Resolvent Behaviour of $\mathscr{R}$-diagonal operators

Todd Kemp
CLE Moore Instructor
MIT

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January 17, 2008

The Ginibre Ensemble

The *Ginibre ensemble* $GinUE(N)$ is the space $\mathrm{Mat}_N(\mathbb{C})$ equipped with the probability measure

$$\alpha_N \, e^{-X^*X} \, dX.$$

Alternatively, it is the set of $N \times N$ random matrices whose entries are all i.i.d. complex normals (Re,Im i.i.d. $N(0,1)$).

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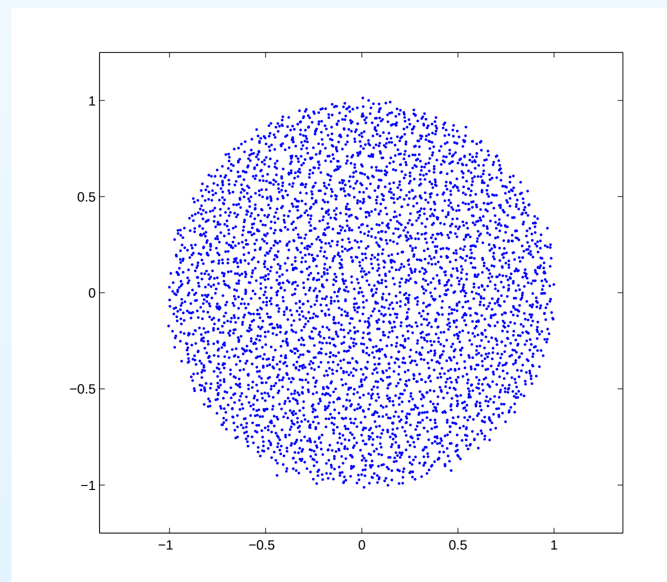

Voiculescu’s Circular Operator

The *circular operator* c is the limit (in the sense of free probability) of the renormalized $GinUE(N)$ as $N \to \infty$.

It can also be realized as $c = \frac{1}{\sqrt{2}}(s_1 + is_2)$, where s_1, s_2 are free semicircular operators.

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Quick advertisement: pick up **Lectures on the Combinatorics of Free Probability** by A. Nica and R. Speicher for everything you need to know about free cumulants.

$\mathscr{R}$-Diagonal Operators

Let u be a Haar unitary operator (renormalized limit of Haar unitary ensemble). As Jamie Mingo showed us on Monday, the only non-zero free cumulants of u, u^* are of the form

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Definition. a is **$\mathscr{R}$-diagonal** if its only non-zero free cumulants are of the forms

$$\kappa_{2n}[a, a^*, \ldots, a, a^*] \quad \kappa_{2n}[a^*, a, \ldots, a^*, a].$$

Alternate characterization. a is **$\mathscr{R}$-diagonal** if, given u Haar unitary $*$-free from a,

$$ua \sim a.$$

Properties of $\mathscr{R}$-Diagonal Operators

- Matrix models: ensembles of the form $\alpha_N \, e^{-V(X^*X)} \, dX$.
- If a, b are $\mathscr{R}$-diagonal and $*$-free, then $a + b$ and a^n are $\mathscr{R}$-diagonal; if x is *anything* $*$-free from a, ax is $\mathscr{R}$-diagonal.
- Never normal except scalar multiples of Haar unitaries.
- Brown measure of a can be computed explicitly from the $\mathscr{S}$-transform of a^*a; rotationally-invariant, analytic density.
- Have continuous families of invariant subspaces.
- Maximize free entropy (χ and χ^*) under distribution constraints.
- Satisfy a *strong Haagerup* inequality.

Strong Haagerup Inequality

Theorem. Let $a_1, \ldots, a_d$ be $*$-free $\mathscr{R}$-diagonal operators. If T is spanned by words of length n in $a_1, \ldots, a_d$ (and *not* $a_1^*, \ldots, a_d^*$), then

$$\|T\| \leq \alpha \sqrt{n}\, \|T\|_2,$$

where α is a constant depending on $\sup(\|a_j\|/\|a_j\|_2)$.

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E.g. for Haar unitaries $u_1, \ldots, u_d$, this is a statement about the free group: if $f\colon \mathbb{F}_d \to \mathbb{C}$ is supported on words of length n in the generators (and *not* their inverses), then

$$\|f\|_* \le \sqrt{e}\,\sqrt{n+1}\,\|f\|_2.$$

(If the inverses are included, the constant is $(n+1)$; this is the classical Haagerup inequality.)

Non-Crossing $*$-Pairings

E.g. with $n = 3$, $r = 4$:

$$u_1 \;\; u_5 \;\; u_2 \;\; u_2^{-1} u_3^{-1} u_3^{-1} u_3 \;\; u_2 \;\; u_1 \;\; u_1^{-1} u_2^{-1} u_1^{-1} u_1 \;\; u_3 \;\; u_3 \;\; u_4^{-1} u_5^{-1} u_4^{-1} u_4 \;\; u_5 \;\; u_4 \;\; u_3^{-1} u_5^{-1} u_1^{-1}$$

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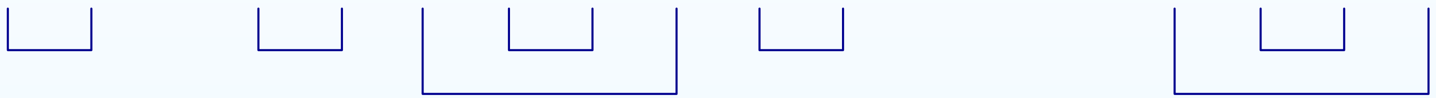

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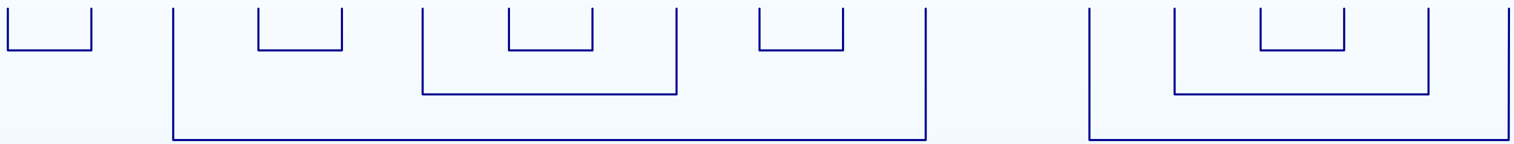

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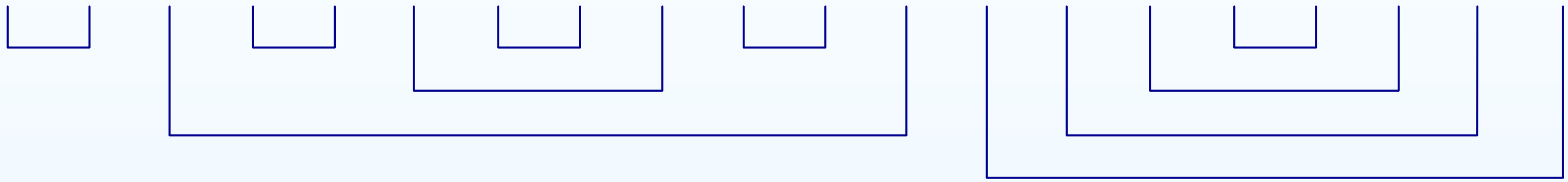

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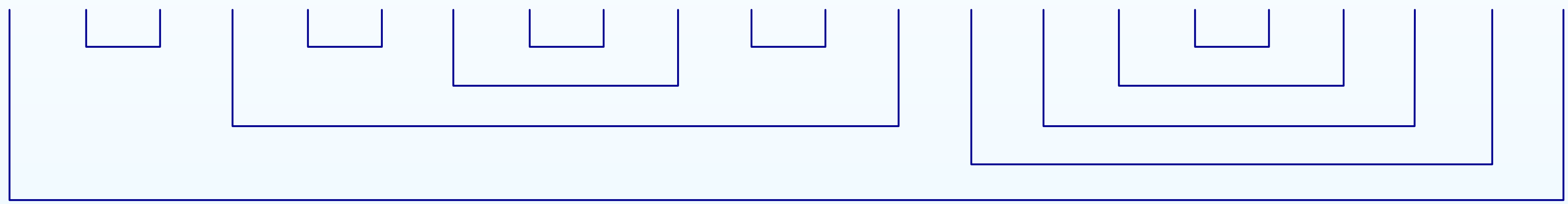

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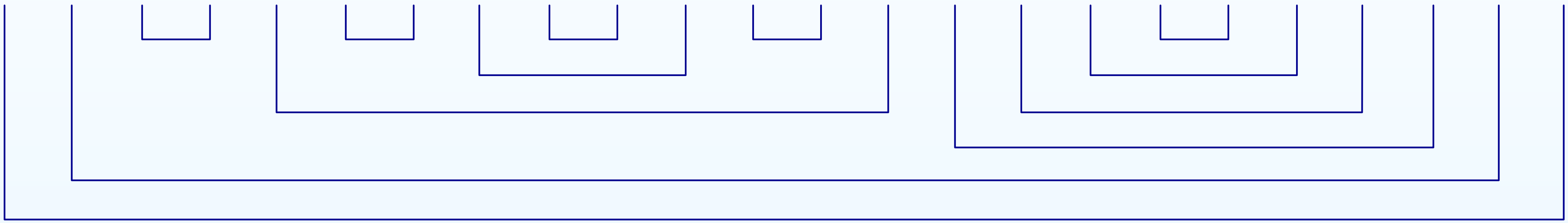

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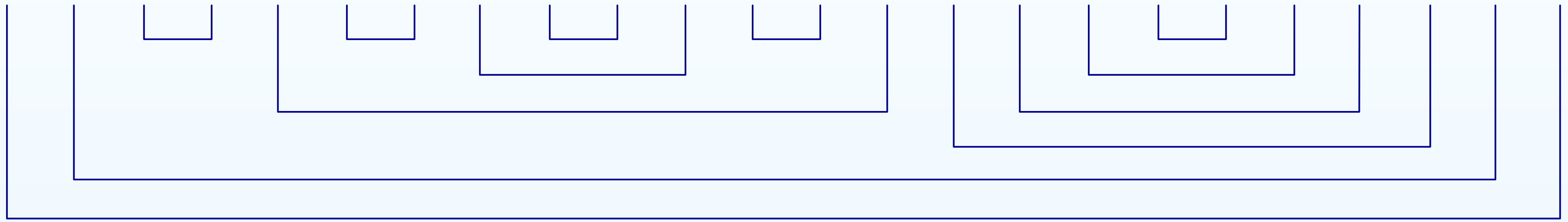

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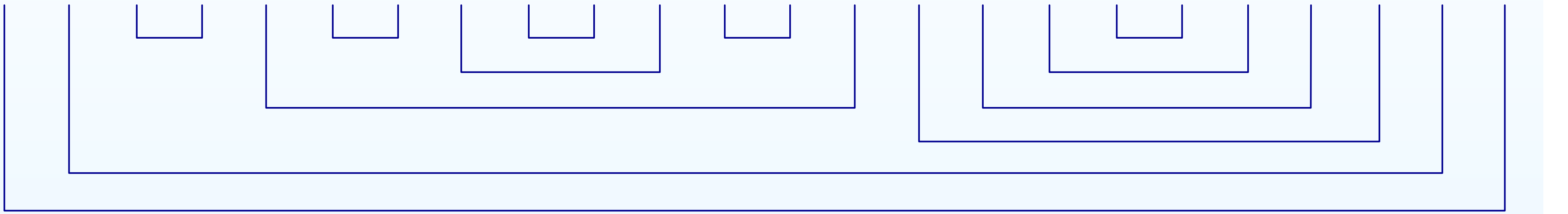

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- π is non-crossing
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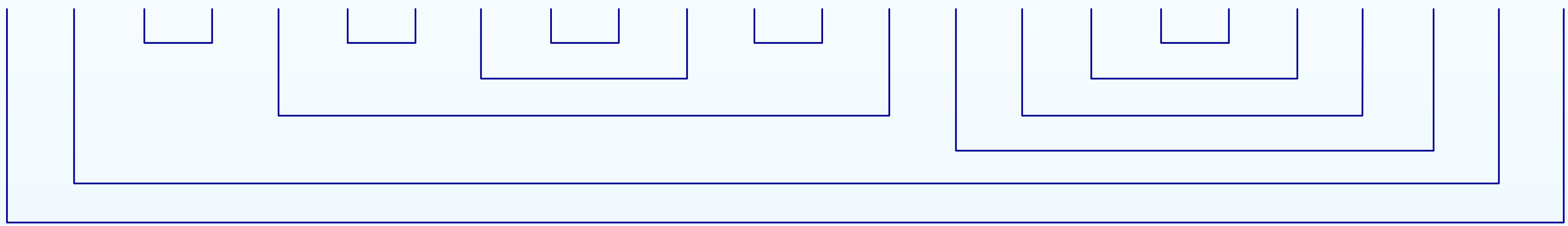

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The set of such $*$-pairings is counted by the *Fuss-Catalan* numbers

$$C_r^{(n)} = \frac{1}{nr+1} \binom{(n+1)r}{r} \sim \left(\sqrt{e}\,\sqrt{n+1}\right)^{2r}.$$

□

A Resolvent Upper Bound

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Write this as a geometric series

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Apply the strong Haagerup inequality term-by-term, use the Cauchy-Schwarz inequality, and arrive at

Proposition. There is a constant $\alpha(a) > 0$ so that, for $1 < \lambda < 2$,

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Question. Is this optimal?

Answer = YES

Theorem. Let a be $\mathscr{R}$-diagonal, and suppose that $\|a\|_2 = 1$ and $\|a\|_4 > 1$ (i.e. a is not Haar unitary). Then

$$\|\rho_a(\lambda)\| \sim \sqrt{\frac{27}{32}}\, v(a)\, \frac{1}{(\lambda - 1)^{3/2}},$$

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E.g. $$\mathscr{R}_{|\lambda-c|^2}(z) = \frac{1}{1-z} + \frac{\lambda^2}{(1-z)^2}.$$

Related Techniques: Negative Moments

The same techniques allow us to calculate (to leading order, at least) the negative moments of μ_λ.

Theorem. For $\lambda \searrow 1$ and $k \geq 0$,

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Let's look at the circular case.

References

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