Structure theorems on quadrature domains for subharmonic functions

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1. $Q(\mu, SL^1)$ for μ with compact support
2. $\overline{Q}(\mu, SL^1)$ for μ whose support may not be compact

μ : a positive Radon measure on $\mathbb{R}^d$

$\mathbb{R}^d$: the d-dimensional Euclidean space, $d \geq 3$

a measure : a positive Radon measure on $\mathbb{R}^d$

an open set : an open subset of $\mathbb{R}^d$

λ : the d-dimensional Lebesgue measure on $\mathbb{R}^d$

1. $Q(\mu, SL^1) : \lambda(\Omega) < +\infty$

μ : a measure with compact support, $\mu \neq 0$

An open set $\Omega \neq \emptyset$ is called a *quadrature domain of* μ *for subharmonic functions* if

(i) $\mu|\Omega^c = 0$;
(ii) $\forall s \in SL^1(\Omega),\ \int s d\mu \leq \int_\Omega s d\lambda$.

$\Omega^c = \mathbb{R}^d \setminus \Omega$

$\mu|\Omega^c$: the restriction of μ onto Ω^c

$S(\Omega) = \{s : -\infty \leq s(x) < +\infty,\ \Delta s \geq 0 \text{ in } \Omega\}$

$SL^1(\Omega) = \{s \in S(\Omega) : \int_\Omega |s| d\lambda < +\infty\}$

$Q(\mu, SL^1)$: the family of quadrature domains Ω of μ for subharmonic functions satisfying $\lambda(\Omega) < +\infty$

s : harmonic in $\Omega \implies \int s d\mu = \int_\Omega s d\lambda$

The Newtonian potential $N\mu$ of μ :

$$N\mu(y) = \int \frac{1}{|x-y|^{d-2}} d\mu(x)$$

$N\mu$ is superharmonic in $\mathbb{R}^d$ and $-(1/A_d)\Delta N\mu = \mu$.

$A_d = d(d-2)\lambda(B_1)$

Proposition 1.1.

Ω : a bounded open set

$\Longrightarrow$

$\Omega \in Q(\mu, SL^1) \Longleftrightarrow$

$N(\lambda|\Omega)(y) \leq N\mu(y)\,(y \in \mathbb{R}^d)$ and

$N(\lambda|\Omega)(y) = N\mu(y)\,(y \in \Omega^c)$

Quadrature measures

$$\mathcal{S} = S(\mathbb{R}^d) - S(\mathbb{R}^d)$$

$$\mathcal{L}_\lambda(N\mu, \mathbb{R}^d) = \{v \in \mathcal{S} : v \leq N\mu, -(1/A_d)\Delta v \leq \lambda\}$$

$$R_\lambda(x; N\mu, \mathbb{R}^d) = \sup\{v(x) : v \in \mathcal{L}_\lambda(N\mu, \mathbb{R}^d)\}$$

Proposition 1.2.

$\mathcal{L}_\lambda(N\mu, \mathbb{R}^d) \neq \emptyset$ and $R_\lambda(x; N\mu, \mathbb{R}^d) \in \mathcal{L}_\lambda(N\mu, \mathbb{R}^d)$. $R_\lambda(x; N\mu, \mathbb{R}^d)$ is superharmonic in $\mathbb{R}^d$ and of class C^1 in $\mathbb{R}^d$.

$\beta_\lambda(\mu, \mathbb{R}^d) = -(1/A_d)\Delta R_\lambda(x; N\mu, \mathbb{R}^d)$

$\beta_\lambda(\mu, \mathbb{R}^d)$: the quadrature measure of μ

$\beta_\lambda = \beta_\lambda(\mu, \mathbb{R}^d)$

$0 \leq \beta_\lambda \leq \lambda$

$\Omega_0(\mu) = \{x \in \mathbb{R}^d : N\beta_\lambda(x) < N\mu(x)\}$

$\Omega^0(\mu) = \cup\{U$: an open set satisfying $\beta_\lambda | U = \lambda | U\}$

$\qquad = (\mathrm{supp}\,(\lambda - \beta_\lambda))^c$

$\Omega_0(\mu) \subset \Omega^0(\mu)$

Proposition 1.3.

$$R_\lambda(x) = R_\lambda(x; N\mu, \mathbb{R}^d)$$
$$\beta_\lambda = \beta_\lambda(\mu, \mathbb{R}^d)$$
$$\Omega_0 = \Omega_0(\mu)$$
$$\Omega^0 = \Omega^0(\mu)$$
$$\Longrightarrow$$

(1) $\operatorname{supp}\beta_\lambda \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x, \operatorname{supp}\mu) \le r(\mu)\}$,
$\|\mu\| = \lambda(B_{r(\mu)})$;

(2) $R_\lambda = N\beta_\lambda$, $\|\beta_\lambda\| = \|\mu\|$;

(3) $\forall\Omega$: an open set satisfying $\Omega_0 \subset \Omega \subset \Omega^0$,
$\beta_\lambda = \lambda|\Omega + \mu|\Omega^c$.

Theorem 1.4.

Ω : an open set

$\beta_\lambda = \beta_\lambda(\mu, \mathbb{R}^d)$

$\Omega_0 = \Omega_0(\mu)$

$\Longrightarrow$

$\Omega \in Q(\mu, SL^1) \Longleftrightarrow \beta_\lambda = \lambda|\Omega \neq 0$ and $\Omega_0 \subset \Omega$

Theorem 1.5.

$\Omega^0 = \Omega^0(\mu)$
$\Omega_0 = \Omega_0(\mu)$
$\Longrightarrow$

(1) $Q(\mu, SL^1) \neq \emptyset \Longleftrightarrow \Omega^0 \neq \emptyset$ and $\mu|(\Omega^0)^c = 0$
(2) $Q(\mu, SL^1) \neq \emptyset \Longrightarrow$

$Q(\mu, SL^1) = \{\Omega$: an open set satisfying
$\Omega_0 \subset \Omega \subset \Omega^0$ and $\lambda(\Omega^0 \setminus \Omega) = 0\}$

2. $\overline{Q}(\mu, SL^1) : \lambda(\Omega) \leq +\infty$

μ : a measure whose support may not be compact, including $\mu = 0$

An open set $\Omega \neq \emptyset$ is called a *quadrature domain of* μ *for subharmonic functions* if

(i) $\mu | \Omega^c = 0$;

(ii) $\forall s \in SL^1(\Omega), \int s d\mu \leq \int_\Omega s d\lambda$.

We don't assume that $\lambda(\Omega) < +\infty$.

It may be $\lambda(\Omega) = +\infty$ even if μ has a compact support.

$\overline{Q}(\mu, SL^1)$: the family of quadrature domains Ω of μ for subharmonic functions.

$Q(0, SL^1) = \emptyset$, whereas there are many $\Omega \in \overline{Q}(0, SL^1)$. They are called null quadrature domains.

$\Omega = \mathbb{R}^d \Longrightarrow SL^1(\mathbb{R}^d) = \{0\}$

$\Omega \neq \mathbb{R}^d \Longrightarrow$ There are many $s \in SL^1(\Omega)$.

$y \in \Omega^c,\ a, b > 0,\ d - 2 \leq p < d \Longrightarrow$

$$s(x) = \max\left\{\frac{a}{|x-y|^p} - b, 0\right\} \in SL^1(\Omega)$$

Lemma 2.1.

$\Omega \in \overline{Q}(\mu, SL^1)$ and $y \in \Omega^c$
$\Longrightarrow$

$$\int_{\{x\in\mathbb{R}^d:|x|>1\}} \frac{d\mu(x)}{|x|^{d+1}} < +\infty,$$

$$\int_{\{x\in\mathbb{R}^d:|x-y|<1\}} \frac{d\mu(x)}{|x-y|^{d-2}} < +\infty$$

Genl. Newtonian potentials

$$N(x,y) = \frac{1}{|x-y|^{d-2}}, \quad N\mu(y) = \int N(x,y)d\mu(x)$$

$n_d = d(d+3)/2$
$\{y_n\} = \{y_1, \ldots, y_{n_d}\}$: a set of n_d distinct points
$\forall B,\ \exists \{y_n\} \subset B$,
$\mathcal{N}(x,y) = N(x,y) + \sum_{n=1}^{n_d} c_n(y) N(x, y_n)$ satisfies
$\mathcal{N}(x,y) = O(\frac{1}{|x|^{d+1}})$ in a neighborhood of ∞
for fixed y.
$c_n(y) = c_n(y; y_1, \ldots, y_{n_d})$ is a harmonic polynomial of degree 2 as a function of y.

We fix $\{y_n\}$ and set

$\mathcal{M}(\{y_n\}) = \{\mu$: a measure satisfying

$$\int_{\{x\in\mathbb{R}^d:|x|>1\}} \frac{d\mu(x)}{|x|^{d+1}} < +\infty,$$

$$\int_{\{x\in\mathbb{R}^d:|x-y_n|<1\}} \frac{d\mu(x)}{|x-y_n|^{d-2}} < +\infty$$

for every $y_n \in \{y_n\}\}$.

The generalized Newtonian potential $\mathcal{N}\mu$ of $\mu \in \mathcal{M}(\{y_n\})$:

$$\mathcal{N}\mu(y) = \int \mathcal{N}(x,y)d\mu(x)$$

$\Omega \in \overline{Q}(\mu, SL^1)$ and $\{y_n\} \subset \Omega^c$
$\Longrightarrow$
$\mu \in \mathcal{M}(\{y_n\})$

$\lambda \in \mathcal{M}(\{y_n\})$

$\lambda|\Omega \in \mathcal{M}(\{y_n\})$ for every open set Ω.

Proposition 2.2.

$\mu \in \mathcal{M}(\{y_n\})$
$\Omega \neq \emptyset$: an open set satisfying $\{y_n\} \subset \Omega^c$
$\Longrightarrow$

$\Omega \in \overline{Q}(\mu, SL^1) \Longleftrightarrow$
$\mathcal{N}(\lambda|\Omega)(y) \leq \mathcal{N}\mu(y)(y \in \mathbb{R}^d)$ and
$\mathcal{N}(\lambda|\Omega)(y) = \mathcal{N}\mu(y)(y \in \Omega^c)$

We define $\beta_\lambda(\mu, \mathbb{R}^d)$, $\Omega_0(\mu)$ and $\Omega^0(\mu)$, replacing $N\mu$ with $\mathcal{N}\mu$.

Theorem 2.3.

$\mathcal{L}_\lambda(\mathcal{N}\mu, \mathbb{R}^d) \neq \emptyset$ and $\{y_n\} \subset \Omega_0(\mu)^c$
$\Longrightarrow$

(1) $\Omega^0(\mu) \neq \emptyset$ and $\mu|\Omega^0(\mu)^c = 0$
$\Longrightarrow \Omega^0(\mu) \in \overline{Q}(\mu, SL^1)$;

(2) $\Omega \in \overline{Q}(\mu, SL^1)$ and $\{y_n\} \subset \Omega^c$
$\Longrightarrow \beta_\lambda(\mu, \mathbb{R}^d)\big|\Omega^c = 0.$
In particular, $\Omega_0(\mu) \subset \Omega$ and $\lambda(\Omega^0(\mu) \setminus \Omega) = 0.$

Proposition 2.4.

$\mathcal{L}_\lambda(\mathcal{N}\mu, \mathbb{R}^d) \neq \emptyset$ and $\{y_n\} \subset \Omega_0(\mu)^c$
$\Longrightarrow$

(1) $\mu \in \mathcal{M}(\{y_n\}) \neq 0$ and λ are mutually singular
$\Longrightarrow \Omega_0(\mu),\ \Omega^0(\mu) \in \overline{Q}(\mu, SL^1)$;

(2) $U \neq \emptyset$: an open set, $\nu \in \mathcal{M}(\{y_n\})$: a measure on U, $\mu = \lambda|U + \nu$
$\Longrightarrow \Omega^0(\mu) \in \overline{Q}(\mu, SL^1)$;

(3) U_1 and U_2 : open sets with $U_1 \cup U_2 \neq \emptyset$,
$\mu = \lambda|U_1 + \lambda|U_2$
$\Longrightarrow \Omega^0(\mu) \in \overline{Q}(\mu, SL^1)$.

Proposition 2.5.

$U \in \overline{Q}(0, SL^1)$
$\mu \in \mathcal{M}(\{y_n\}) \neq 0$: a measure on U or
a measure such that $\mu|U^c$ and λ are mutually singular
$\mathcal{L}_\lambda(\mathcal{N}(\mu + \lambda|U), \mathbb{R}^d) \neq \emptyset$ and $\{y_n\} \subset \Omega_0(\mu + \lambda|U)^c$
$\Longrightarrow$

$$\Omega_0(\mu + \lambda|U),\ \Omega^0(\mu + \lambda|U) \in \overline{Q}(\mu, SL^1)$$

Proposition 2.6.

$\Omega \in \overline{Q}(\mu, SL^1)$ and $\{y_n\} \subset \Omega^c$
$\Longrightarrow$

$\exists U \subset \Omega$: an open set, $\lambda|\Omega = \beta_\lambda(\mu + \lambda|U, \mathbb{R}^d)$.

If $U \neq \emptyset$, then U is very close to a null quadrature domain.