(notes by Jaap Top)

Lenny Taelman Banff 28-9-09

Mordell-Weil and Shafarevic modules of Drinfeld modules

<u>notation:</u> k = finite field of cardinality q

$A = k[t]$ $\quad$ $\mathrm{Spec}\,A \hookrightarrow \mathbb{P}^1_t$

X/k curve,

function field $K = k(X)$ $\quad$ $Y \hookrightarrow X$ $\qquad$ $k(t) \hookrightarrow K$, $t \mapsto \theta$

$$Y = \mathrm{Spec}(R) \quad (R = \text{normalization of } A \text{ in } K)$$

A D.M. is $E = (\mathbb{G}_{a,R}, \varphi)$ with $\varphi : A \to \mathrm{End}(\mathbb{G}_a)$

$$t \mapsto \theta + a_1\tau + \cdots + a_r\tau^r$$

with $a_i \in R$, $a_r \in R^*$

(i.e., the module has everywhere good reduction)

problem: $E(K) = K$ as additive group is not a fin. gen. A-module.

neither is $E(R)$ (result of Bjorn Poonen)

§1 Extensions.

$M :=$ t-motive corresp. to E, i.e.,

$M = \mathrm{Hom}(E, \mathbb{G}_a)$, an $R[t,\tau]$-module

$$\left(E(-) = \mathrm{Hom}_{R[\tau]\text{-mod}}(M, -) \right)$$

M has rank r as module over $R[t]$.

Thm $$E(R) = \mathrm{Ext}_{R[t,\tau]}(M, \underset{\text{action: } \tau t = t}{R[t]})$$

Pf: apply $\mathrm{Hom}_{R[t,\tau]}(M, -)$ to $0 \to R[t] \to R((t^{-1})) \to \mathrm{Quot} \to 0$

this gives

$$\dots \to \mathrm{Hom}(M, R((t^{-1}))/R[t]) \to \mathrm{Ext}^1(M, R[t])$$

which turns out to be $\cong$.

now $\mathrm{Hom}(M, R((t^{-1}))/R[t]) \xrightarrow{\sim} \mathrm{Hom}_{R[t]}(M,R) = E(R)$

res $=$ coeff of t^{-1}.

$\boxtimes$

Construction:

step 1: linearise

we have $\sigma : M \to M : m \mapsto \tau m$

which is semi-linear

it gives $\sigma_{lin} : \tau^* M \to M : m \otimes f \mapsto f(\tau m)$

$\tau^* M = M \otimes_{R[t]} R[t]$

with $R[t] \xrightarrow[\tau]{} R[t]$ the module structure

Then $\mathrm{coker}(\sigma_{lin})^\vee = \mathrm{Lie}_E(R)$

step 2: compactify. naive: extend σ_{lin} to

$\sigma_{lin} : \tau^* \mathcal{M} \to \mathcal{M}$,

$\mathcal{M}$ a vector bundle on $X_A = X \times \mathrm{Spec} A$.

Proposition $\exists$: 1) vector bundles $\mathcal{M}, \mathcal{M}'$ on X_A extending $M, \tau^* M$, respectively.

2) ..

2) a lin. map $\sigma : \mathcal{M}' \to \mathcal{M}$ extending σ_{lin}

3) a lin. map $j : \mathcal{M}' \to \tau^* \mathcal{M}$
extending $id : \tau^* M \to \tau^* M$

s.t. $\sigma|_{X-Y}$ is an isomorphism. $\boxtimes$

Notation. $\underline{\mathcal{M}}$ = the data $(\mathcal{M}, \mathcal{M}', \sigma, j)$

$\underline{\mathbb{1}} = (\mathcal{O}_{X_A}, \mathcal{O}_{X_A}, 1, 1)$

<u>facts</u>: $\mathrm{Hom}(\underline{\mathcal{M}}, \underline{\mathbb{1}}) = 0$
$\mathrm{Ext}(\underline{\mathcal{M}}, \underline{\mathbb{1}})$ is fin. gen. as A-module
$\mathrm{Ext}^2(\underline{\mathcal{M}}, \underline{\mathbb{1}})$ is finite

(ideas also used by Lafforgue)

<u>Step 3</u> take limits.
<u>Thm</u> the Ext^i are indep. of $\underline{\mathcal{M}}$,
if $\mathcal{M}$ is "large enough".

<u>Def</u>:
$$\widetilde{MW} = \varprojlim_{\underline{\mathcal{M}}} \mathrm{Ext}^1(\underline{\mathcal{M}}, \underline{\mathbb{1}})$$

$$MW := \mathrm{image}\,(\widetilde{MW} \to E(R) = \mathrm{Ext}(M, R[t]))$$

$$\underline{Ш} := \varprojlim \mathrm{Ext}^2(\underline{\mathcal{M}}, \underline{\mathbb{1}})$$

Example. Carlitz module, then $Ш = (0)$
over $R = k[\theta]$ (& MW depends on $q = 2, q \neq 2$)

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Prop: $\operatorname{rk}_A \widetilde{MW} = \operatorname{rk}_A \operatorname{Lie}_E(R) = r \cdot [K : k(\theta)]$

§3 Regulators

L = number field

$$\begin{array}{ccc}
\text{finite} & & \text{finite} \\
\uparrow & & \uparrow \\
O_L^* & \subseteq & (L\otimes\mathbb{R})^*_{Norm=1} \\
\uparrow \exp & & \uparrow \exp \\
\exp^{-1}(O_L^*) & \subseteq & (L\otimes\mathbb{R})_{tr=0} \\
\uparrow & & \uparrow \\
(2\pi i\mathbb{Z})^{\oplus r_2} & = & (2\pi i\mathbb{Z})^{\oplus r_2} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

cocompact, discrete.

Now the same for DM's:

$K_\infty := K \otimes k((\theta^{-1}))$ is a product of fields.

$$\begin{array}{ccccccc}
0 \to & \ker(\exp) \hookrightarrow & \operatorname{Lie}(K_\infty) & \xrightarrow{\exp_E} & E(K_\infty) & \to \text{torsion} \to 0 \\
& \| & & & \cup & \\
0 \to & \ker(\exp) \hookrightarrow & \exp_E^{-1}(MW) & \xrightarrow{\exp_E} & MW & \to \text{finite} \to 0
\end{array}$$

Thm

$$\operatorname{rk}_A \exp^{-1}(MW) = \operatorname{rk}_A \operatorname{Lie}_E(R) = r \cdot [K : k(\theta)]$$

Conjecture (Lenny!) $\exp^{-1}(MW) \otimes k((t^{-1})) \longrightarrow \operatorname{Lie}_E(K_\infty)$ is an isomorphism.

if conj. true, then can take determinant,

which gives elt. in $k((t^{-1}))^* / k[t]^*$.

the monic generator of image here is called <u>Reg</u>.

<u>§4</u> special values

notation. $\left| \prod A/(f_i) \right| := \prod f_i$ (assuming the f_i monic)

Question:

$$L(M^\vee, 0) \overset{?}{=} \mathrm{Reg} \cdot |Ш| \quad \text{in } k((t^{-1}))$$

" Goss' L-function

if $E = (\mathbb{G}_a,\ t \mapsto \theta + \tau)$, then this says

$$\zeta_R(1) = \sum_{\underline{m} \subset R} \frac{1}{|R/m|} = \mathrm{Reg} \cdot |Ш|$$

which is OK for $R = k[\theta]$