

Quantum Computation with Topological Phases of Matter

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1 Overview of the Field

Quantum computers, if realized in practice, promise exponential speed-up of some of the computational tasks that conventional classical algorithms are intrinsically incapable of handling efficiently. Perhaps even more intriguing possibility arises from being able to use quantum computers to simulate the behavior of other physical systems – an exciting idea dating back to Feynman.

Unfortunately, realizing a quantum computer in practice proves to be very difficult, chiefly due to the debilitating effects of decoherence plaguing all possible schemes which use microscopic degrees of freedom (such as nuclear or electronic spins) as basic building blocks. From this perspective, topological quantum computing offers an attractive alternative by encoding quantum information in nonlocal topological degrees of freedom that are intrinsically protected from decoherence due to local noise.

2 Recent Developments and Open Problems

Over the last few years there have been substantial advances in both our understanding of how to compute with topological phases, and our understanding of topological phases themselves in various physical manifestations. The important features of these phases include topological order – the ground state degeneracy that depends on the topology of the underlying 2D surface – as well as nontrivial, often fractional statistics of their excitations. The goal of this workshop was to both study the physical nature of topological phases as well as to address the most important theoretical issues connected with any attempt to practically realize a topological quantum computer. One of the main objectives of the proposed workshop was to bring together experts from different fields including condensed matter physics, quantum optics and theory of quantum information and quantum computation. The exchange of ideas between researchers working on these subjects has

resulted in making new inroads into this broad, inter-disciplinary field. Specifically, the discussions focused on the following open questions:

1. Physical systems with topological order. So far, of experimentally observed systems, only Fractional Quantum Hall systems are believed to possess topological order. Numerous other possibilities have been discussed recently, ranging from atomic systems in optical lattices to Josephson junction arrays to frustrated magnetic systems. How to engineer systems supporting a topological phase remains an open question.

2. Fractional Quantum Hall Effect and related systems. Recently, new experiments purported to probe the quasiparticle statistics in the Fractional Quantum Hall systems have been reported by Goldman's group. However, their results appear to be open to interpretation. Hence a clear, unambiguous experimental evidence of fractional statistics remains to be confirmed. Experimental detection of non-Abelian anyons and related topological order remains an open problem despite several recent theoretical suggestions. How to control topological excitations, a crucial part of building a topological quantum computer, also remains to be explored.

3. Topological Algorithms and Computational Architectures Given that such topological phases do exist, and are capable of performing universal quantum computation, one must ask how to build efficient architecture and how one would actually go about performing such computations. The structure of topological phases is extremely complex and imposing the traditional qubit architecture may not be the most efficient scheme.

3 Presentation Highlights

The five days of the workshop were devoted to five different topics. Here, we give a brief outline of the presentations given by the participants.

3.1 Day 1: Topological phases and topological order

Paul Fendley: "Topological order from quantum loops and nets" — I define models of quantum loops and nets that have ground states with topological order. These make possible excited states comprised of deconfined anyons with non-abelian braiding. With the appropriate inner product, these quantum loop models are equivalent to net models whose topological weight involves the chromatic polynomial. A simple Hamiltonian preserving the topological order is found by exploiting quantum self-duality. For the square lattice, this Hamiltonian has only four-spin interactions.

C. Castelnovo: "Topological phases at finite temperature" — We study topological order in a toric code in three spatial dimensions, or a 3+1D Z_2 gauge theory, at finite temperature. We compute exactly the topological entropy of the system, and show that it drops, for any infinitesimal temperature, to half its value at zero temperature. The remaining half of the entropy stays constant up to a critical temperature T_c , dropping to zero above T_c . These results show that topologically ordered phases exist at finite temperatures, and we give a simple interpretation of the order in terms of fluctuating strings and membranes, and how thermally induced point defects affect these extended structures. Finally, we discuss the nature of the topological order at finite temperature, and its quantum and classical aspects.

X.-G. Wen: "Classifying FQH states through pattern of zeros" — A large class of fractional quantum Hall (FQH) states can be classified according to their pattern of zeros, which describes the way ideal ground state wave functions go to zero as various clusters of electrons are brought together. In this paper we generalize this approach to classify multilayer FQH states. Such a clas-

sification leads to the construction of a class of non-Abelian multilayer FQH states that are closely related to \hat{g}_k parafermion conformal field theories, where \hat{g}_k is an affine simple Lie algebra. We discuss the possibility of some of the simplest of these non-Abelian states occurring in experiments on bilayer FQH systems at $\nu = 2/3, 4/5, 4/7$, etc.

J. Slingerland: "Condensation induced transitions between topological phases" — We investigate transitions between topologically ordered phases in two spatial dimensions induced by the condensation of a bosonic quasiparticle. To this end, we formulate an extension of the theory of symmetry breaking phase transitions which applies to phases with topological excitations described by quantum groups or modular tensor categories. This enables us to deal with phases whose quasiparticles have non-integer quantum dimensions and obey braid statistics. Many examples of such phases can be constructed from two-dimensional rational conformal field theories and we find that there is a beautiful connection between quantum group symmetry breaking and certain well-known constructions in conformal field theory, notably the coset construction, the construction of orbifold models and more general conformal extensions. Besides the general framework, many representative examples are worked out in detail.

P. Bonderson: "Measurement-Only Topological Quantum Computation" — We describe measurement only topological quantum computation using both projective and interferometrical measurement of topological charge. We demonstrate how anyonic teleportation can be achieved using "forced measurement" protocols for both types of measurement. Using this, it is shown how topological charge measurements can be used to generate the braiding transformations used in topological quantum computation, and hence that the physical transportation of computational anyons is unnecessary. We give a detailed discussion of the anyonics for implementation of topological quantum computation (particularly, using the measurement-only approach) in fractional quantum Hall systems.

3.2 Day 2: *p*-wave superconductors and superfluids, Topological insulators

*C. Kallin: "Is Sr_2RuO_4 a chiral *p*-wave superconductor?"* — Much excitement surrounds the possibility that strontium ruthenate exhibits chiral *p*-wave superconducting order. Such order would be a solid state analogue of the A phase of He-3, with the potential for exotic physics relevant to quantum computing. We take a critical look at the evidence for such time-reversal symmetry breaking order. The possible superconducting order parameter symmetries and the evidence for and against chiral *p*-wave order are reviewed, with an emphasis on the most recent theoretical predictions and experimental observations. In particular, attempts to reconcile experimental observations and theoretical predictions for the spontaneous supercurrents expected at sample edges and domain walls of a chiral *p*-wave superconductor and for the polar Kerr effect, a key signature of broken time-reversal symmetry, are discussed.

*V. Gurarie: "Creating stable *p*-wave atomic superfluids"* — We study theoretically a dilute gas of identical fermions interacting via a *p*-wave resonance. We show that, depending on the microscopic physics, there are two distinct regimes of *p*-wave resonant superfluids, which we term "weak" and "strong". Although expected naively to form a BCS-BEC superfluid, a strongly-resonant *p*-wave superfluid is in fact unstable towards the formation of a gas of fermionic triplets. We examine this instability and estimate the lifetime of the *p*-wave molecules due to the collisional relaxation into triplets. We discuss consequences for the experimental achievement of *p*-wave superfluids in both weakly- and strongly-resonant regimes.

J. Moore: "Topological insulators: overview and relevance to quantum computing" — Spin-orbit coupling in some materials leads to the formation of surface states that are topologically pro-

tected from scattering. Theory and experiments have found an important new family of such materials. Topological insulators are materials with a bulk insulating gap, exhibiting quantum-Hall-like behaviour in the absence of a magnetic field. Such systems are thought to provide an avenue for the realization of fault-tolerant quantum computing because they contain surface states that are topologically protected against scattering by time-reversal symmetry.

S.-S. Lee: "Many-body generalization of the Z2 invariant in the time reversal symmetric topological insulator" — We propose a many-body generalization of the Z2 topological invariant for the quantum spin Hall insulator, which does not rely on single-particle band structures. The invariant is derived as a topological obstruction that distinguishes topologically distinct many-body ground states on a torus. It is also expressed as a Wilson-loop of the SU(2) Berry gauge field, which is quantized due to the time-reversal symmetry.

N. Read: "Adiabatic statistics, quasiparticle spin, and Hall viscosity in quantum Hall states and $p_x + ip_y$ paired superfluids" — Many trial wavefunctions for fractional quantum Hall states in a single Landau level are given by functions called conformal blocks, taken from some conformal field theory. Also, wavefunctions for certain paired states of fermions in two dimensions, such as $p_x + ip_y$ states, reduce to such a form at long distances. Here we investigate the adiabatic transport of such many-particle trial wavefunctions using methods from two-dimensional field theory. One context for this is to calculate the statistics of widely-separated quasiholes, which has been predicted to be non-Abelian in a variety of cases. The Berry phase or matrix (holonomy) resulting from adiabatic transport around a closed loop in parameter space is the same as the effect of analytic continuation around the same loop with the particle coordinates held fixed (monodromy), provided the trial functions are orthonormal and holomorphic in the parameters so that the Berry vector potential (or connection) vanishes. We show that this is the case (up to a simple area term) for paired states (including the Moore-Read quantum Hall state), and present general conditions for it to hold for other trial states (such as the Read-Rezayi series). We argue that trial states based on a non-unitary conformal field theory do not describe a gapped topological phase, at least in many cases. By considering adiabatic variation of the aspect ratio of the torus, we calculate the Hall viscosity, a non-dissipative viscosity coefficient analogous to Hall conductivity, for paired states, Laughlin states, and more general quantum Hall states. Hall viscosity is an invariant within a topological phase, and is generally proportional to the "conformal spin density" in the ground state.

3.3 Day3: Fractionalization of charge and statistics in FQHE and other systems

W. Kang: "Experimental Study of Mesoscopic Quantum Hall Interferometers" — The fractional quantum Hall effect is presently the most promising platform for a topological quantum computer among various candidate systems. The global topological protection afforded by fractional quantum Hall effect produces fault-tolerance in a topological quantum computer. A topological quantum computer consequently becomes immune against the effects of local quantum decoherence. A fault-tolerant qubit can be constructed by taking advantage of the non-Abelian braiding statistics of elementary excitations (called anyons) which are thought to exist in certain, exotic fractional quantum Hall states. Experimental goals include detection and manipulations of the postulated non-Abelian anyons in quantum interferometers constructed from high quality semiconductor heterostructures.

S. Vishveshwara: "Correlators and Fractional Statistics in the quantum Hall bulk" — We derive single-particle and two-particle correlators of anyons in the presence of a magnetic field in the lowest Landau level. We show that the two-particle correlator exhibits signatures of fractional statistics which can distinguish anyons from their fermionic and bosonic counterparts. These signatures include the zeroes of the two-particle correlator and its exclusion behavior. We find that the single-

particle correlator in finite geometries carries valuable information relevant to experiments in which quasiparticles on the edge of a quantum Hall system tunnel through its bulk.

S. Simon: "Other Quantum Hall States, Hamiltonians, and Conformal Field Theories" — We construct a family of BCS paired composite fermion wavefunctions that generalize, but remain in the same topological phase as, the Moore-Read Pfaffian state for the half-filled Landau level. It is shown that for a wide range of experimentally relevant inter-electron interactions the groundstate can be very accurately represented in this form.

B. Seradjeh: "Zero modes and fractionalization in bilayer-graphene exciton condensate" — A real-space formulation is given for the recently discussed exciton condensate in a symmetrically biased graphene bilayer. We show that in the continuum limit an oddly-quantized vortex in this condensate binds exactly one zero mode per valley index of the bilayer. In the full lattice model the zero modes are split slightly due to intervalley mixing. We support these results by an exact numerical diagonalization of the lattice Hamiltonian. We also discuss the effect of the zero modes on the charge content of these vortices and deduce some of their interesting properties.

3.4 Day 4: FQH States

M. Fisher: "Tunneling and edge transport in non-Abelian quantum Hall states" — We analyze charge- $e/4$ quasiparticle tunneling between the edges of a point contact in a non-Abelian model of the $\nu = 5/2$ quantum Hall state. We map this problem to resonant tunneling between attractive Luttinger liquids and use the time-dependent density-matrix renormalization group (DMRG) method to compute the current through the point contact in the presence of a *finite voltage difference* between the two edges. We confirm that, as the voltage is decreases, the system is broken into two pieces coupled by electron hopping. In the limits of small and large voltage, we recover the results expected from perturbation theory about the infrared and ultraviolet fixed points. We test our methods by finding the analogous non-equilibrium current through a point contact in a $\nu = 1/3$ quantum Hall state, confirming the Bethe ansatz solution of the problem.

E. Ardonne: Non-abelian quantum Hall states: what can be learned from the thin-torus limit — We analyze the non-abelian Read-Rezayi quantum Hall states on the torus, where it is natural to employ a mapping of the many-body problem onto a one-dimensional lattice model. On the thin torus—the Tao-Thouless (TT) limit—the interacting many-body problem is exactly solvable. The Read-Rezayi states at filling $\nu = \frac{k}{kM+2}$ are known to be exact ground states of a local repulsive $k + 1$ -body interaction, and in the TT limit this is manifested in that all states in the ground state manifold have exactly k particles on any $kM + 2$ consecutive sites. For $M \neq 0$ the two-body correlations of these states also imply that there is no more than one particle on M adjacent sites. The fractionally charged quasiparticles and quasiholes appear as domain walls between the ground states, and we show that the number of distinct domain wall patterns gives rise to the nontrivial degeneracies, required by the non-abelian statistics of these states. In the second part of the paper we consider the quasihole degeneracies from a conformal field theory (CFT) perspective, and show that the counting of the domain wall patterns maps one to one on the CFT counting via the fusion rules. Moreover we extend the CFT analysis to topologies of higher genus.

J. Eisenstein: "Edge State Heat Transport in the Quantum Hall Regime" — Heat transport in the quantum Hall regime is investigated using micron-scale heaters and thermometers positioned along the edge of a millimeter-scale two dimensional electron system (2DES). The heaters rely on localized current injection into the 2DES, while the thermometers are based on the thermoelectric effect. In the $\nu = 1$ integer quantized Hall state, a thermoelectric signal appears at an edge thermometer only when it is "downstream," in the sense of electronic edge transport, from the heater.

When the distance between the heater and the thermometer is increased, the thermoelectric signal is reduced, showing that the electrons cool as they propagate along the edge.

G. Ortiz: Symmetries, dimensional reduction and topological quantum order — Are systems that display Topological Quantum Order (TQO), and have a gap to excitations, hardware fault-tolerant at finite temperatures? We show that in surface code models that display low d-dimensional Gauge-Like Symmetries, such as Kitaev’s and its generalizations, the expectation value of topological symmetry operators vanishes at any non-zero temperature, a phenomenon that we coined thermal fragility. The autocorrelation time for the non-local topological quantities in these systems may remain finite even in the thermodynamic limit. We provide explicit expressions for the autocorrelation functions in Kitaev’s model. If temperatures far below the gap may be achieved then these autocorrelation times, albeit finite, can be made large. The physical engine behind the loss of correlations at large spatial and/or temporal distance is the proliferation of topological defects at any finite temperature as a result of a dimensional reduction. This raises an important question: How may we best quantify the degree of protection of quantum information in a topologically ordered system at finite temperature?

A. Stern: “Possible future experiments with non-abelian quantum Hall systems” — We show that non-abelian quantum Hall states can be identified by experimental measurements of the temperature dependence of either the electrochemical potential or the orbital magnetization. The predicted signals of non-abelian statistics are within experimental resolution, and can be clearly distinguished from other contributions under realistic circumstances. The proposed measurement technique also has the potential to resolve spin-ordering transitions in low density electronic systems in the Wigner crystal and strongly-interacting Luttinger liquid regimes.

3.5 Day 5: Interacting anyons etc.

S. Trebst: “Interacting anyons in topological quantum liquids: Things golden” — We show that chains of interacting Fibonacci anyons can support a wide variety of collective ground states ranging from extended critical, gapless phases to gapped phases with ground-state degeneracy and quasiparticle excitations. In particular, we generalize the Majumdar-Ghosh Hamiltonian to anyonic degrees of freedom by extending recently studied pairwise anyonic interactions to three-anyon exchanges. The energetic competition between two- and three-anyon interactions leads to a rich phase diagram that harbors multiple critical and gapped phases. For the critical phases and their higher symmetry endpoints we numerically establish descriptions in terms of two-dimensional conformal field theories. A topological symmetry protects the critical phases and determines the nature of gapped phases.

L. Fidkowski: “Infinite randomness in the Golden chain” — Topological insulators supporting non-abelian anyonic excitations are at the center of attention as candidates for topological quantum computation. In this paper, we analyze the ground-state properties of disordered non-abelian anyonic chains. The resemblance of fusion rules of non-abelian anyons and real space decimation strongly suggests that disordered chains of such anyons generically exhibit infinite-randomness phases. Concentrating on the disordered golden chain model with nearest-neighbor coupling, we show that Fibonacci anyons with the fusion rule $\tau \otimes \tau = \mathbf{1} \oplus \tau$ exhibit two infinite-randomness phases: a random-singlet phase when all bonds prefer the trivial fusion channel, and a mixed phase which occurs whenever a finite density of bonds prefers the τ fusion channel. Real space RG analysis shows that the random-singlet fixed point is unstable to the mixed fixed point. By analyzing the entanglement entropy of the mixed phase, we find its effective central charge, and find that it increases along the RG flow from the random singlet point, thus ruling out a c-theorem for the

effective central charge.

J. Pachos: "Exact demonstration of non-Abelian statistics for the honeycomb lattice model" — The classification of loop symmetries in Kitaev's honeycomb lattice model provides a natural framework to study the abelian topological degeneracy. We derive a perturbative low-energy effective Hamiltonian, that is valid to all orders of the expansion and for all possible toroidal configurations. Using this form we demonstrate at what order the system's topological degeneracy is lifted by finite size effects and note that in the thermodynamic limit it is robust to all orders. Further, we demonstrate that the loop symmetries themselves correspond to the creation, propagation and annihilation of fermions. Importantly, we note that these fermions, made from pairs of vortices, can be moved with no additional energy cost.

N. Bonesteel: "Topological Quantum Computing with Read-Rezayi States" — Read-Rezayi fractional quantum Hall states are among the prime candidates for realizing non-Abelian anyons which in principle can be used for topological quantum computation. We present a prescription for efficiently finding braids which can be used to carry out a universal set of quantum gates on encoded qubits based on anyons of the Read-Rezayi states with $k > 2$, $k \neq 4$. This work extends previous results which only applied to the case $k = 3$ (Fibonacci) and clarifies why in that case gate constructions are simpler than for a generic Read-Rezayi state.

G. Moller: Paired Composite Fermion States — We provide numerical evidence for composite fermion pairing in quantum Hall bilayer systems at filling $\nu = 1/2 + 1/2$ for intermediate spacing between the layers. We identify the phase as $p_x + ip_y$ pairing, and construct high accuracy trial wavefunctions to describe the groundstate on the sphere. For large distances between the layers, and for finite systems, a competing "Hund's rule" state, or composite fermion liquid, prevails for certain system sizes. We argue that for larger systems, the pairing phase will persist to larger layer spacing.

4 Scientific Progress Made

The chief obstacle facing any practical realization of a quantum computer is decoherence; interactions with the environment tend to scramble the fragile quantum state that is needed to effect a useful computation. One strategy, pursued by most researchers in this field, is to minimize the effect of decoherence by clever engineering and use of error correcting codes. Whether or not this will ultimately lead to a usable device is an open question. In 1997 Kitaev proposed a radically different approach to solve this problem: use a quantum state encoded in global topological properties of a system that are inherently immune to the debilitating effects of decoherence. Kitaev's proposal relies on the remarkable occurrence in nature of states of matter that possess topological order. The ground state of such systems exhibits degeneracy, which however, one cannot detect by any local measurement. To do so one must place the system onto a manifold with nontrivial topology (e.g. torus) and perform a highly nonlocal measurement of some quantity. By contrast, the environment, which couples only to the local degrees of freedom, cannot distinguish between the degenerate ground states and thus cannot affect the quantum state encoded as the linear superposition of the individual ground states. Such a system then represents an ideal quantum bit (qubit) in which all errors are corrected automatically at the physical level.

The Workshop summarized worldwide progress made to date towards the ultimate realization of a topological quantum computer. It has been concluded that this beautiful idea faces considerable but not insurmountable practical hurdles. Of all experimentally realizable systems only FQHE is known to unambiguously possess topological order. Abelian anyons have indeed been experimentally confirmed for certain filling fractions ν . However, in order to implement a universal

quantum computer non-abelian anyons are needed. These are expected to occur at $\nu = 5/2$ in the Moore-Read Pfaffian state, but have not yet been experimentally observed. Even if the existence of non-abelian anyons is confirmed it is not at all clear how to create and manipulate them in a controlled fashion. In a sense the very property which makes anyons potentially useful their complete decoupling from the environment makes their manipulation difficult.

Participants discussed other emerging new physical systems that could potentially be used as a platform for topological quantum computation. These include string-net condensates in strongly correlated electron systems, exotic superconductors and superfluids, and the recently discovered topological insulators. The expectations were high especially regarding the latter due to the fact that these materials manifest their topologically non-trivial behavior even at room temperature. This is in contrast to FQHE or exotic superconductors which exhibit interesting behavior only at cryogenic temperatures and therefore expensive and complicated cooling machinery would typically be required to operate any device based on these systems. Participants were optimistic that the advances in material synthesis, experimental manipulation and detection, combined with the recent progress in quantum algorithms will lead to major breakthroughs in the coming years, culminating in a working prototype of topological quantum computer.