Higher Order Freeness: A Survey

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Second order freeness and fluctuations of random matrices:

Mingo + Speicher:
I. Gaussian and Wishart matrices and cyclic Fock spaces
JFA 235 (2006), 226-270

Mingo + Sniady + Speicher:
II. Unitary random matrices
Adv. Math. 209 (2007), 212-240

Collins + Mingo + Sniady + Speicher:
III. Higher order freeness and free cumulants
Documenta Math. 12 (2007), 1-70

Kusalik + Mingo + Speicher: CRELLES 604 (2007), 1-46

Mingo + Speicher + Tan: arXiv:0708.0586 (to appear in TAMS)

Warning

We deal only with **complex** random matrices.

Higher order freeness for the real case still has to be worked out!

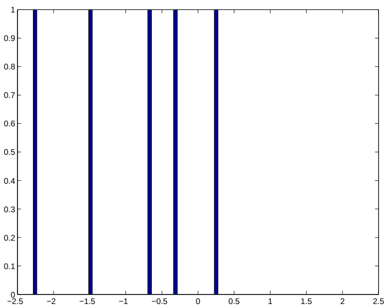
We want to consider $N \times N$ random matrices A_N in the limit $\mathbf{N} \to \infty$.

Which kind of information about the random matrices do we want to keep in the limit $N = \infty$?

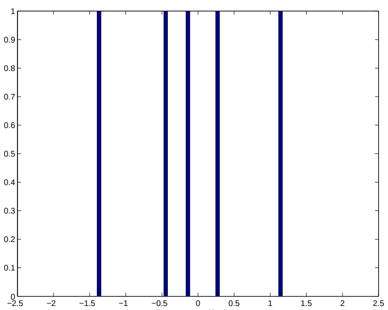
Consider selfadjoint Gaussian $N \times N$ random matrices X_N. One knows:

- empirical eigenvalue distribution of X_N converges almost surely to deterministic limit distribution μ_X

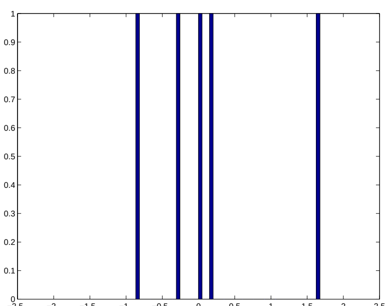
- one has a large deviation principle for convergence towards μ_X



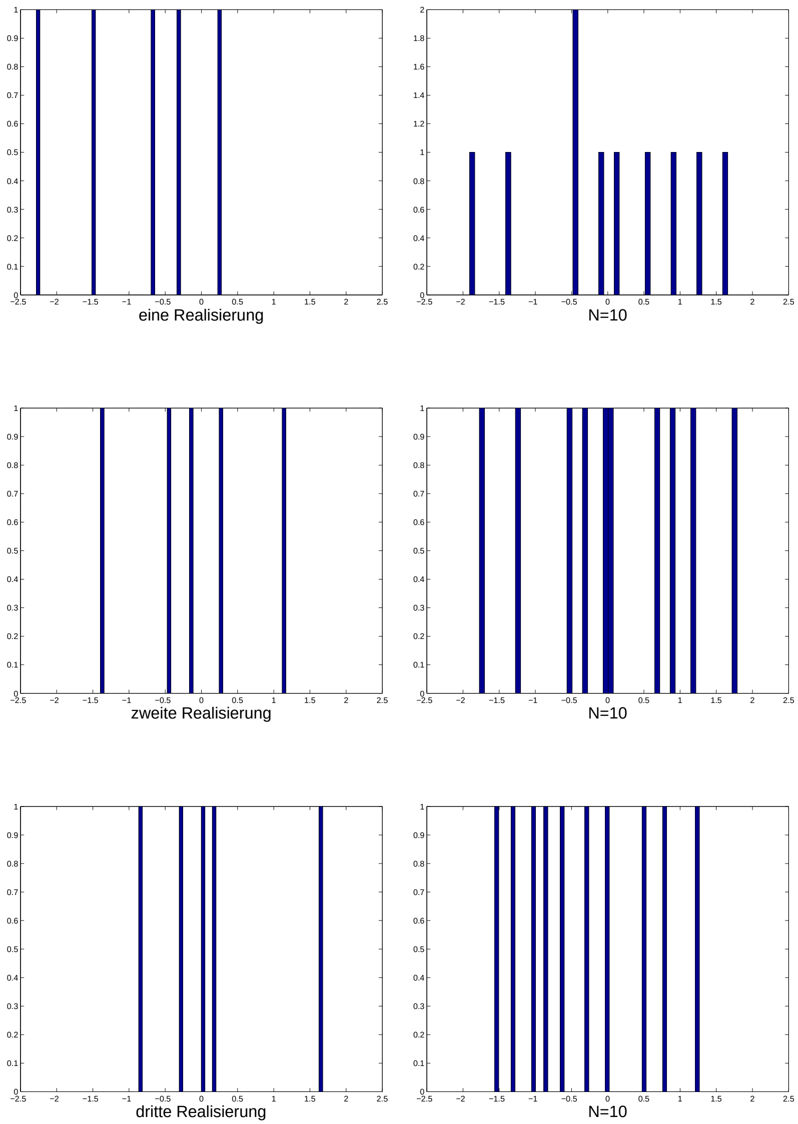
eine Realisierung



zweite Realisierung



dritte Realisierung


eine Realisierung
N=10
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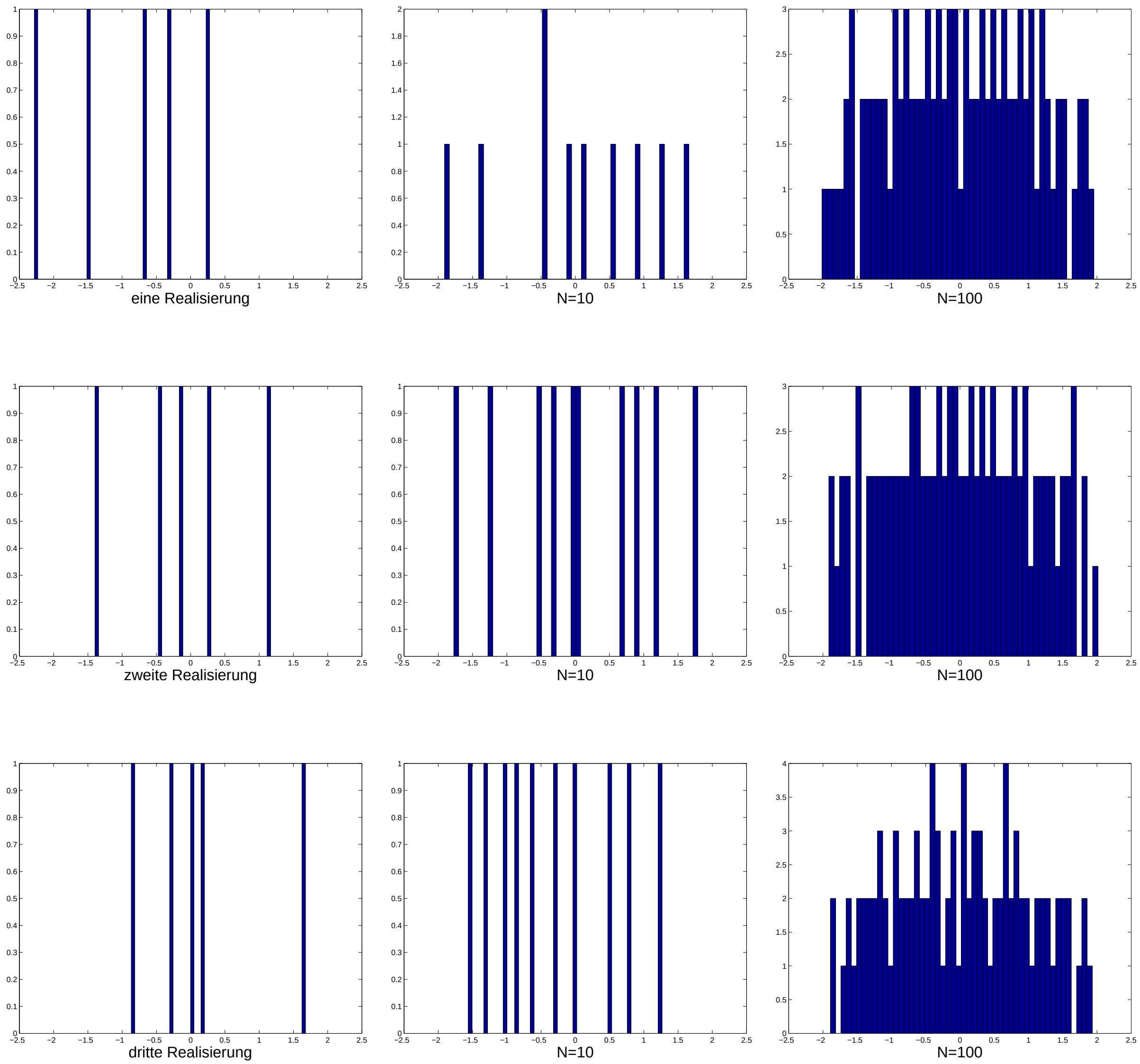
eine Realisierung
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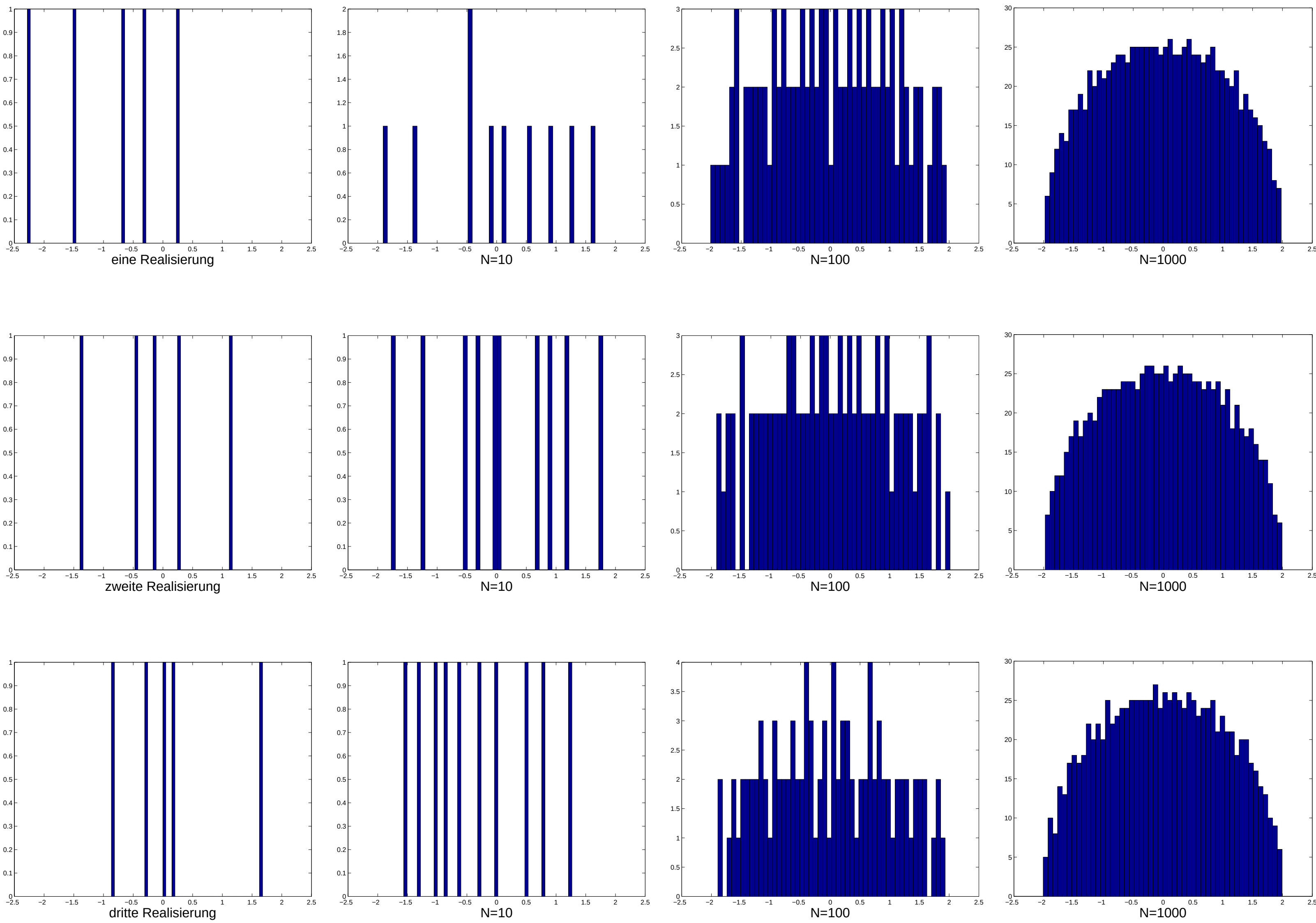
dritte Realisierung
N=10
N=100


eine Realisierung
N=10
N=100
N=1000
zweite Realisierung
N=10
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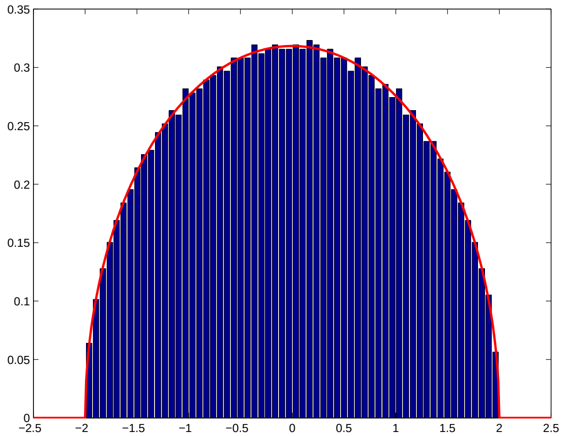
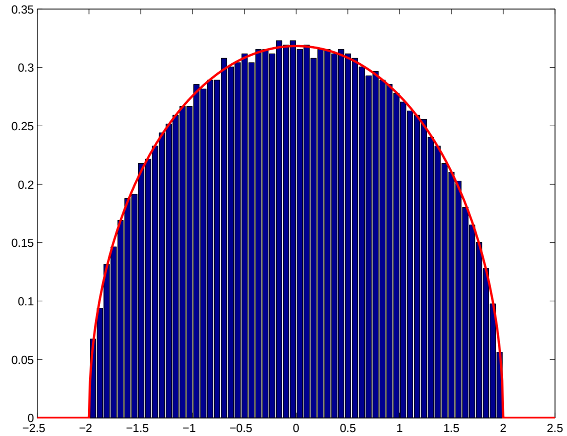
N=1000
dritte Realisierung
N=10
N=100
N=1000

Wigner's semicircle law

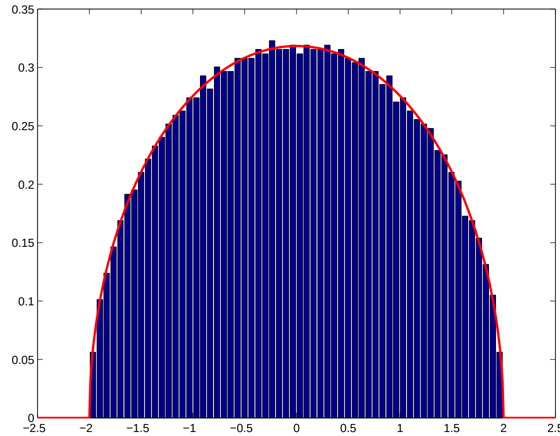
$$N = 4000$$



... one realization ...



... another realization ...



... yet another one ...

Convergence of μ_{X_N} towards μ_X is governed by **large deviation principle**:

$$\text{Prob}(\mu_{X_N} \approx \nu) \sim e^{-N^2 I(\nu)},$$

where rate function $\nu \mapsto I(\nu)$ is given as Legendre transform of

$$\mathbb{C}\langle X\rangle \ni p \mapsto \lim_{N\to\infty} \frac{1}{N^2} \log E\Big\{e^{-N^2 \text{tr}(p(X_N))}\Big\}$$

Note:

$$\log E\Big\{e^{-N^2 \text{tr}(p(X_N))}\Big\} = \sum_r \frac{(-1)^r}{r!} N^{2r} \cdot k_r\Big(\text{tr}(p(X_N)), \dots, \text{tr}(p(X_N))\Big)$$

where

k_r are classical cumulants

This motivates our **general assumption** on the considered random matrices A_N:

For all $r \in \mathbb{N}$ and all $k_1, \dots, k_r \in \mathbb{N}$ the following limits exists

$$\lim_{N\to\infty} N^{2r-2} \underbrace{k_r\Big(\mathrm{tr}(A_N^{k_1}), \dots, \mathrm{tr}(A_N^{k_r})\Big)}_{\substack{\text{classical cumulants} \\ \text{of traces of powers}}} =: \alpha^A_{k_1,\dots,k_r}$$

The α's are the asymptotic **correlation moments** of our random matrix ensemble A_N and constitute its

limiting distribution of all orders.

Typical examples for random matrices where limiting distribution of all orders exists: Gaussian random matrices, Wishart random matrices, Haar unitary random matrices, and combinations of independent copies of such ensembles

Note: We are looking on random matrix ensembles whose eigenvalues have a correlation as for Gaussian random matrices:

$$\text{tr}(A_N^k) = \frac{\lambda_1^k + \cdots + \lambda_N^k}{N}$$

Eigenvalues $\lambda_1, \ldots, \lambda_N$ of A_N are **not independent**, but feel some interaction

Contrast this with following situation:

$$D_N = \begin{pmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_N \end{pmatrix},$$

where $\eta_1, \eta_2, \ldots$ are **independent** and identically distributed according to η.

Then

$$\mathrm{tr}(D_N^k) = \frac{\eta_1^k + \cdots + \eta_N^k}{N} \to E[\eta^k]$$

with large deviation principle $\sim e^{-\mathbf{N}H(\nu)}$; not $\sim e^{-N^2 I(\nu)}$

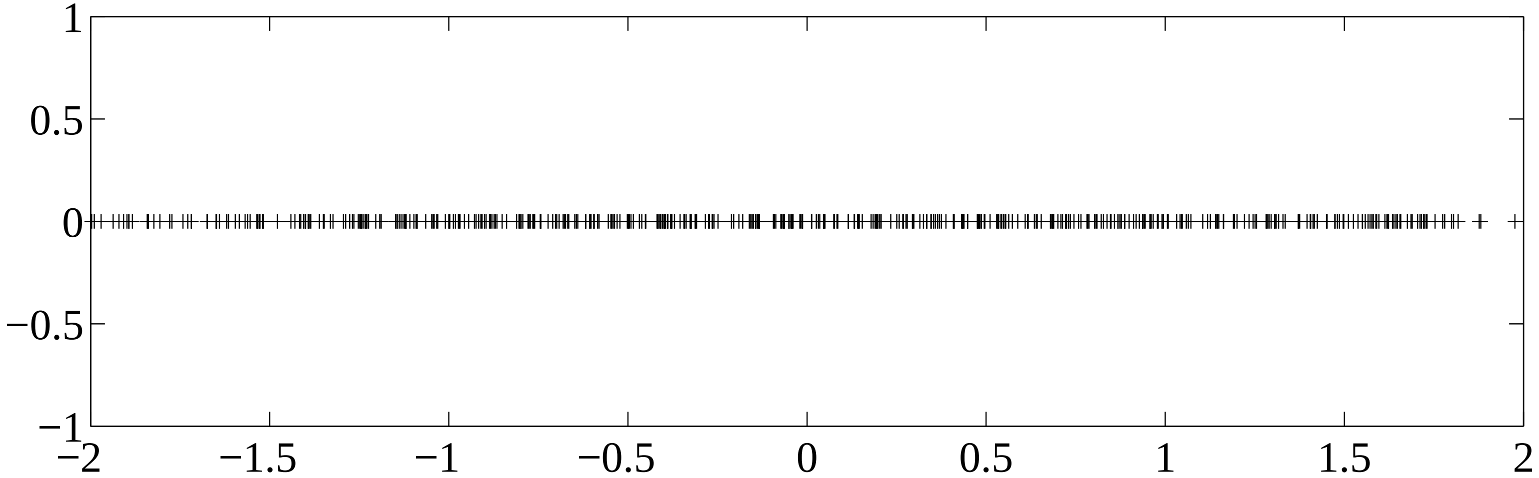
In this case:

$$k_r\Big(\text{tr}(D_N^{k_1}),\dots,\text{tr}(D_N^{k_r})\Big)= N^{1-r}k_r(\eta^{k_1},\dots,\eta^{k_r}),$$

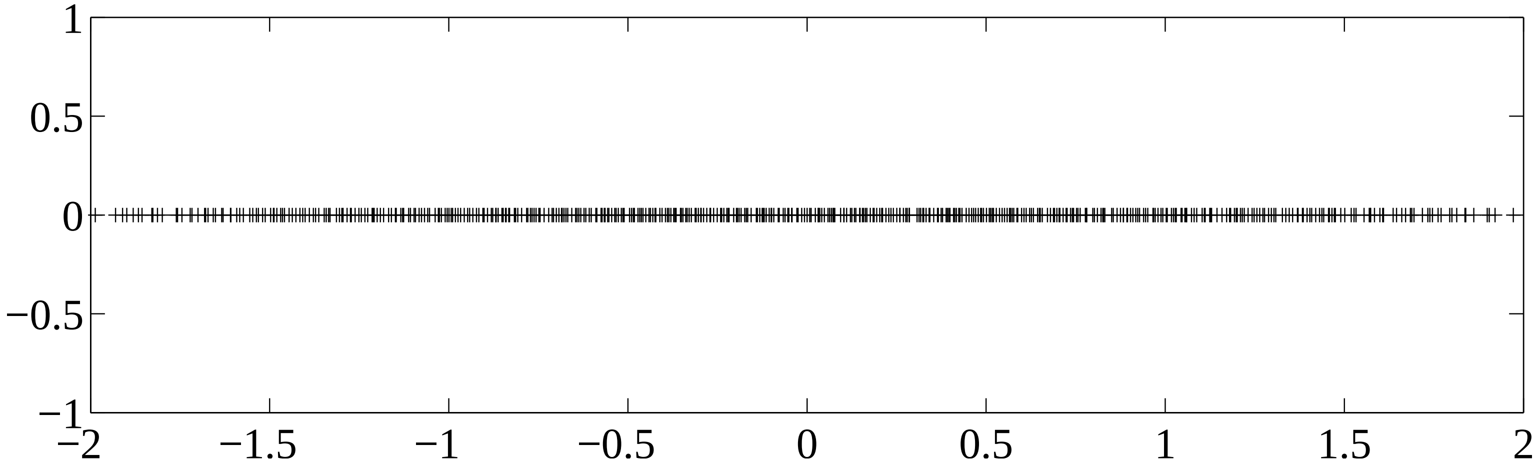
and thus: D_N has no limiting distribution of all orders in our sense.

The Gaussian random matrices A_N and the above ensemble with a semicircle distribution for η have the same asymptotic eigenvalue distribution, but a quite different type of convergence towards the semicircle

500 independent eigenvalues with semicircular distribution



eigenvalues of a 500 x 500 Gaussian random matrix



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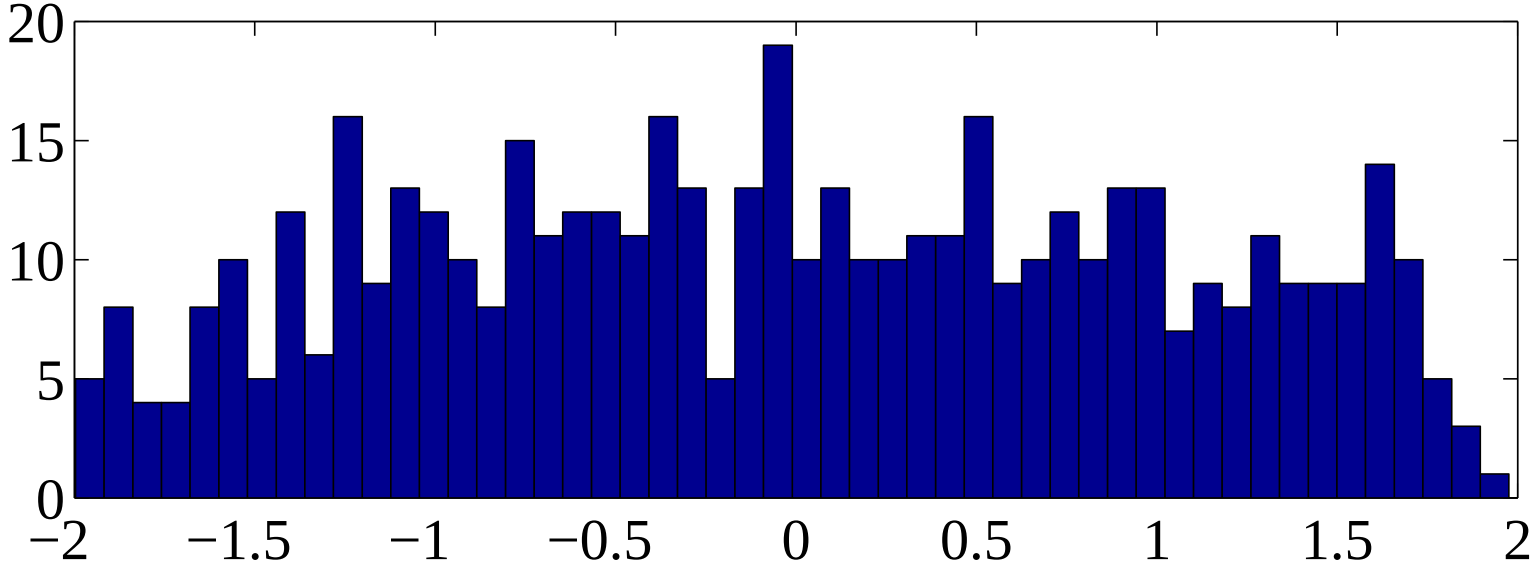
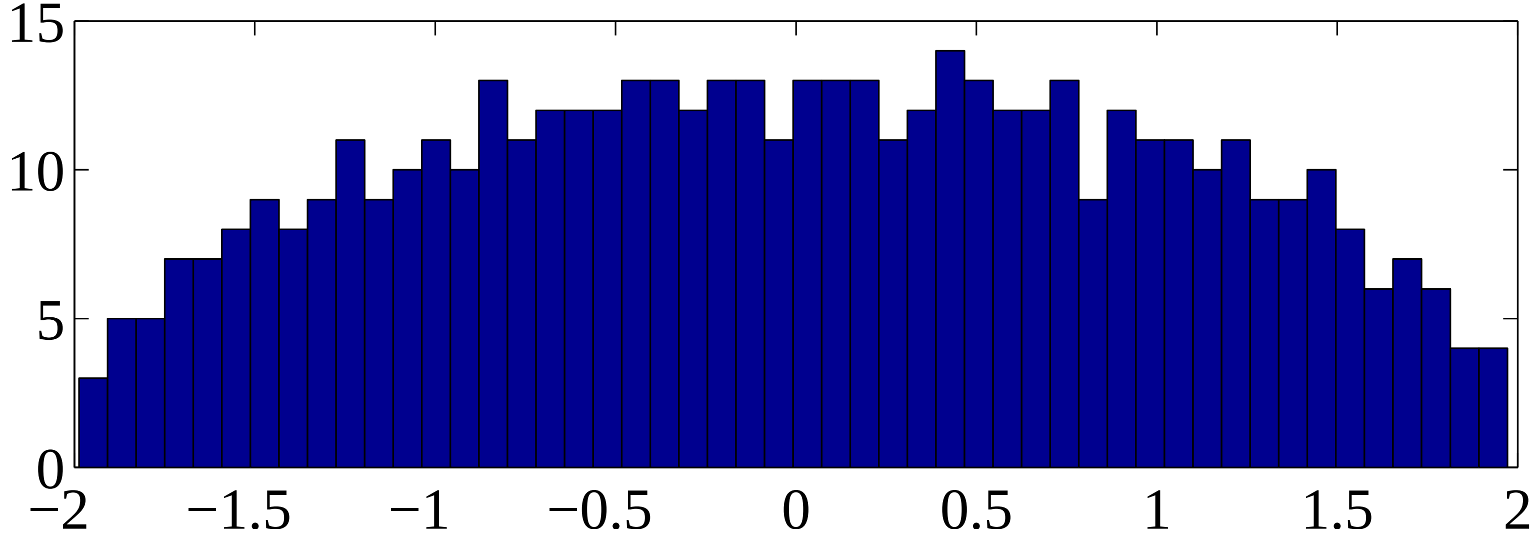


eigenvalues of a 500 x 500 Gaussian random matrix

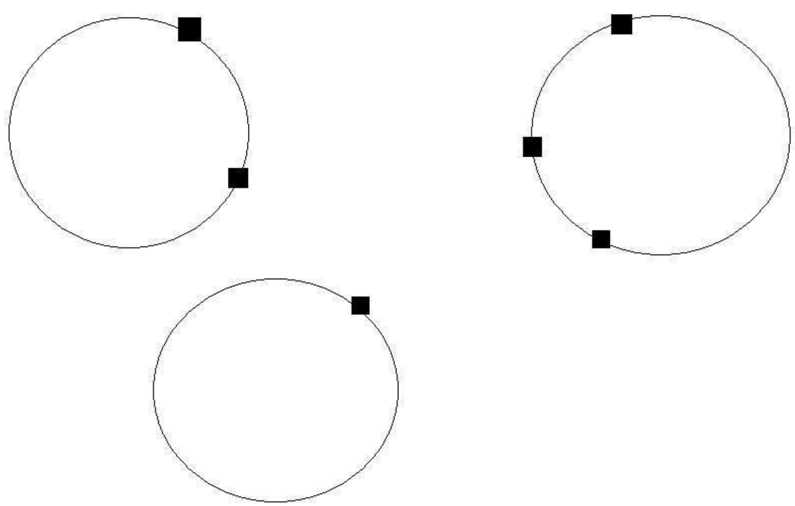


Remarks:

1) For Gaussian (and also for Wishart) random matrices there are nice combinatorial descriptions of the higher order limit distributions in terms of planar pictures

$$\alpha^{Gaussian}_{k_1,\dots,k_r} = \#\text{NC-pairings of } r \text{ circles,}$$
with k_1 points on first circle,
k_2 points on second circle, etc.
such that all circles are connected by pairing

Consider $\alpha_{2,3,1}$



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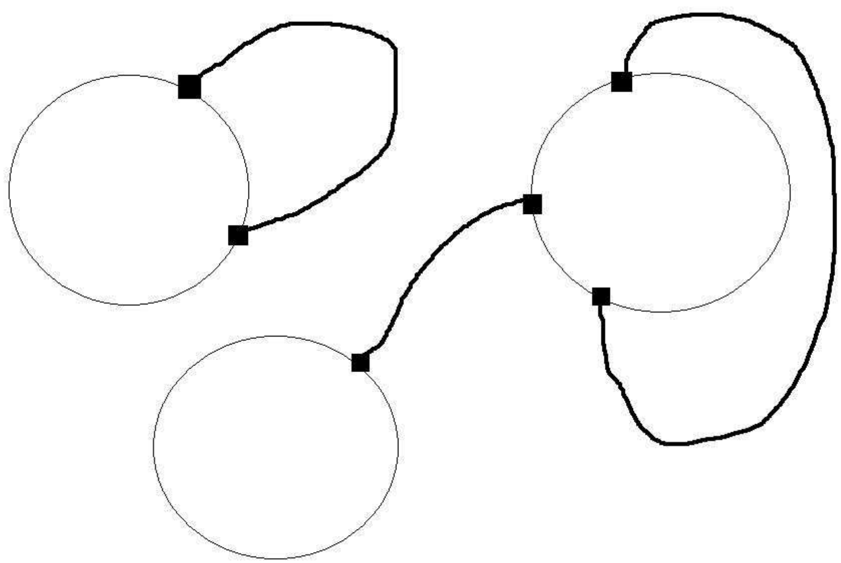
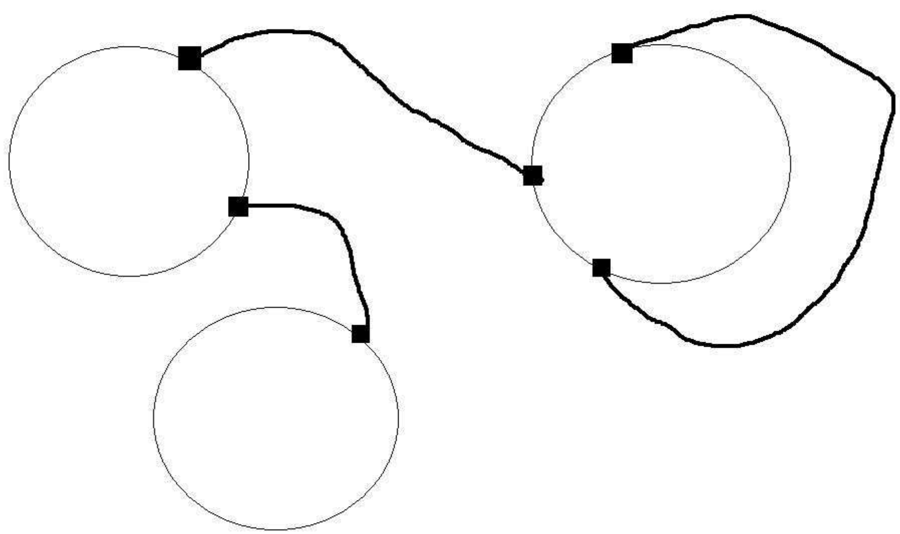


does not count!

Consider $\alpha_{2,3,1}$



counts!

2) Specialize general theory to second order: An $N \times N$ random matrix ensemble $(A_N)_{N\in\mathbb{N}}$ has a **second order limit distribution** if for all $m, n \geq 1$ the limits

$$\alpha_n := \lim_{N\to\infty} E[\mathrm{tr}(A_N^n)]$$

and

$$\alpha_{m,n} := \lim_{N\to\infty} \mathrm{cov}\Big(\mathrm{Tr}(A_N^m), \mathrm{Tr}(A_N^n)\Big)$$

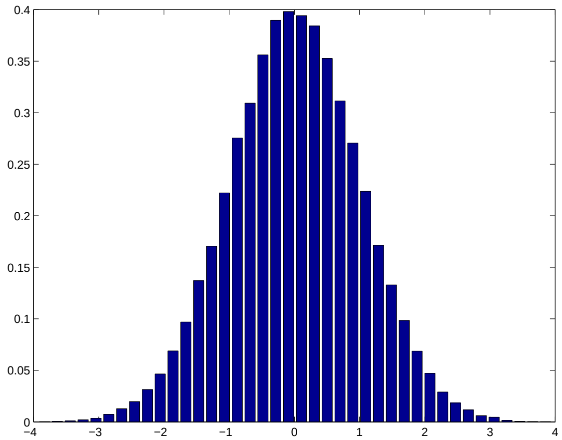
exist and if all higher classical cumulants of $\mathrm{Tr}(A_N^m)$ go to zero.

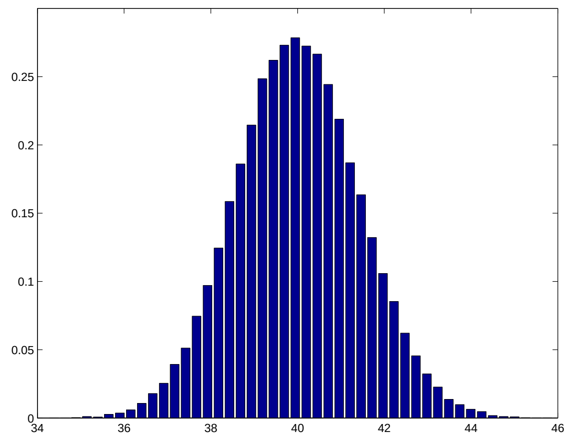
This means that the family

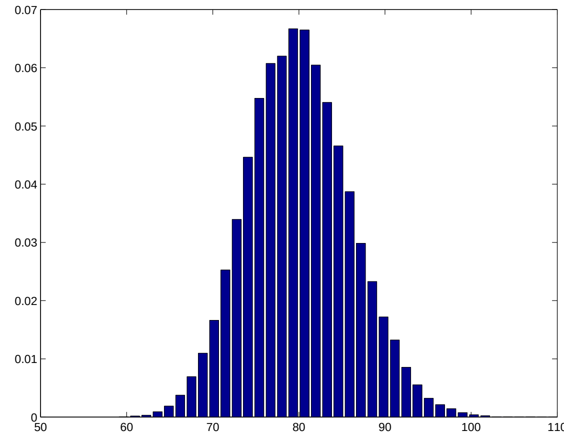
$$\Big(\mathrm{Tr}(A_N^m) - E[\mathrm{Tr}(A_N^m)]\Big)_{m\in\mathbb{N}}$$

converges to a Gaussian family.

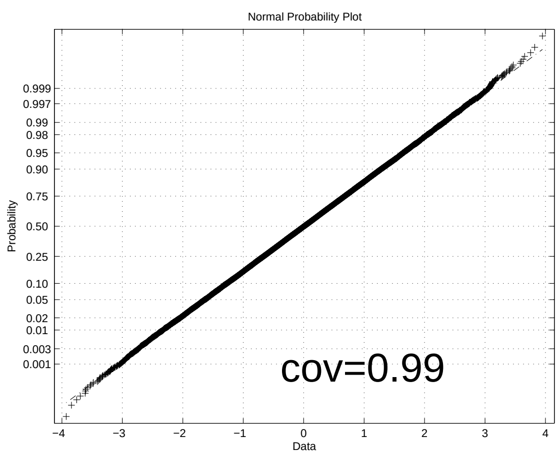
Example: Gaussian random matrix A ($N = 40$, trials=50.000)

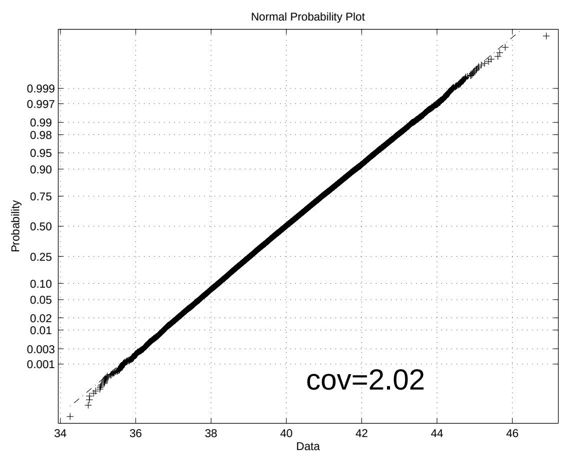


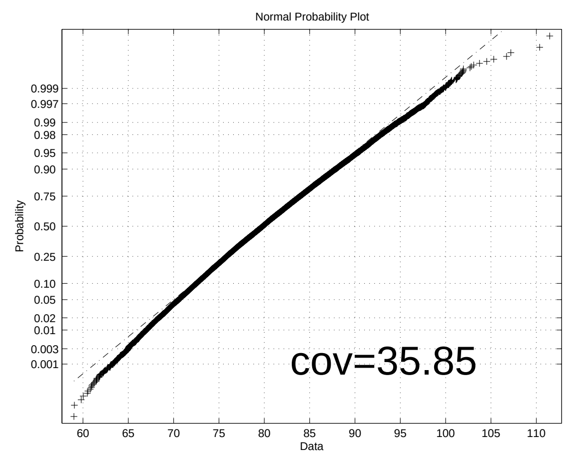




. $\mathrm{Var}(\mathrm{Tr}(A)) = 1$ $\quad$ $\mathrm{Var}(\mathrm{Tr}(A^2)) = 2$ $\quad$ $\mathrm{Var}(\mathrm{Tr}(A^4)) = 36$







Now consider **two random matrix ensembles** A_N, B_N

Relevant quantities are all joint correlation moments

$$\lim_{N\to\infty} N^{2r-2} k_r\Big(\text{tr}(p_1(A_N, B_N)), \dots, \text{tr}(p_r(A_N, B_N))\Big)$$

for all $r \in \mathbb{N}$ and all polynomials $p_1, \dots, p_r$

asymptotic joint distribution of all orders of A_N, B_N

Theorem: If A_N and B_N are in **generic position**, i.e.,

- A_N and B_N are independent
- at least one of them is unitarily invariant

and if A_N as well as B_N have asymptotic distributions of all orders then also the asymptotic joint distribution of all orders of A_N, B_N exists and it is, furthermore, determined uniquely and in a universal way by the joint distribution of A and the joint distribution of B.

This universal calculation rule is the essence of

freeness (of all orders)

$$\lim_{N\to\infty} \mathrm{cov}\Big(\mathrm{Tr}(A_N B_N), \mathrm{Tr}(A_N B_N)\Big)$$

$$= \lim_{N\to\infty} \Big\{ E\Big[\mathrm{tr}(A_N A_N)\Big] \cdot E\Big[\mathrm{tr}(B_N B_N)\Big]$$

$$- E\Big[\mathrm{tr}(A_N A_N)\Big] \cdot E\Big[\mathrm{tr}(B_N)\Big] \cdot E\Big[\mathrm{tr}(B_N)\Big]$$

$$- E\Big[\mathrm{tr}(A_N)\Big] \cdot E\Big[\mathrm{tr}(A_N)\Big] \cdot E\Big[\mathrm{tr}(B_N B_N)\Big]$$

$$+ E\Big[\mathrm{tr}(A_N)\Big] \cdot E\Big[\mathrm{tr}(A_N)\Big] \cdot E\Big[\mathrm{tr}(B_N)\Big] \cdot E\Big[\mathrm{tr}(B_N)\Big]$$

$$+ \mathrm{cov}\Big(\mathrm{tr}(A_N), \mathrm{tr}(A_N)\Big) \cdot E\Big[\mathrm{tr}(B_N)\Big] \cdot E\Big[\mathrm{tr}(B_N)\Big]$$

$$+ E\Big[\mathrm{tr}(A_N)\Big] \cdot E\Big[\mathrm{tr}(A_N)\Big] \cdot \mathrm{cov}\Big(\mathrm{tr}(B_N), \mathrm{tr}(B_N)\Big) \Big\}$$

In order to understand this universal calculation rule use the

idea of cumulants!

Write our correlation moments

$$k_r\Big(\text{tr}(A^{k_1}),\dots,\text{tr}(A^{k_r})\Big)$$

in terms of cumulants of entries of our matrix,

$$k_r(a_{i(1)j(1)},\dots,a_{i(r)j(r)}).$$

Asymptotically, the later will give the cumulants in our theory.

To make this connection explicit, consider

$$\mathbf{unitarily\ invariant}\ \mathbf{A_N = (a_{ij})},$$

i.e., the joint distribution of the entries of A_N is the same as the joint distribution of UA_NU^*, for any unitary $N \times N$ matrix U.

Then, the only contributing cumulants of the entries are those with cycle structure in their indices!

We have a **Wick type formula**:

$$k_r(a_{i(1)j(1)}, \ldots, a_{i(r)j(r)}) = \sum_{\pi \in S_n} \delta_{i,j\circ\pi} \kappa(\pi)$$

Examples:

$$k_1(a_{79}) = ?$$

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Note: κ depends actually on N

$$k_1(a_{77}) = \kappa^{(N)}((1))$$

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$$\pi \in S_r : \qquad \kappa^{(N)}(\pi) \sim N^{-r+2-\#\pi}$$

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$$\pi \in S_r : \qquad \kappa^{(N)}(\pi) \sim N^{-r+2-\#\pi}$$

$$\kappa(\pi) := \lim_{N\to\infty} N^{r-2+\#\pi}\kappa^{(N)}(\pi)$$

Consider

$$\alpha_{2,1} = \lim_{N\to\infty} N^2 k_2\Big(\mathrm{tr}(A^2), \mathrm{tr}(A)\Big)$$

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$$k_3(a_{\mathbf{ij}}, a_{\mathbf{ji}}, a_{\mathbf{kk}})$$
$$+\, k_2(a_{ij}, a_{kk})k_1(a_{ji})$$
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$$= \kappa((1,2)(3)) +$$

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$$= \kappa((1,2)(3)) + \kappa((1,2,3)) + \kappa((1,3,2)) +$$

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$$= \kappa((1,2)(3)) + \kappa((1,2,3)) + \kappa((1,3,2)) + \kappa((1)(3))\kappa((2))$$
$$+ \kappa((1,3))\kappa((2)) + \kappa((2)(3))\kappa((1)) + \kappa((2,3))\kappa((1))$$

Thus

$$\begin{aligned}\alpha_{2,1} = \quad & \kappa((1,2)(3)) \\ & + \kappa((1,2,3)) \\ & + \kappa((1,3,2)) \\ & + \kappa((1)(3))\kappa((2)) \\ & + \kappa((1,3))\kappa((2)) \\ & + \kappa((2)(3))\kappa((1)) \\ & + \kappa((2,3))\kappa((1))\end{aligned}$$

Thus

$$
\begin{aligned}
\alpha_{2,1} = \quad & \kappa((1,2)(3)) && \kappa_{2,1} \\
& + \kappa((1,2,3)) && \kappa_3 \\
& + \kappa((1,3,2)) && \kappa_3 \\
& + \kappa((1)(3))\kappa((2)) && \kappa_{1,2}\kappa_1 \\
& + \kappa((1,3))\kappa((2)) && \kappa_2\kappa_1 \\
& + \kappa((2)(3))\kappa((1)) && \kappa_{1,1}\kappa_1 \\
& + \kappa((2,3))\kappa((1)) && \kappa_2\kappa_1
\end{aligned}
$$

Thus

$$
\begin{aligned}
\alpha_{2,1} = \quad & \kappa((1,2)(3)) && \kappa_{2,1} && \kappa\Big(\{1,2,3\},(1,2)(3)\Big) \\
& + \kappa((1,2,3)) && \kappa_3 && \kappa\Big(\{1,2,3\},(1,2,3)\Big) \\
& + \kappa((1,3,2)) && \kappa_3 && \kappa\Big(\{1,2,3\},(1,3,2)\Big) \\
& + \kappa((1)(3))\kappa((2)) && \kappa_{1,2}\kappa_1 && \\
& + \kappa((1,3))\kappa((2)) && \kappa_2\kappa_1 && \\
& + \kappa((2)(3))\kappa((1)) && \kappa_{1,1}\kappa_1 && \\
& + \kappa((2,3))\kappa((1)) && \kappa_2\kappa_1 &&
\end{aligned}
$$

Thus

$$
\begin{aligned}
\alpha_{2,1} = \quad & \kappa((1,2)(3)) && \kappa_{2,1} && \kappa\Big(\{1,2,3\},(1,2)(3)\Big) \\
& + \kappa((1,2,3)) && \kappa_3 && \kappa\Big(\{1,2,3\},(1,2,3)\Big) \\
& + \kappa((1,3,2)) && \kappa_3 && \kappa\Big(\{1,2,3\},(1,3,2)\Big) \\
& + \kappa((1)(3))\kappa((2)) && \kappa_{1,2}\kappa_1 && \kappa(\Big(\{1,3\}\{2\},(1)(3)(2)\Big) \\
& + \kappa((1,3))\kappa((2)) && \kappa_2\kappa_1 && \kappa(\Big(\{1,3\}\{2\},(1,3)(2)\Big) \\
& + \kappa((2)(3))\kappa((1)) && \kappa_{1,1}\kappa_1 && \kappa(\Big(\{1\}\{2,3\},(1)(2)(3)\Big) \\
& + \kappa((2,3))\kappa((1)) && \kappa_2\kappa_1 && \kappa(\Big(\{1\}\{2,3\},(1)(2,3)\Big)
\end{aligned}
$$

general combinatorial object

partitioned permutation $(\mathcal{V}, \pi) \in \mathcal{P}S_n$

$$\pi \in S_n, \qquad \mathcal{V} \in \mathcal{P}_n, \qquad \text{with} \qquad \mathcal{V} \geq \pi$$

Index both correlation moments $\varphi(\mathcal{V}, \pi)$ and cumulants $\kappa(\mathcal{V}, \pi)$ with $(\mathcal{V}, \pi)$:

product of moments/cumulants according to blocks of $\mathcal{V}$, distribution into slots for arguments according to cycles of π:

Let $C_1, \ldots, C_9 \in \{A, B\}$, with $C_k = (c_{ij}^{(k)})_{i,j=1}^{N}$.

$$\varphi\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\}, (1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1, \ldots, C_9]$$

$$\kappa\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\}, (1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1, \ldots, C_9]$$

Let $C_1, \dots, C_9 \in \{A, B\}$, with $C_k = (c_{ij}^{(k)})_{i,j=1}^N$.

$$\varphi\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\}, (1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1, \dots, C_9]$$

$$= \lim_{N\to\infty} N^6 \cdot k_3\Big(\mathrm{tr}(C_1 C_3 C_4), \mathrm{tr}(C_6), \mathrm{tr}(C_7)\Big) \cdots$$

$$\kappa\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\}, (1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1, \dots, C_9]$$

$$= \lim_{N\to\infty} N^9 \cdot k_5\Big(c_{12}^{(1)}, c_{23}^{(3)}, c_{31}^{(4)}, c_{44}^{(6)}, c_{55}^{(7)}\Big) \cdots$$

Let $C_1, \ldots, C_9 \in \{A, B\}$, with $C_k = (c_{ij}^{(k)})_{i,j=1}^N$.

$$\varphi\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\},(1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1,\ldots,C_9]$$

$$= \lim_{N\to\infty} N^6 \cdot k_3\Big(\text{tr}(C_1C_3C_4),\text{tr}(C_6),\text{tr}(C_7)\Big)\cdot k_2\Big(\text{tr}(C_2C_8),\text{tr}(C_5)\Big)\cdots$$

$$\kappa\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\},(1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1,\ldots,C_9]$$

$$= \lim_{N\to\infty} N^9 \cdot k_5\Big(c_{12}^{(1)},c_{23}^{(3)},c_{31}^{(4)},c_{44}^{(6)},c_{55}^{(7)}\Big)\cdot k_3\Big(c_{12}^{(2)},c_{21}^{(8)},c_{33}^{(5)}\Big)\cdots$$

Let $C_1, \dots, C_9 \in \{A, B\}$, with $C_k = (c_{ij}^{(k)})_{i,j=1}^N$.

$$\varphi\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\}, (1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1, \dots, C_9]$$

$$= \lim_{N\to\infty} N^6 \cdot k_3\Big(\text{tr}(C_1C_3C_4), \text{tr}(C_6), \text{tr}(C_7)\Big) \cdot k_2\Big(\text{tr}(C_2C_8), \text{tr}(C_5)\Big) \cdot k_1\Big(\text{tr}(C_9)\Big)$$

$$\kappa\Big(\{1,3,4,6,7\}\{2,5,8\}\{9\}, (1,3,4)(2,8)(5)(6)(7)(9)\Big)[C_1, \dots, C_9]$$

$$= \lim_{N\to\infty} N^9 \cdot k_5\Big(c_{12}^{(1)}, c_{23}^{(3)}, c_{31}^{(4)}, c_{44}^{(6)}, c_{55}^{(7)}\Big) \cdot k_3\Big(c_{12}^{(2)}, c_{21}^{(8)}, c_{33}^{(5)}\Big) \cdot k_1\Big(c_{33}^{(9)}\Big)$$

Define **length function**

$$|(\mathcal{V},\pi)| := n - (2\#\mathcal{V} - \#\pi)$$

We have triangle inequality ($(\mathcal{V},\pi),(\mathcal{W},\sigma) \in \mathcal{P}S_n$)

$$|(\mathcal{V}\vee\mathcal{W},\pi\sigma)| \leq |(\mathcal{V},\pi)| + |(\mathcal{W},\sigma)|$$

Define product

$$(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma) = \begin{cases} (\mathcal{V}\vee\mathcal{W},\pi\sigma), & |(\mathcal{V}\vee\mathcal{W},\pi\sigma)| = |(\mathcal{V},\pi)| + |(\mathcal{W},\sigma)| \\ 0, & \text{otherwise} \end{cases}$$

Asymptotically, for $\mathbf{N} \to \infty$, only the **geodesic terms** corresponding to equality in the triangle inequality contribute.

In particular, the relation between correlation moments and cumulants is given by the

moment-cumulant formula for all orders

$$\varphi(\mathcal{U},\gamma)[C_1,\dots,C_n] = \sum_{\substack{(\mathcal{V},\pi)\in\mathcal{P}S_n \\ (\mathcal{V},\pi)\cdot(0,\gamma\pi^{-1})=(\mathcal{U},\gamma)}} \kappa(\mathcal{V},\pi)[C_1,\dots,C_n]$$

If A_N and B_N are in generic position (i.e., asymptotically free of all orders), then we have for their asymptotic distribution

- **the vanishing of mixed cumulants**

  $\kappa(1_n,\pi)[C_1,\dots,C_n] = 0$, whenever $C_1,\dots,C_n$ contain A as well as B

- **convolution formula for cumulants of products**

$$\kappa(\mathcal{U},\gamma)[AB,AB,\dots,AB]$$

$$= \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \kappa(\mathcal{V},\pi)[A,A,\dots,A]\cdot\kappa(\mathcal{W},\sigma)[B,B,\dots,B]$$

Restrict now to special situation

Consider only first and second order, and restrict to problem of the sum of A and B

If A and B are free, then the second order distribution (covariances) of $A+B$ depends only on the expectations and covariances of A and of B.

Example: We have

$$\alpha_{1,2}^{A+B} = \alpha_{1,2}^{A} + \alpha_{1,2}^{B} + 2\alpha_{1}^{A} \cdot \alpha_{1,1}^{B} + 2\alpha_{1}^{B} \cdot \alpha_{1,1}^{A},$$

i.e.,

$$\begin{aligned}\mathrm{cov}\Big(\mathrm{Tr}(A+B), &\mathrm{Tr}\big((A+B)^2\big)\Big) \\ &= \mathrm{cov}\Big(\mathrm{Tr}(A), \mathrm{Tr}(A^2)\Big) + \mathrm{cov}\Big(\mathrm{Tr}(B), \mathrm{Tr}(B^2)\Big) \\ &\quad + 2E[\mathrm{tr}(A)] \cdot \mathrm{cov}\Big(\mathrm{Tr}(B), \mathrm{Tr}(B)\Big) \\ &\quad + 2E[\mathrm{tr}(B)] \cdot \mathrm{cov}\Big(\mathrm{Tr}(A), \mathrm{Tr}(A)\Big)\end{aligned}$$

Moment-cumulant formulas for first and second order say

$$
\begin{aligned}
\alpha_1 &= \kappa_1 \\
\alpha_2 &= \kappa_2 + \kappa_1\kappa_1 \\
\alpha_3 &= \kappa_3 + \kappa_1\kappa_2 + \kappa_2\kappa_1 + \kappa_2\kappa_1 + \kappa_1\kappa_1\kappa_1 \\
\alpha_4 &= \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_1^2\kappa_2 + \kappa_1^4 \\
&\vdots \\
\alpha_{1,1} &= \kappa_{1,1} + \kappa_2 \\
\alpha_{1,2} &= \kappa_{1,2} + 2\kappa_1\kappa_1 + 2\kappa_3 + 2\kappa_1\kappa_2 \\
\alpha_{2,2} &= \kappa_{2,2} + 4\kappa_1\kappa_{1,2} + 4\kappa_1^2\kappa_{1,1} + 4\kappa_4 \\
&\quad + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_1^2\kappa_2 \\
&\vdots
\end{aligned}
$$

Vanishing of mixed cumulants gives

additivity of free cumulants for free A, B

$$\kappa_m^{A+B} = \kappa_m^A + \kappa_m^B \qquad \forall\, m$$

and

$$\kappa_{m,n}^{A+B} = \kappa_{m,n}^A + \kappa_{m,n}^B \qquad \forall\, m, n$$

Combinatorial relation between moments and cumulants can be rewritten in terms of generating power series

Recall: first order case (Voiculescu)

$$G(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_n}{x^{n+1}} \qquad \text{Cauchy transform}$$

and

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \kappa_n x^{n-1} \qquad \mathcal{R}\text{-transform}$$

are related by the relation

$$\frac{1}{G(x)} + \mathcal{R}(G(x)) = x.$$

Second order R-transform formula

$$G(x,y) := \sum_{m,n\geq 1} \alpha_{m,n} \frac{1}{x^{m+1}} \frac{1}{y^{n+1}}$$

and

$$\mathcal{R}(x,y) = \sum_{m,n\geq 1} \kappa_{m,n} x^{m-1} y^{n-1}$$

are related by the equation

$$G(x,y) = G'(x) \cdot G'(y) \cdot \mathcal{R}\Big(G(x), G(y)\Big) + \frac{\partial^2}{\partial x \partial y} \left[\log \left(\frac{G(x) - G(y)}{x - y} \right) \right]$$

If second order free cumulants are zero, then formula reduces to

$$G(x,y) = \frac{\partial^2}{\partial x \partial y}\left[\log\left(\frac{G(x)-G(y)}{x-y}\right)\right],$$

i.e. the fluctuations in such a case are determined by the eigenvalue distribution.

This is the formula of **Bai and Silverstein (2004)** for the fluctuations of general Wishart matrices.

$$G(x,y) = \frac{\partial^2}{\partial x \partial y}\left[\log\left(\frac{G(x)-G(y)}{x-y}\right)\right],$$

Second order free cumulants are zero for example for

- Gaussian random matrices
- Wishart matrices
- independent sums of Gaussian and Wishart

How do Wishart matrices fit in this theory?

Consider

$$A_N = X_N T_N X_N^*$$

where

- X_N are $N \times N$ non-selfadjoint Gaussian random matrices
- T_N are random matrix ensemble such that second order limit distribution exists
- X_N and T_N are independent (for example, T_N are deterministic)

Then, in first order,

$$A_N = X_N T_N X_N^*$$

converges to

$$A = CTC^*$$

where

- C is circular
- T has the limit distribution of the T_N
- C and T are $*$-free

And

$$A = CTC^*$$

is a

free compound Poisson element,

determined by the fact that

$$\kappa_n^A = \alpha_n^T \qquad \text{for all } n$$

In terms of transforms this gives the fixed point equation of Marchenko-Pastur for the Cauchy transform of A in terms of the Cauchy transform of T.

In second order, the situation is exactly the same: The limit

$$A = CTC^*$$

of

$$A_N = X_N T_N X_N^*$$

is a

free compound Poisson element of second order,

determined by the fact that

$$\kappa_n^A = \alpha_n^T \qquad \text{for all } n$$

and

$$\kappa_{m,n}^A = \alpha_{m,n}^T \qquad \text{for all } m, n$$

$$\kappa_n^A = \alpha_n^T, \qquad \kappa_{m,n}^A = \alpha_{m,n}^T \qquad \text{for all } m, n$$

In terms of transforms this gives:

$$G^A(x,y) = \frac{G'(x) \cdot G'(y)}{G(x)^2 G(y)^2} \cdot G^T\Big(1/G(x), 1/G(y)\Big) + \frac{\partial^2}{\partial x \partial y}\left[\log\left(\frac{G(x) - G(y)}{x - y}\right)\right]$$

If T_N are deterministic (i.e., $\kappa_{m,n}^A = \alpha_{m,n}^T = 0$)), then this reduces to the formula of Bai-Silverstein