Amalgamation properties for types in stable theories and beyond

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Introduction

We ask: when can certain systems of types be amalgamated, and when is the result unique?
This yields properties such as n-existence (or the n amalgamation property) and n-uniqueness.
Many natural algebraic examples have these properties, and they have nice consequences, such as:

Theorem

(De Piro, Kim, Young) If T is simple and has 5 complete amalgamation over models, then the existence of a hyperdefinable group configuration implies the existence of a hyperdefinable group.

More recently, Hrushovski showed that in stable T, 4-existence is equivalent to the eliminability of "generalized imaginary sorts" as well as the collapsing of certain definable groupoids.

The 3-amalgamation problem

3-amalgamation is about the following question:

Question

Given complete types $p_{12}(x_1, x_2)$, $p_{23}(x_2, x_3)$, and $p_{13}(x_1, x_3)$, when is $p_{12} \cup p_{23} \cup p_{13}$ consistent?

Equivalently: given any realization (a_1, a_2) of p_{12}, is there a common realization of the two types $p_{23}(a_2, x_3)$ and $p_{13}(a_1, x_3)$?

A minimal necessary requirement is <u>coherence</u>: $p_{12} \restriction x_1 = p_{13} \restriction x_1$, $p_{12} \restriction x_2 = p_{23} \restriction x_2$, and $p_{13} \restriction x_3 = p_{23} \restriction x_3$.

Failures of 3-amalgamation

Question

Given complete types $p_{12}(x_1, x_2)$, $p_{23}(x_2, x_3)$, and $p_{13}(x_1, x_3)$, when is $p_{12} \cup p_{23} \cup p_{13}$ consistent?

But many coherent triples of types cannot be amalgamated, e.g.:

If the universe is linearly ordered by "$<$," $x_1 < x_2 \in p_{12}$, $x_2 < x_3 \in p_{23}$, and $x_3 < x_1 \in p_{13}$;

Or in a theory with an equivalence relation E with exactly two classes, if $\neg E(x_i, x_j) \in p_{ij}$.

3-amalgamation in stable theories

Theorem

Suppose that T is stable, $B = \operatorname{acl}^{eq}(B)$, $a_1 \mathop{\smile\!\!\!\!\!\!\,|}_B a_2$, and the types $p_1(a_1, x_3)$ and $p_2(a_2, x_3)$ are nonforking extensions of a common type $p(x_3) \in S(B)$. Then there is a realization a_3 of $p_1(a_1, x_3) \cup p_2(a_2, x_3)$ such that $a_3 \mathop{\smile\!\!\!\!\!\!\,|}_B a_1 a_2$.

Proof.

Pick any a_3 realizing $p_1(a_1, x_3)$ such that $a_3 \mathop{\smile\!\!\!\!\!\!\,|}_{Ba_1} a_2$. By stationarity of p, $a_3 \models p_2(a_2, x_3)$. □

3-amalgamation in simple theories

Kim and Pillay generalized this to simple theories:

Theorem

Suppose that T is simple, $B = \mathrm{bdd}^{heq}(B)$, $a_1 \mathop{\smile\!\!\!\!\!\!\,|}_B a_2$, and the types $p_1(a_1, x_3)$ and $p_2(a_2, x_3)$ are nonforking extensions of a common type $p(x_3) \in S(B)$. Then there is a realization a_3 of $p_1(a_1, x_3) \cup p_2(a_2, x_3)$ such that $a_3 \mathop{\smile\!\!\!\!\!\!\,|}_B a_1 a_2$.

If T has elimination of hyperimaginaries (e.g. if T is supersimple), then $B = \mathrm{acl}^{eq}(B)$ is enough.

From 3 to n

In the terminology we are about to define, we have shown that all stable theories have 3-existence (or the 3-amalgamation property).

Now we will generalize this property from 3 to n.

n-amalgamation problems

Notation: $\mathscr{P}^-(n) = \{s : s \subsetneq \{1, \ldots, n\}\}$.

Definition

1. An <u>n-amalgamation problem</u> is a functor $A : \mathscr{P}^-(n) \to \mathscr{P}(\mathfrak{C})$, where the maps on the right are elementary.
2. A <u>solution</u> to an n-amalgamation problem A is an extension to a functor $A' : \mathscr{P}(n) \to \mathscr{P}(\mathfrak{C})$ (again with elmentary maps on the right).

With A as above and $s \subseteq t \subsetneq n$, let $\tau_t^s : A(s) \to A(t)$ be the image of the inclusion $s \subseteq t$.

Fuctoriality says: $\tau_u^t \circ \tau_t^s = \tau_u^s$ whenever this makes sense.

[Draw picture of 3-amalgamation problem]

Bases of amalgamation

Definition

If A is an n-amalgamation problem, then A is over B if $B = A(\emptyset)$ and for every $s \subsetneq n$, $\tau_s^{\emptyset}$ fixes B pointwise.

If we are looking at the solutions of A, clearly we may assume that A is over $A(\emptyset)$ (just shift the $A(s)$'s by appropriate autmorphisms).

From now on we always assume A is over $A(\emptyset)$.

Independent amalgamation problems

We write "i" for $\{i\}$ to simplify notation.

Definition

An n-amalgamation problem A is <u>independent</u> if for every s s.t. $\emptyset \neq s \subsetneq n$,

1. $\left\{\tau_s^i(A(i)) : i \in s\right\}$ is an $A(\emptyset)$-independent set;
2. If $t \subseteq s$, then $\tau_s^t(A(t)) = \text{bdd}^{heq}\left(A(\emptyset) \cup \left\{\tau_s^i(A(i)) : i \in t\right\}\right)$.

(If T is stable, replace "bdd^{heq}" by "acl^{eq}.")

So if the τ-maps are all inclusions, then $A(t) \downarrow_{A(t \cap u)} A(u)$.

Independent solutions to A are defined in a similar way.

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n-existence, n-uniqueness

Assume T is simple.

Definition

1. T has <u>n-existence</u> if every independent n-amalgamation problem has an independent solution.
2. T has <u>n-uniqueness</u> if every independent n-amalgamation problem A has at most one independent solution up to isomorphism over A.
3. T has <u>n-complete amalgamation</u> if for every k with $3 \leq k \leq n$, T has k-existence.

n-existence and n-uniqueness continued

So n-existence and n-uniqueness give two different ways to classify simple theories:

2-existence is true in any simple theory, by the existence of nonforking extensions;
3-existence is true in any stable theory, and all known examples of simple theories;
4-existence can fail even in stable theories (we'll see an example).

2-uniqueness is true in any stable theory (by stationarity of strong types), but fails for unstable simple T;
3-uniqueness can fail even for stable T.

Example: the random graph

The theory of a random graph is simple and has n-existence for all $n \geq 2$.

But if $A(i) = a_i$ (for $i = 1, 2$), then there are two solutions to the 2-amalgamation problem A: one with an edge between the points and one with no edge. So the random graph does not have 2-uniqueness.

Example: random hypergraphs

A hypergraph is a set with a symmetric ternary relation R.

The theory of a random "tetrahedron-free hypergraph" (where R cannot hold of every 3-element subset of a 4-element set) turns out to be simple.

However, it fails 4-existence: consider a 4-amalgamation problem where $A(\{i, j, k\})$ is a triple of points on which R holds.

Similarly, the n-simplex-free hyper^{n-3}graph is simple and has $(n-1)$-complete amalgamation but not n-existence.

“Q-example”

We now give an example of a stable T which fails 3-uniqueness.

Let I be some infinite set,
$[I]^2$ is all 2-element subsets of I,
$E \subseteq I \times [I]^2$ is set membership,
$P = \{0, 1\} \times [I]^2$, with projection map $\pi : P \to [I]^2$,
And $Q \subseteq P \times P \times P$ be the set of all $((i, s), (j, t), (k, u))$ such that:

1. s, t, u are all distinct sets,
2. $|s \cup t \cup u| = 3$, and
3. $i + j + k$ is even.

[Draw picture on blackboard]

$T = \mathrm{Th}(I, [I]^2, E, P, \pi, Q)$.

Q-example continued

$T = \mathrm{Th}(I, [I]^2, E, P, \pi, Q)$

Note that if $a, b \in I$, then $|\pi^{-1}(\{a, b\})| = 2$, so $\pi^{-1}(\{a, b\}) \subseteq \mathrm{acl}(a, b)$.

It turns out that T is totally categorical, hence stable.

Note that if $Q(x, y, z)$ holds, then $z \in \mathrm{dcl}(x, y)$.

Therefore, for any three distinct elements $a_1, a_2, a_3 \in I$, note that

$$\pi^{-1}(\{a_1, a_2\}) \subseteq \mathrm{dcl}(\pi^{-1}(\{a_1, a_3\}) \cup \pi^{-1}(\{a_2, a_3\})).$$

Q-example continued

Given three distinct elements $a_1, a_2, a_3 \in I$, let A be the 3-amalgamation problem given by $A(\{i\}) = a_i$ and $A(\{i,j\}) = \mathrm{acl}(a_i, a_j)$.

There are two solutions A_1, A_2 to A, defined by:

$A_1(\{1,2,3\}) = A_2(\{1,2,3,\}) = \mathrm{acl}(\{a_1, a_2, a_3\})$;

All transition maps in A_1 are inclusion maps.

In A_2, the transition maps $A(\{1,3\}) \to A_2(\{1,2,3\})$ and $A(\{2,3\}) \to A_2(\{1,2,3\})$ are inclusions, but the transition map $A(\{1,2\}) \to A_2(\{1,2,3\})$ fixes a_1 and a_2 but switches the two elements of $\pi^{-1}(\{a_1, a_2\})$.

$A_1 \not\cong A_2$ because of the relation Q on the fibers.

$(n+1)$-existence from $\leq n$-uniqueness

Theorem

Suppose T is stable and T has k-uniqueness for all $2 \leq k \leq n$ (where $n \geq 2$). Then T has $(n+1)$-existence.

Proof.

Suppose A is an independent $(n+1)$-amalgamation problem.
Let $A'(\{1,\ldots,n+1\})$ be the algebraic closure of independent copies of $A(\{1,\ldots,n\})$ and $A(\{n+1\})$.
Define maps $\tau^{i}_{1,\ldots,n+1} : A(\{i\}) \to A'(\{1,\ldots,n+1\})$ in the natural way.
For any $i \leq n$, there is only one way to define the transition map $\tau^{i,n+1}_{1,\ldots n+1} : A(\{i,n+1\}) \to A'(\{1,\ldots,n+1\})$ (by 2-uniqueness).
If $n > 3$, 3-uniqueness implies there is a unique way to extend these transition maps to "faces." Repeat using induction.

□

Characterizing 3-uniqueness in stable T

Theorem

(Hrushovski) If T is stable, then TFAE:

1. *T has 3-uniqueness;*
2. *T has 4-existence;*
3. *Every connected definable groupoid in T with finite automorphism groups is "equivalent" to a group.*

Retractable groupoids

Definition

1. A groupoid is a category $\mathscr{G}$ in which every morphism has a (unique, 2-sided) inverse.
2. A groupoid is connected if there is a morphism between any two objects.

In a connected groupoid, any two automorphism groups $\mathrm{Mor}_{\mathscr{G}}(a, a)$ and $\mathrm{Mor}_{\mathscr{G}}(b, b)$ are isomorphic. (Conjugate by $f \in \mathrm{Mor}_{\mathscr{G}}(a, b)$.)

Definition

A connected definable groupoid $\mathscr{G}$ is retractable if there is a definable family of commuting morphisms $\{f_{ab} \in \mathrm{Mor}_{\mathscr{G}}(a, b) : a, b \in \mathrm{Ob}_{\mathscr{G}}\}$.

Symmetric witnesses to non-3-uniqueness

Lemma

Suppose that T is stable. T does not have 3-uniqueness if and only if there is a set A, elements a_1, a_2, and a_3, and elements f_{12}, f_{23}, and f_{31} such that:

1. a_1, a_2, a_3 *is a Morley sequence over* A;
2. $f_{ij} \in \mathrm{acl}(Aa_ia_j) \setminus \mathrm{dcl}(Aa_ia_j)$;
3. $a_1a_2f_{12} \equiv_A a_2a_3f_{23} \equiv_A a_3a_1f_{31}$;
4. *If* (i, j, k) *is a cyclic permutation of* $(1, 2, 3)$, *then* $f_{ij} \in \mathrm{dcl}(Af_{jk}f_{ki})$.

$\{a_1, a_2, a_3, f_{12}, f_{23}, f_{31}\}$ as above is called a symmetric witness to non-3-uniqueness.

Non-retractable groupoids from failure of 3-uniqueness

Theorem

(G.-Kolesnikov) Suppose T is stable and $\{a_1, a_2, a_3, f_{12}, f_{23}, f_{31}\}$ is a symmetric witness to non-3-uniqueness over A.
Then $\mathrm{tp}(\mathrm{acl}(Aa_i)/\mathrm{acl}(A))$ defines the object class of a connected $\star$-definable non-retractable groupoid $\mathscr{G}$, with
$\mathrm{Mor}_{\mathscr{G}}(a_1, a_2) = \{f' : f' \equiv_{Aa_1a_2} f_{12}\}$.

Corollary

If T is stable, then T does not have 3-uniqueness if and only if there is a connected $\star$-definable groupoid with algebraically closed objects which is not retractable.

Generalizations?

Question

Does failure of n-uniqueness in stable T corresponded to the definability of a certain kind of "higher-dimensional groupoid" for $n \geq 4$?

There are various different notions of "*n*-category" and "*n*-groupoid" in the literature, and it is not clear which one is appropriate here.

Question

In stable T, is $(n+1)$-existence equivalent to n-uniqueness (for $n \geq 4$)?

Generalized imaginaries

For stable T, Hrushovski proves there is an expansion $\mathfrak{C}^*$ of the monster model $\mathfrak{C}$ such that:

1. $\mathfrak{C}^*$ is $\mathfrak{C}$ plus a bounded collection of new sorts;
2. $\mathfrak{C}$ is stably embedded in $\mathfrak{C}^*$;
3. Each sort $S \in \mathfrak{C}^*$ admits a definable map into $\mathfrak{C}$ with finite fibers;
4. $\mathfrak{C}^*$ has n-uniqueness and n-existence for all n.

However, we lack an "explicit" description of the new sorts in $\mathfrak{C}^*$ – presumably they are related to higher groupoids definable in $\mathfrak{C}$.

Forcing amalgamation for simple theories?

Question

If T is simple, is there an expansion $\mathfrak{C}^* \supseteq \mathfrak{C}$ with n-existence into which $\mathfrak{C}$ is stably embedded?

Consistent amalgamation in rosy theories

What kinds of amalgmation can we expect in rosy theories?

O-minimal structures can't have 3-existence (we can't amalgamate $x_1 < x_2$, $x_2 < x_3$, and $x_3 < x_1$). But they do have the following property:

Definition

(T rosy) T has consistent n-amalgamation if any thorn-independent n-amalgamation problem with a solution has a thorn-independent solution.

Consistent amalgamation continued

Theorem

(Onshuus) There is a rosy theory which does not have consistent 3-amalgamation.

The example he constructs is a variation of Hrushovski's *ab initio* construction, and has U-thorn-rank 1, but it is not dependent.

Conjecture

If T is rosy and NIP, then T has consistent 3-amalgamation.

Question

What about consistent n-amalgamation?

Works cited

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