NTP1 theories

Byunghan Kim

BIRS workshop

Dept. Math. Yonsei University

Feb. 9-13, 2009

Outline

1. The tree properties
2. Type counting criteria
3. Discussion/Suggestion

NTP1 theories

Byunghan Kim

BIRS workshop

Dept. Math. Yonsei University
Feb. 9-13, 2009

Definition

- Recall $\psi(x,y)$ has the *k-tree property (k-TP)* if there is some set of tuples $\{c_\beta | \beta \in \omega^{<\omega}\}$ such that
  - for each $\beta \in \omega^\omega$, $\{\psi(x, c_{\beta\lceil n}) | n \in \omega\}$ is consistent, and
  - for each $\beta \in \omega^{<\omega}$, $\{\psi(x, c_{\beta n}) | n \in \omega\}$ is k-inconsistent.
- $\psi(x,y)$ has TP if it has k-TP for some k.
- T has TP if some formula has TP.

Fact

- *T is simple iff T does not have TP.*
- *If $\psi(x,y)$ has k-TP then $\psi(x,y_1) \wedge ... \wedge \psi(x,y_n)$ for some n has 2-TP.*

Definition

- $\psi(x, y)$ has the *k-tree property 1 (k-TP1)* if there is some set of tuples $\{c_\beta | \beta \in \omega^{<\omega}\}$ such that
  - for each $\beta \in \omega^\omega$, $\{\psi(x, c_{\beta\lceil n}) | n \in \omega\}$ is consistent,
  - for any pairwise incomparable $\{\beta_1, ..., \beta_k\} \subseteq \omega^{<\omega}$, $\{\psi(x, c_{\beta_i}) |\ 1 \leq i \leq k\}$ is inconsistent.
- T has TP1 if some formula has 2-TP1.
- T has k-TP1 if some formulas has k-TP1.

Question

Are TP1 and k-TP1 equivalent?
In paticular, if φ has k-TP1, then does its some conjunction have 2-TP1?

Definition

- $\psi(x, y)$ has the *k-tree property 1 (k-TP1)* if there is some set of tuples $\{c_\beta | \beta \in \omega^{<\omega}\}$ such that
  - for each $\beta \in \omega^\omega$, $\{\psi(x, c_{\beta\lceil n}) | n \in \omega\}$ is consistent,
  - for any pairwise incomparable $\{\beta_1, ..., \beta_k\} \subseteq \omega^{<\omega}$, $\{\psi(x, c_{\beta_i}) | \ 1 \leq i \leq k\}$ is inconsistent.
- T has TP1 if some formula has 2-TP1.
- T has k-TP1 if some formulas has k-TP1.

Question

Are TP1 and k-TP1 equivalent?
In paticular, if φ has k-TP1, then does its some conjunction have 2-TP1?

Both yes.

Definition

T has the *tree property 2 (TP2)* if there is some set of tuples $\{c^i_j | i, j < \omega\}$ such that for some ψ,

- for any $f : \omega \to \omega$, $\{\psi(x, c^i_{f(i)}) | i \in \omega\}$ is consistent, and
- for each $i \in \omega$, $\{\psi(x, c^i_j) | j \in \omega\}$ is 2-inconsistent.

Fact

T has TP iff T has either TP1 or TP2.

Definition

$\psi(x, y)$ has the *binary tree property (BTP=SOP$_2$)* if there is some set of tuples $\{c_\beta | \beta \in 2^{<\omega}\}$ such that

- for each $\beta \in 2^\omega$, $\{\psi(x, c_{\beta\lceil n}) | n \in \omega\}$ is consistent,
- for any incomparable $\alpha, \beta \in \omega^{<\omega}$, $\psi(x, c_\alpha) \wedge \psi(x, c_\beta)$ is inconsistent.

Similarly we define k-BTP.

Fact

Strict Order Property $\Rightarrow$ .. $SOP_4 \Rightarrow SOP_3 \Rightarrow SOP_2$=$BTP \Rightarrow SOP_1 \Rightarrow TP$=*nonsimple.*

Observation

T has TP1 iff T has BTP.

examples

The prototypical example of NTP1

The prototypical example with NTP1: The model companion of the theory with sorts P, E and a ternary $x \sim_z y$ on $P^2 \times E$ saying that for each $e \in E$, $x \sim_e y$ forms an equivalence relation on P. It is complete, ω-categorical having QE.

The prototypical example of NTP1

The prototypical example with NTP1: The model companion of the theory with sorts P, E and a ternary $x \sim_z y$ on $P^2 \times E$ saying that for each $e \in E$, $x \sim_e y$ forms an equivalence relation on P. It is complete, ω-categorical having QE.

Stable Simple NTP1

examples

The prototypical example of NTP1

The prototypical example with NTP1: The model companion of the theory with sorts P, E and a ternary $x \sim_z y$ on $P^2 \times E$ saying that for each $e \in E$, $x \sim_e y$ forms an equivalence relation on P. It is complete, ω-categorical having QE.

Stable	Simple	NTP1
Infinite set	The random graph	The random equi. rel.s

examples

The prototypical example of NTP1

The prototypical example with NTP1: The model companion of the theory with sorts P, E and a ternary $x \sim_z y$ on $P^2 \times E$ saying that for each $e \in E$, $x \sim_e y$ forms an equivalence relation on P. It is complete, ω-categorical having QE.

Stable	Simple	NTP1
Infinite set	The random graph	The random equi. rel.s
ACF	Bounded PAC fields	ω-free PAC fields

examples

The prototypical example of NTP1

The prototypical example with NTP1: The model companion of the theory with sorts P, E and a ternary $x \sim_z y$ on $P^2 \times E$ saying that for each $e \in E$, $x \sim_e y$ forms an equivalence relation on P. It is complete, ω-categorical having QE.

Stable	Simple	NTP1
Infinite set	The random graph	The random equi. rel.s
ACF	Bounded PAC fields	ω-free PAC fields
V = vector sapce	$(V, \langle , \rangle)$ / a finite F	$(V, \langle , \rangle)$ / an infinite F

Theorem

(Shelah) TFAE.

1. *T has TP.*
2. *Some formula has 2-TP.*
3. *There are a cardinal κ and a family $\mathcal{F}$ of types over A such that*
   - $|\mathcal{F}| > |A|^{|T|} + 2^{|T|+\kappa}$,
   - $|p| \leq \kappa$ *for each* $p \in \mathcal{F}$,
   - *whenever* $\mathcal{G} \subseteq F$ *and* $|\mathcal{G}| > \kappa$, *then* $\bigcup \mathcal{G}$ *is inconsistent.*

Proof. (1)⇒(3)⇒(2)⇒(1).

Theorem

TFAE.

1. *T has k-TP1 for some k.*
2. *Some formula has BTP.*
3. *Some formula has 2-TP1.*
4. *There are a regular cardinal κ and a family $\mathcal{F}$ of types over A such that*
   - $|p| = \kappa$ *for each* $p \in \mathcal{F}$,
   - $|\mathcal{F}| = \lambda^+$ *where* $\lambda = |A|^{|T|} + |T|^\kappa$, *and*
   - *given any subfamily* $\mathcal{G} = \{q_i | i < \lambda^+\}$ *of* $\mathcal{F}$, *there are disjoint subsets* τ_1, τ_2 *of* λ^+ *with* $|\tau_j| = \lambda^+$ $(j = 0, 1)$, *and* $q_i' \subseteq q_i$ *with* $|q_i - q_i'| < \kappa$ $(i < \lambda^+)$, *such that* ${}^\vee\mathcal{G}_0 \cap {}^\vee\mathcal{G}_1 = \emptyset$, *where* $\mathcal{G}_j = \{q_i' |\ i \in \tau_j\}$, *and* ${}^\vee\mathcal{G}_j = \bigcup\{\varphi(\mathcal{M}) | \varphi \in \bigcup \mathcal{G}_j\}$.

Proof. (1)⇒(2)⇒(3)⇒(1) (Džamonja, Shelah, Usvyatsov)[1].
(3)⇒(4)⇒(2).

[1] M. Džamonja, S. Shelah, 'On $\lhd^*$-maximality' APAL 2004; S. Shelah, A. Usvyatsov, 'More on SOP_1 and SOP_2', APAL

Hence T has TP1 iff so does T^{eq}. (Expansive way of proving. Cheap way: Consider preimages in the home-sort.)

Hence T has TP1 iff so does T^{eq}. (Expansive way of proving. Cheap way: Consider preimages in the home-sort.)

Key idea of Džamonja, Shelah, Usvyatsov

If $\mathcal{C} = \{c_\beta | \beta \in 2^{<\omega}\}$ witnesses k-BTP of φ, then one can additionally assume that $\mathcal{C}$ is tree-indiscernible. Namely,

$$c_{\alpha_1} \ldots c_{\alpha_n} \equiv c_{\beta_1} \ldots c_{\beta_n}$$

whenever both $\{\alpha_1, \ldots, \alpha_n\}, \{\beta_1, \ldots, \beta_n\} (\subseteq 2^\omega)$ are

- closed under $\cap$, and $\lhd$-order isomorphic.

Then it follows that some conjunction of φ has 2-BTP.

The rest are all tentative with possible naivety.

Definition

- $\psi(x, a)$ strongly divides over A if for any $A_0(\subseteq A)$, and any Morley I of $\mathrm{tp}(a/A)$, $\{\psi(x, a') \mid a' \in I\}$ is inconsistent.
- Write $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}^{s}$ = non-strong dividing.
- T is *subtle* if $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}^{s}$ satisfies local character.

The rest are all tentative with possible naivety.

Definition

- $\psi(x,a)$ strongly divides over A if for any $A_0 (\subseteq A)$, and any Morley I of $\mathrm{tp}(a/A)$, $\{\psi(x,a') | \; a' \in I\}$ is inconsistent.
- Write $\downfree^s$ = non-strong dividing.
- T is *subtle* if $\downfree^s$ satisfies local character.

Stable $\subseteq$ Simple (there $\downfree = \downfree^s$) $\subseteq$ Subtle.

The rest are all tentative with possible naivety.

Definition

- $\psi(x, a)$ strongly divides over A if for any $A_0(\subseteq A)$, and any Morley I of $\mathrm{tp}(a/A)$, $\{\psi(x, a') | \ a' \in I\}$ is inconsistent.
- Write $\mathop{\smile\hskip -0.9em ^|}^{s}$ = non-strong dividing.
- T is *subtle* if $\mathop{\smile\hskip -0.9em ^|}^{s}$ satisfies local character.

Stable $\subseteq$ Simple (there $\mathop{\smile\hskip -0.9em ^|} = \mathop{\smile\hskip -0.9em ^|}^{s}$) $\subseteq$ Subtle.

Question

- (We may additionally assume forking=dividing) NTP1$\Rightarrow$ Subtle (even are both equivalent)?
- Does symmetry over $\emptyset$ hold?
- Note that different from simple case, $A \mathop{\smile\hskip -0.9em ^|}^{s}_{B} C$ is *not* equivalent to $A \mathop{\smile\hskip -0.9em ^|}^{s} C$ in $\mathcal{L}(B)$!! Indeed in the examples of NTP1, possibly independence notions are *not* invariant under naming elements, so we may end up need quarternary relation rather than ternary $\mathop{\smile\hskip -0.9em ^|}$?