New inequalities for the von Neumann entropy (?)

Andreas Winter (U. Bristol & NUS)

[with Noah Linden, CMP 259, 129–138 (2005)

Ben Ibinson, PhD Thesis, Bristol (2007)
[unpublished]]

Outline

I. Entropy + Inequalities

II. "Laws of Info. Th." for 2 & 3 parties

III. 4 parties: constrained inequality

IV. Quest for an unconstr. inequality...

V. Open problems

I. Entropy + Inequalities

$S(\rho) = -\operatorname{Tr} \rho \log \rho$ von Neumann entropy

for state $\rho \geq 0$, $\operatorname{Tr} \rho = 1$

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Universal relations between entropies of multiparty systems?

I.e., for density operator $\rho_{ABC\ldots Z}$ on Hilbert space $A \otimes B \otimes C \ldots \otimes Z$:

Consider marginals
$$\rho_A = \mathrm{Tr}_{B\ldots Z}\,\rho$$
$$\rho_B = \mathrm{Tr}_{AC\ldots Z}\,\rho$$
$$\rho_{AB} = \mathrm{Tr}_{C\ldots Z}\,\rho$$
etc.

& entropies
$$S(A) = S(\rho_A)$$
$$S(B) = S(\rho_B)$$
$$S(AB) = S(\rho_{AB})$$
etc.

For n-party state: 2^n values, one for each subset $\mathcal{X}$

Relations:

$$S(\mathcal{H}) \geq 0 \quad \text{and} \quad S(\emptyset) = 0 \tag{0}$$

$$I(A:B) := S(A) + S(B) - S(AB) \geq 0 \tag{1}$$

$$I(A:C|B) := S(AB) + S(BC) - S(ABC) - S(B) \geq 0 \tag{2}$$

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... but there is more — purely quantum mechanically:

(i) If $\rho = |\psi\rangle\langle\psi|$ is pure (i.e. $S(\rho)=0$), then
$S(\mathcal{X}) = S(\overline{\mathcal{X}})$ [since $\rho_{\mathcal{X}}$ and $\rho_{\overline{\mathcal{X}}}$ have the same spectrum]

(ii) Any (n-party) state ρ is the marginal of an ($n+1$-party) <u>pure</u> state $|\varphi\rangle\langle\varphi|$.

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Hence:

$$S(AB) + S(A) - S(B) \geq 0 \qquad (3)$$

$$\left[\mathcal{X} \cap \mathcal{Y} = \emptyset \Rightarrow S(\mathcal{X} \cup \mathcal{Y}) + S(\mathcal{X}) - S(\mathcal{Y}) \geq 0 \right]$$

$$S(AB) + S(BC) - S(A) - S(C) \geq 0 \qquad (4)$$

$$\left[S(\mathcal{X}) + S(\mathcal{Y}) - S(\mathcal{X} \setminus \mathcal{Y}) - S(\mathcal{Y} \setminus \mathcal{X}) \geq 0 \right]$$

"weak monotonicity"

I.e., all entropy vectors $\left[S(\mathcal{X}) : \emptyset \neq \mathcal{X} \subset \{1, \dots, n\}\right]$

$\in \mathbb{R}^{2^n - 1}$

are contained in the convex cone Σ_n, defined by

set of all: Σ_n^*

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Pippenger IEEE-IT '03: $\overline{\Sigma_n^*}$ also a convex cone, contained in Σ_n: i.e. with $\vec{v}, \vec{w}$ & $\lambda, \mu \geq 0$, contains $\lambda \vec{v} + \mu \vec{w}$.

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{ Classical case: $\overline{\Delta_n^*} \subset \Delta_n$ ← Shannon cone
R.W. Yeung; Z. Zhang; R. Dougherty et al.; ...
- Δ_n^* not closed
- $\overline{\Delta_4^*} \neq \Delta_4$: $\exists$ non-Shannon-type ineq's! }

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$\boxed{n=1}$ $\Sigma_1^* = \Sigma_1$: all non-negative values allowed.

$\boxed{n=2}$ $\Sigma_2^* = \Sigma_2$: all non-negative values of $S(A), S(B), S(AB)$ that obey

$$S(AB) \leq S(A) + S(B)$$
$$S(A) \leq S(AB) + S(B)$$
$$S(B) \leq S(AB) + S(A)$$

are allowed.

$\boxed{n=3}$ Pippenger IEEE-IT '03 computed the extremal rays of Σ_3: spanned by

A	B	C	AB	AC	BC	ABC
1	1	0	0	1	1	0
1	0	0	1	1	0	1
1	1	1	1	1	1	1
1	1	1	2	2	2	1

(up to permutations)

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(up to permutations)

... sufficient to find states realizing them!

$n=3$ cont'd

Useful to introduce purifying system D, s.t. $S(D) = S(ABC)$, $S(AD) = S(BC)$, etc.

A	B	C	D	AB	AC	BC
1	1	0	0	0	1	1
1	1	1	1	1	1	1
1	1	1	1	2	2	2

$\boxed{\text{n=3 cont'd}}$

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A	B	C	D	AB	AC	BC	
1	1	0	0	0	1	1	(i)
1	1	1	1	1	1	1	(ii)
1	1	1	1	2	2	2	(iii)

States:

(i) $|EPR\rangle_{AB}|0\rangle_C|0\rangle_D = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1100\rangle\right)$

(ii) $|GHZ\rangle_{ABCD} = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$

(iii) $\frac{1}{3}\sum_{a,b\in\mathbb{Z}_3}|a\rangle_A|b\rangle_B|a+b\rangle_C|a+2b\rangle_D =: |\psi\rangle_{ABCD}$ "max. entangled state"

III. 4 parties...

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With purifying system E, there are 76 rays. Up to permutation symmetry, only 8 classes, as follows...

	A	B	C	D	E ABCD	AB	AC	AD	AE BCD	BC	BD	BE ACD	CD	CE ABD	DE ABC
(i)	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0
(ii)	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
(iii)	1	1	1	1	0	2	2	2	1	2	2	1	2	1	1
(iv)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
(v)	2	1	1	1	1	3	3	3	3	2	2	2	2	2	2
(vi)	1	1	2	2	2	2	3	3	3	3	3	3	2	2	2
(vii)	3	3	2	2	2	4	3	3	3	3	3	3	3	4	4
(viii)	3	3	3	3	2	4	4	4	5	4	4	5	6	5	5

(i), (ii), (iii) we know already: realized by

$|EPR\rangle_{AB}|000\rangle_{CDE}$

$|GHZ\rangle_{ABCD}|0\rangle_E$

$|M\rangle_{ABCD}|0\rangle_E$

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(iv): realized by $|GHZ\rangle_{ABCDE}$

(v), (vi): realized by code or secret sharing states

e.g. $|\psi_5\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_{A'} |0_L\rangle_{ABCDE} + |1\rangle_{A'} |1_L\rangle_{ABCDE}\right]$

logical 0,1 in 5-qubit code

$$|\psi_6\rangle = \frac{1}{\sqrt{27}} \sum_{a,b,c \in \mathbb{Z}_3} |a\rangle_A |b\rangle_B |a+b\rangle_C |a+c\rangle_{C'} |a+b\rangle_D |a+c\rangle_{D'} |a+b\rangle_E |a+c\rangle_{E'}$$

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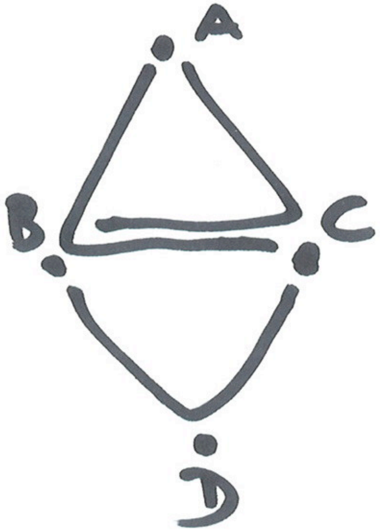
(vii), (viii): ???

Thm. (Linden/AW '05) If ρ_{ABCD} satisfies

$$I(A:B|C)=0$$
$$I(A:C|B)=0$$
$$I(B:C|D)=0$$



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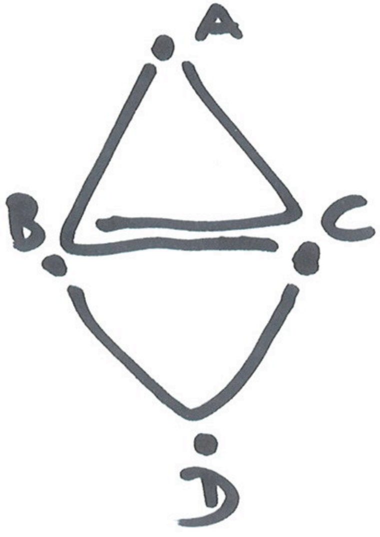

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<u>Proof</u> uses characterization of "="-case of SSA [Hayden/Jozsa/Petz/AW, CMP '03] as "quantum Markov chain" & exploiting the ensuing classical information ...

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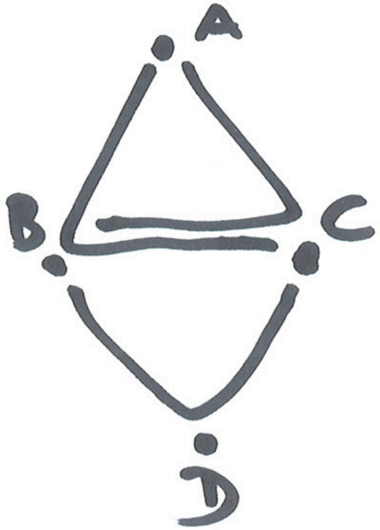

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then $I(A:D) \geq I(A:BC)$.

Both ways (vii) & (viii) contradict the Thm.

$\Longrightarrow \Sigma_4^* \neq \Sigma_4$ (but leaves open $\overline{\Sigma_4^*} \overset{?}{=} \Sigma_4$)

IV. An unconstrained inequality?

New (constrained) inequality cannot be derived from "standard" ones: it excludes part of a face of Σ_4.

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Here some attempts to find one (→ Ben Ibinson, PhD thesis, Bristol 2007)

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Hence, there are more extremal rays to $\overline{\Sigma_4^*}$.

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So, perhaps there exist $k_i \geq 0$ s.t. for all states,

$$k_1 I(A:C|B) + k_2 I(B:C|A) + k_3 I(A:B|D) + [I(C:D) - I(C:AB)] \geq 0$$

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... However, difficult to test since κ_i unknown

[see also Ibinson/Linden/AW, CMP '08]

(6) Yeung – Zhang inequality

Yeung/Zhang, IEEE-IT '98 showed that the following is true for classical densities:

$$2I(A:B) + 3I(A:B|C) + I(A:B|D) \leq I(C:D) + I(C:AB)$$

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For quantum states:

- violated by rays (vii) & (viii)
- no counterexamples known (neither in special states nor in numerical search)
- YZ proof (via custom-made conditional indep. structure) doesn't work, however...

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* ... but as of now have only weak indications what they might be
* Strongest candidate: Yeung-Zhang inequality
* Interesting to note: All known parts of the cones $\overline{\Sigma}_n^*$ spanned by stabilizer states.