Come Fly With Me:
A Look at Dependence from Above

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Flight Plan

1. Superstability and Limit Models
2. Limit Models and the Generic Pairs Conjecture
3. Dependence in AECs
4. Splintering in Dependent (f.o.) theories

$n_\lambda(T)$

Fix M a saturated model of cardinality λ^+. M is unique up to isomorphism, but M may have several non-isomorphic decompositions into $\langle M_i \mid i < \lambda^+\rangle$ each M_i of cardinality λ. The function $n_\lambda(T)$ counts these.

Defn. (Shelah)

$$n_\lambda(T) = \min\left\{ |M_i/\cong| \;\middle|\; \begin{array}{l} E \text{ is a club on } \lambda^+ \text{ and} \\ \langle M_i \mid i < \lambda^+\rangle \text{ is an inc} \\ \text{and continuous decomp} \\ \text{of } M \end{array} \right\}$$

Superstability for T first order

Thm. Let T be a complete first order theory. TFAE:

1. T is superstable.
2. $\kappa(T) = \aleph_0$.
3. Any increasing union of saturated models is saturated.
4. $n_\lambda(T) = 1$ for λ regular and $> |T|^+$.

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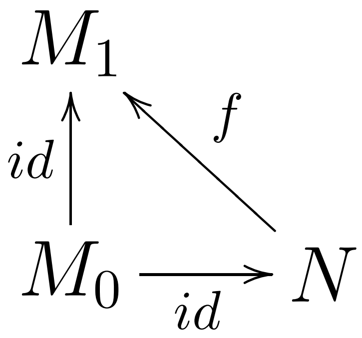
5. $\Rightarrow$ 4. Sketch on board.

Limit Models

Defn. M_1 is said to be universal over M_0 provided that for every N extending M_0 of cardinality $\|M_0\|$, there exists a $\mathcal{K}$-mapping $f : N \to M_1$ with $f \upharpoonright M_0 = id_{M_0}$.



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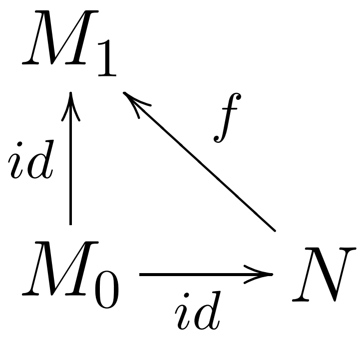

Defn. M^* is a (μ, α)-limit model over M provided there exists a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $\langle M_i \in \mathcal{K} \mid i < \alpha \rangle$ such that $M_0 = M$, $\bigcup_{i<\alpha} M_i = M^*$, $\|M_i\| = \mu$ and M_{i+1} is universal over M_i.

$\mathbf{n}_\lambda(T)$ for T Strictly Stable

$\mathbf{n}_\lambda(T) = 2.$
The two generic models are a saturated model and a $(\lambda, \aleph_0)$-limit model.

Generic Pairs Conjecture

Conj. (Shelah 877) Let T be a complete first order theory. Suppose that $\lambda = \lambda^{<\lambda} > |T|$ and $2^\lambda = \lambda^+$. T has NIP iff $\mathbf{n}_{\lambda,\lambda}(T) = 1$.

Define $\mathbf{n}_{\lambda,\lambda}(T)$ on board.

The unique pair of models in the cases that $\mathbf{n}_{\lambda,\lambda}(T) = 1$ is called the generic pair.

Dependence in AECs

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Fact: λ-stability (in the sense of counting Galois-types) implies λ-dependence.

In this case, the generic pair is (M_1, M_0) where M_1 is a (λ, λ)-limit model over M_0 and M_0 is also a (λ, λ)-limit model. If λ is regular, then these models are also saturated.

One use of dependence

Question. Does categoricity in λ and few models in λ^+ imply the existence of a model if cardinality λ^{++}?

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Some Propaganda. Excellence and The Main Gap

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Thm. (Sh 576) Let $\mathcal{K}$ be an AEC and $\lambda \geq \mathrm{LS}(\mathcal{K})$. Suppose little more than $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. If $\mathcal{K}$ is categorical in λ and λ^+ and in addition we know that $1 \leq \mathrm{I}(\lambda^{++}, \mathcal{K}) < 2^{\lambda^{++}}$ then $\mathcal{K}$ has a model of cardinality λ^{+++}.

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Thm. (GrVaVi '08) Suppose that $\lambda^+ < 2^\lambda < 2^{\lambda^+}$, then if $\mathcal{K}$ is λ-dependent, the answer is yes.

The mechanics

Defn. A type $p = \mathrm{tp}(a/M)$ is said to λ-splinter over N if and only if there are models $N_1, N_2 \in \mathcal{K}_\lambda$ and a $\mathcal{K}$-mapping f satisfying:

- $N \prec_{\mathcal{K}} N_1, N_2, \prec_{\mathcal{K}} M$
- $f : N_1 \cong N_2$
- f fixes N setwise and
- $\mathrm{tp}(a/N) = \mathrm{tp}(f(a)/N)$.
- $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$.

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Strong-splitting $\longrightarrow$ Splitting $\overset{\longrightarrow}{\not\leftarrow}$ Splintering

Compare this definition to splitting.

Splintering in NIP Theories p1

Invariance

Monotonicity: with respect to the domain

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Existence:
Let (M, N) be a generic pair of cardinality λ and $p \in S(M)$ non-algebraic so that the size of p's conjugacy class with respect to $\mathrm{Aut}(M)$ is $\leq \lambda$ (weaker than assuming invariant).
Then there exists $N' \prec M$ so that (M, N') is generic and p does not λ-splinter over N'.

Splintering in NIP Theories p2

Uniqueness and Extension: For $p \in S(M)$ which does not λ-splinter over N, if (M, N) contains a copy of the generic pair and M' is an extension of M of cardinality λ, then there is a unique $q \in S(M')$ extending p which does not λ-splinter over N.

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Continuity: For $\langle M_i \mid i < \alpha \rangle$ an increasing chain of models of length $< \lambda^+$ so that each pair of models contains a copy of the generic pair, then for $p \in S(\bigcup_{i<\alpha} M_i)$ if $p \upharpoonright M_i$ does not λ-splinter over M_0 for every i, then p does not λ-splinter over M_0.