It was considered by Gustafsson, Sakai and others (including the author). Here we specify and use the concept of a dynamical, time dependent, mother body for the internal Hele-Shaw problem.

Parametrization of the Loewner-Kufarev evolution in Sato's Grassmannian

Irina Markina and Alexander Vasil'ev

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The Virasoro algebra $\mathfrak{vir}$ plays a prominent role in modern mathematical physics, both in field theories and solvable models. It appears in physics literature as an algebra obeyed by the stress-energy tensor and as a Lie algebra of the conformal group, called the Virasoro-Bott group Vir, of the worldsheet in two dimensions. It is a unique central extension of the Lie algebra $\mathfrak{diff}S^1 \simeq VectS^1$ for the Lie-Fréchet group $\mathcal{D}iffS^1$ of sense-preserving diffeomorphisms of the unit circle S^1, and it is an infinite-dimensional real vector space. The group $\mathcal{D}iffS^1$ can be thought of as a group of reparametrizations of a closed string. The exponential map from the Lie algebra $\mathfrak{diff}S^1$ to the Lie-Fréchet group $\mathcal{D}iffS^1$ is neither injective nor surjective. However Kirillov and Yuriev in 1986 proposed to consider a Fréchet homogeneous manifold $\mathcal{D}iffS^1/S^1$ and proved that there exists a local bijective exponential map $\exp\colon Vect_0S^1 \to \mathcal{D}iffS^1/S^1$, where $Vect_0S^1 = VectS^1/\text{const}$.

Our main objects of interest will be the complexification of $\mathcal{D}iffS^1/S^1$ and $\mathcal{D}iffS^1$ given as

- $(\mathcal{D}iffS^1, H^{(1,0)})$, where $H^{(1,0)} \oplus H^{(0,1)} = \text{corank}_1(VectS^1 \otimes \mathbb{C})$;
- $(\mathcal{D}iffS^1/S^1, T^{(1,0)})$, where $T^{(1,0)} \oplus T^{(0,1)} = Vect_0S^1 \otimes \mathbb{C}$.

An important fact is the following relation of these complexified objects to analytic functions. Let us denote by $\mathcal{F}$ the class of all smooth univalent functions in the unit disk $\mathbb{D}$, $f(z) = z(a_0 + a_1 z + \dots)$, by $\mathcal{F}_1$ its subclass defined by the normalization of the conformal radius of $f(\mathbb{D})$ to be 1, and we denote by $\mathcal{F}_0$ the subclass of $\mathcal{F}_1$ of normalized univalent functions in $\mathbb{D}$, $f(z) = z(1 + c_1 z + \dots)$. Then there is the biholomorphic equivalence.

- $(\mathcal{D}iffS^1/S^1, T^{(1,0)}) \leftrightarrow \mathcal{F}_0$,

and the bijective Cauchy-Riemann map

- $(\mathcal{D}iffS^1, H^{(1,0)}) \leftrightarrow \mathcal{F}_1$;

The Fourier basis vectors $v_n = -ie^{in\theta}\frac{d}{d\theta}$ in $Vect_0S^1 \otimes \mathbb{C}$, therefore, are mapped onto the vectors $L_n[f](z)$ of the tangent space $T_f\mathcal{F}_0$, $n \in \mathbb{Z}$. The vectors $L_n[f](z)$ are simple for $n > 0$, $L_n[f](z) = z^{n+1}f'(z)$, and for $n = 0$, $L_0[f](z) = zf'(z) - f(z)$. In the first canonical quantization they are the Virasoro generators

$$L_n = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k},$$

where $\partial_k = \partial/\partial c_k$, for $n > 0$, and $L_0 = \sum\limits_{k=1}^{\infty} kc_k\partial_k$.

The next part of our research deals with the Löwner-Kufarev evolution given as an infinite-dimensional control system. Any univalent function $f : U \to \Omega$, $f(z) = z + c_1z^2 + \dots$ (from the class $Sb \supset \mathcal{F}_0$) can be represented as the limit $f(z) = \lim\limits_{t\to\infty} e^t w(z,t)$. The function $\zeta = w(z,t)$ normalized as $w(z,t) = e^{-t}z\left(1 + \sum\limits_{n=1}^{\infty} c_n(t)z^n\right)$, satisfies the Löwner-Kufarev equation

$$\frac{dw}{dt} = -wp(w,t),$$

with the initial condition $w(z,0) = z$. The control function $p(z,t) = 1 + u_1z + \dots$ is measurable in $t \in [0,\infty)$ for every $z \in \mathbb{D}$, analytic in $\mathbb{D}$ for almost all $t \in [0,\infty)$, and has positive real part in $\mathbb{D}$. We shall consider the functions p which are integrable in $t \in [0,\infty)$ and smooth on S^1. Then the contour evolution given by the curve $f(z,t) = e^t w(z,t)$ in $\mathcal{F}_0$ generates an evolution of smooth contours $f(S^1,t)$.

Theorem. *Consider the Hamiltonian function on the cotangent bundle to* $\mathcal{F}_0$

$$H(f,\psi) = \int\limits_{z\in S^1} f(z,t)\big(1 - p(e^{-t}f(z,t),t)\big)\bar{\psi}(z,t)\frac{dz}{iz}, \quad f \in \mathcal{F}_0, \quad \psi \in T^*\mathcal{F}_0.$$

Then the conserved quantities of the Löwner-Kufarev evolution defined by H, *in their first quantizations coincide with the Virasoro generators* $L_n[f]$.

The final part is devoted to a parametrization of the Löwner-Kufarev evolution in Sato's Grassmannian. The idea consists of considering a trivial bundle of smooth Grassmannians Gr_∞ over the cotangent bundle $T^*\mathcal{F}_0$ to $\mathcal{F}_0$. Inside each Grassmannian we construct special points which form a principal bundle $\mathfrak{E} = (\mathcal{F}_0, \mathfrak{W})$ over $\mathcal{F}_0$ with fiber $\mathfrak{W}$ which is a countable family $\mathfrak{W} = \{W^{(-n)}\}_{n=0}^{\infty}$ of linear subspaces of a Hilbert space $L^2(S^1 \to \mathbb{C}) \cap C^\infty$. These special points $W^{(-n)}$ are graphs of rapidly decaying Hilbert-Schmidt operators. The graphs are defined by the Virasoro generators L_{-n}, $n \geq 0$. Making use of the standard basis $\{z^k\}_{k=-\infty}^{+\infty}$ of $L^2(S^1 \to \mathbb{C})$ we consider the polarization

$$\begin{aligned}\mathcal{H}_+ &= spn_{\mathbb{C}}\{z, z^2, z^3, \dots\},\\ \mathcal{H}_- &= spn_{\mathbb{C}}\{1, z^{-1}, z^{-2}, \dots\}.\end{aligned}$$

Theorem. *The Löwner-Kufarev evolution in $\mathcal{F}_0$ gives a curve in the principal bundle $\mathfrak{E}$, such that it traces sections in each connected component $U^{(-n)}_{\mathcal{H}_+}$ of the neighbourhood $U_{\mathcal{H}_+}$ of the point $\mathcal{H}_+ \in Gr_\infty$.*

"Fingerprints" of the Two Dimensional Shapes and Lemniscates

Dmitry Khavinson

The newly emerging field of vision and pattern recognition often focuses on the study of two dimensional "shapes", i.e. simple, closed smooth curves. A common approach to describing shapes consists in defining a "natural" embedding of the space of curves into a metric space and studying the mathematical structure of the latter. Another idea that has been pioneered by Kirillov and developed recently among others by Mumford and Sharon consists of representing each shape by its "fingerprint", a diffeomorphism of the unit circle. Kirillov's theorem states that the correspondence between shapes and fingerprints is a bijection modulo conformal automorphisms of the disk. In this talk we discuss the recent joint work with P. Ebenfelt and Harold S. Shapiro outlining an alternative interpretation of the problem of shapes and Kirillov's theorem based on finding a set of natural and simple fingerprints that is dense in the space of all diffeomorphisms of the unit circle. This approach is inspired by the celebrated theorem of Hilbert regarding approximation of smooth curves by lemniscates. We shall outline proofs of the main results and discuss some interesting function-theoretic ramifications and open questions regarding possibilities of numerical applications of this idea (joint with Peter Ebenfelt and Harold S. Shapiro).

Non-univalent solutions of the Polubarinova-Galin equation

Björn Gustafsson

We discuss solutions of the Polubarinova-Galin (PG)

$$\mathrm{Re}\,[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}] = q(t) \quad (|\zeta| = 1),$$

and Löwner-Kufarev (LK)

$$\dot{f}(\zeta,t) = \zeta f'(\zeta,t) P_f(\zeta,t) \quad (|\zeta| < 1),$$

equations in the case that the "mapping" function $f(\zeta, t)$ from the unit disk to the fluid domain of a Hele-Shaw blob, subject to injection ($q(t) > 0$) or suction ($q(t) < 0$) at the origin, is no longer univalent. Here $P_f(\zeta, t)$ denotes the Poisson-Schwarz integral of $q(t)|f'(\zeta, t)|^{-2}$. Of special interest is the "Abelian" case, i.e., when

$$f'(\zeta, t) = b(t)\frac{\prod_{k=0}^{m}(\zeta - \omega_k(t))}{\prod_{j=0}^{n}(\zeta - \zeta_k(t))}$$

($m \geq n$), in other words, when $f(\zeta, t)$ is an Abelian integral on the Riemann sphere.

When f is univalent, or locally univalent, the PG and LK equations are equivalent. However, when f' has zeros inside the unit disk LK is strictly stronger than PG, in fact LK is equivalent to PG plus the additional condition that the functions $f(\cdot, t)$ make up a subordination chain, equivalently lift to a fixed (independent of t) Riemann surface R over $\mathbb{C}$. On R, which inherits a Riemannian metric from $\mathbb{C}$, the variational inequality weak solution (see reference below) makes sense. By exploiting this solution we are able to show, in the injection case, that the LK equation has a global in time solution in the appropriate weak sense. The main difficulty lies in the fact that the Riemann surface R does not exist in advance, but has to be constructed along with the solution. It has to be continuously enlarged, and every time a zero of f' reaches the unit circle it has to be updated with new branch points in order to prevent the solution domain to become multiply connected, and then outside the scope of the LK equation. In particular, the weak solution will not be smooth when zeros of f' cross the unit circle (as can also be realized directly from the PG and LK equations).

Abelian solutions remain Abelian for all time, but the structure may change during the evolution, namely when zeros $\omega_k(t)$ cross the unit circle. The process is not fully understood at present, but the examples we have been able to elaborate give the impression that the appearence of a new branch point on R is accompanied by the creation of a pair of a zero and a pole of f'.

The talk is based on joint work with Yu-Lin Lin, Taipei.

Reference: B. Gustafsson, A. Vasilév, *Conformal and Potential Analysis in Hele-Shaw Cells*, Birkhäuser, 2006.

Topological recursion, β-ensembles and quantum algebraic geometry

Marchal Olivier

During this talk, I presented the topological recursion recently developped by Eynard and Orantin to solve the general expansion of matrix models. In particular, I reviewed the method in the case of the one-matrix model where I presented first the general recursion scheme and then an application on a example given during a previous talk by P. Bleher, namely, the case where the potential is cubic: $V(x) = \frac{x^2}{2} - ux^3$.

In this case, when the parameter u is close to zero, the spectral curve obtained is of genus 0 and I showed how the topological recursion could be easily implemented concretely. Then I naturally switched to the case of the two-matrix model, gave the general stting and made the connection with Laplacian growth. In particular, since some cases of Laplacian growth can be modeled by a formal two-normal matrix model, I presented the way to derive the spectral curve in this context, by application of the Eyanrd Orantin formula. In order to illustrate this setting, I gave the spectral curve corresponding to the simple case presented earlier during the week by Seung Lee and given by $V(z) = -c\ln(z-a)$. In this example, getting the boundary of the fluid is trivial and I showed on a picture of such aircraft wings contours.

Then I switched to the second part of my talk which consisted in the generalization of the topological recursion scheme in the case of β-ensemble. First I recalled to the audience the definition of the model, which can be viewed as an extension of the traditionnal hermitian matrix model. Then I gave the so-called loop equations in that case and presented a method to solve them. In this context, the loop equations are no longer algebraic, but diffential. Therefore, the solution implies the introduction of a "quantum" Riemann surface and to generalize in this context all standard tool of algebraic geometry. Hence I presented the links between Stokes phenomenon, subdominant solutions and the way to define: genus, cycles, holomorphic differentials, third kind differential, Bergman kernel in the context of a "quantum" curve. Eventually, I gave the generalization of the topological recursion scheme for β ensemble providing a solution to the loop equations for β-ensembles. To conclude, I gave some unsolved problems that still miss to have a complete understanding of β ensmbles.

An assembly of steadily translating bubbles in a Hele-Shaw channel

Christopher Green

New solutions for any finite number of steadily translating bubbles in a Hele–Shaw channel were presented. The problem in consideration is a paradigmatic example of a Laplacian growth process and can be shown to be a special type of Riemann-Hilbert problem on a multiply-connected circular domain. The solutions can be written down explicitly, and elegantly, in terms of a special transcendental function known as the SchottkyKlein prime function. In doing so, our solutions generalize exact single bubble solutions found first by Taylor & Saffman, Tanveer, and Vasconcelos and Crowdy (references below).

We presented various plots showing different bubble configurations at some prescribed speed. We qualitatively observe the interaction effects of the bubbles and also how the configuration adjusts due to the effects from the two channel sidewalls. As a check on these solutions, we verified the Taylor-Saffman limit by measuring the aspect ratios for our small bubbles and we found that they are indeed in agreement with Taylor and Saffman's result.

This work has a number of important aspects. We have solved another version of a classical Laplacian growth nonlinear free boundary value problem and therefore adds to the body of prior work on bubbles in Hele–Shaw channels and cells. Our problem formulation makes no *a priori* assumptions on the geometrical arrangement of the bubbles, and is therefore very general. Our solution is neatly expressed in terms of the Schottky-Klein prime function which is both mathematically concise and also convenient for computational purposes, particularly now that software is freely available for its computation (reference to website below) [joint work with D. Crowdy].

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Laplacian growth, elliptic growth, and singularities of the Schwarz potential

Erik Lundberg

The Schwarz function has played an elegant role in understanding and in generating new examples of exact solutions to the Laplacian growth (or "Hele-Shaw") problem in the plane. The guiding principle in this connection is the fact that "non-physical" singularities in the "oil domain" of the Schwarz function are stationary, and the "physical" singularities obey simple dynamics.

This talk discussed the same principle for Laplacian growth in higher dimensions and in a non-homogeneous environment ("elliptic growth"), Each case has simple dynamics of singularities of the *Schwarz potential*, a generalization of the Schwarz function introduced by D. Khavinson and H.S. Shapiro,

Given a domain Ω bounded by a non-singular, algebraic hypersurface Γ, the *Schwarz potential* is the unique solution to the following Cauchy problem for Laplace's equation:

$$\begin{cases} \Delta w = 0 \text{ near } \Gamma \\ w|_\Gamma = \frac{1}{2}||\mathbf{x}||^2 \\ \nabla w|_\Gamma = \mathbf{x}, \end{cases}$$

Based on this connection new exact solutions were described for both cases: the higher-dimensional Laplacian growth and the case of Elliptic growth. $\mathbb{C}^n$-techniques for holomorphic PDEs were presented as an important means of locating singularities of the Schwarz potential.