Recursive and On-line Coloring

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Problem

Given a class $\mathcal{C}$ of graphs (closed under isomorphism) does there exist a function f such that every recursive graph $G \in \mathcal{C}$ satisfies $f(\chi(G)) \leq \chi_{rec}(G)$-coloring.

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Definition

A graph G is perfect if every induced subgraph H satisfies $\chi(H) = \omega(H)$, where ω(G) is the number of vertices of the largest complete subgraph of G.

Highly Recursive Graphs

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Definition (Bean)

A recursive graph $G = (V, E)$ is highly recursive if each vertex x has degree $d(x) < \infty$ and $d : V \longrightarrow \mathbf{N}$ is a recursive function.

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- *(HK 1981 — Recursive Vizing's Theorem) $\chi'_{rec}(G) \leq \chi'(G) + 1$.*

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$1, \ldots, k$ $\quad$ $k, \ldots, 2k-1$

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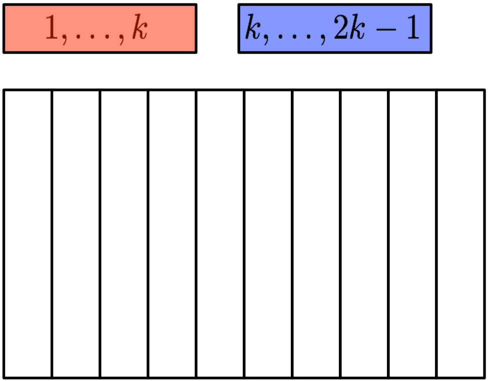

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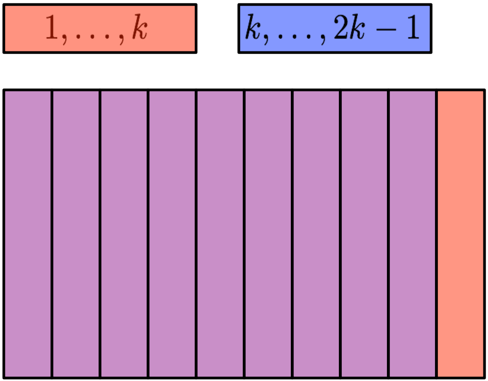

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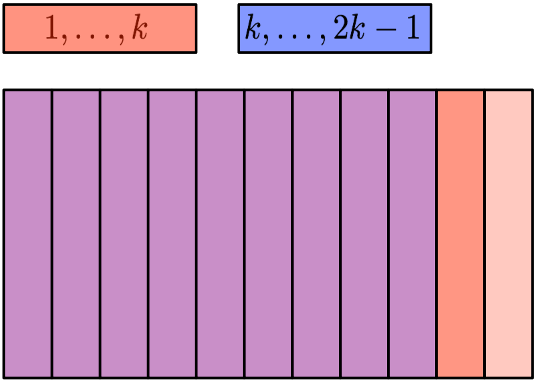

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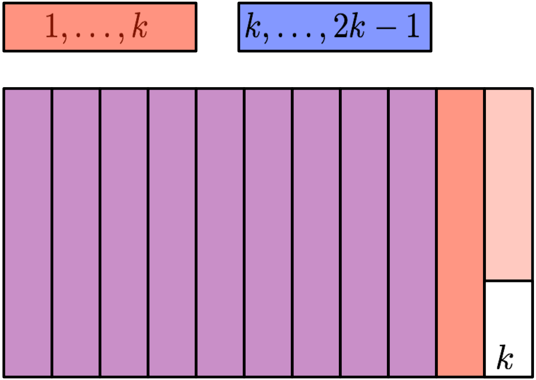

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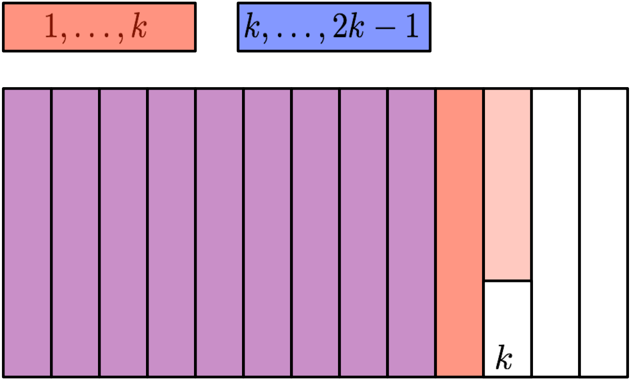

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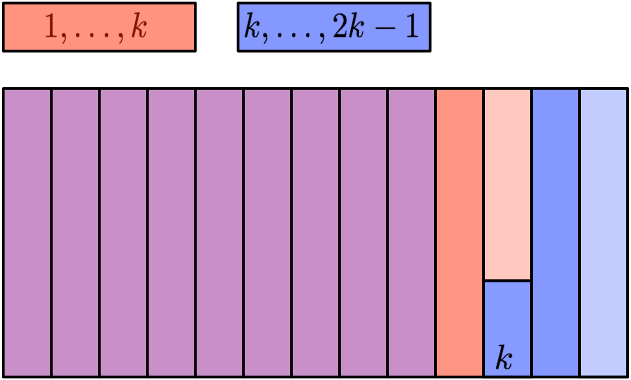

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- on-line coloring algorithms give rise to recursive colorings.

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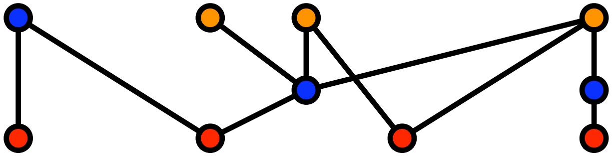
Theorem

Every poset P can be partitioned into height(P) *antichains.*



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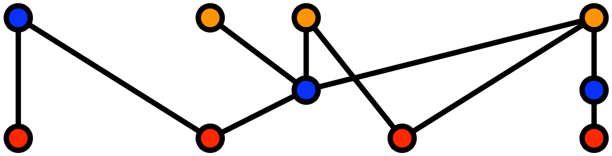

Theorem (Schmerl 1978)

There is an on-line algorithm for partitioning every on-line poset P into

$$\binom{\text{height}(P)+1}{2} \text{ antichains.}$$

Moreover, this is best possible.

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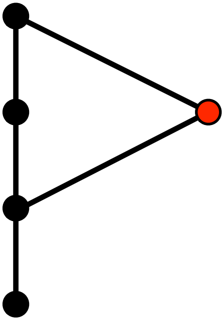

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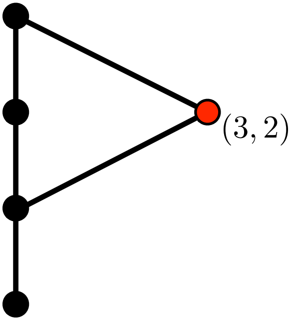

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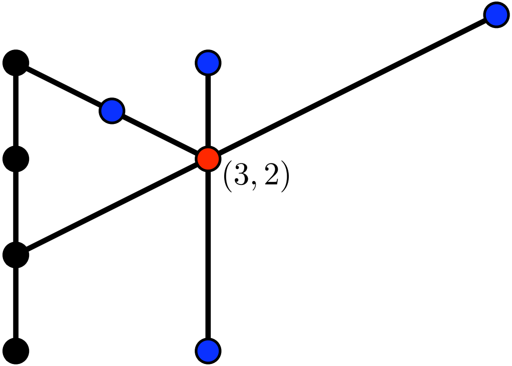

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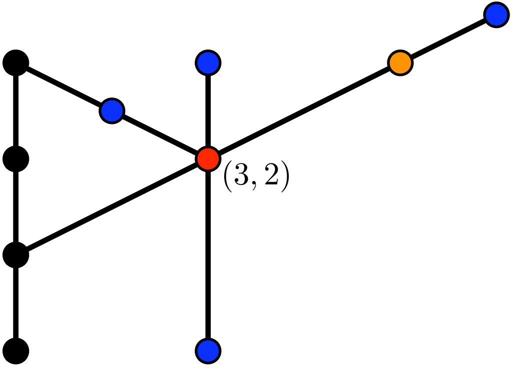

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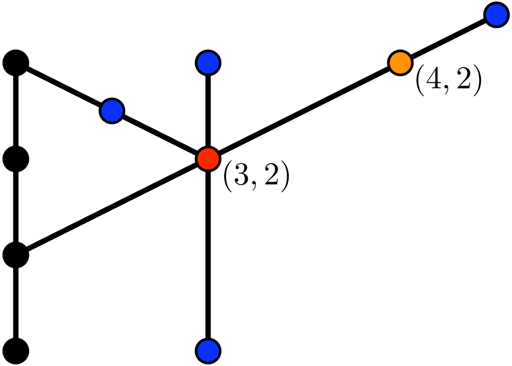

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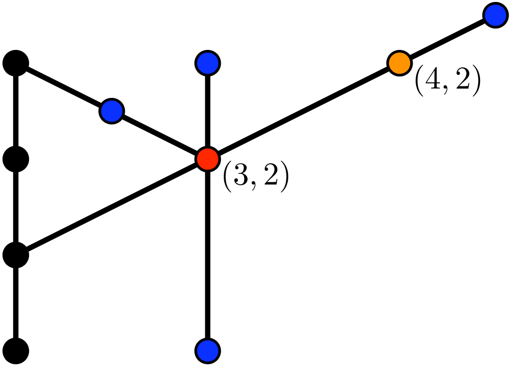

$$\sum_{i+j\leq h+1} 1 = \sum_{j=1}^{h} j = \binom{h+1}{2}$$

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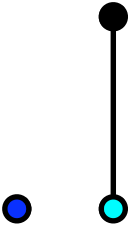

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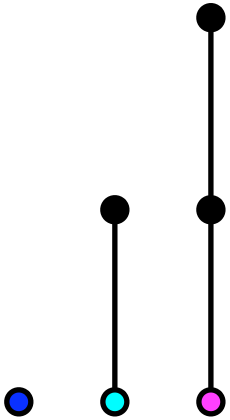

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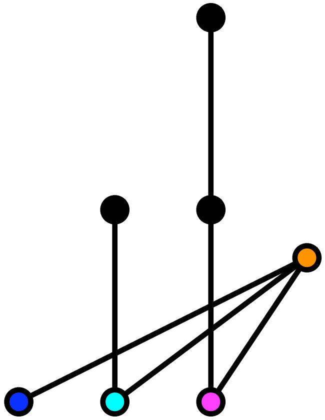

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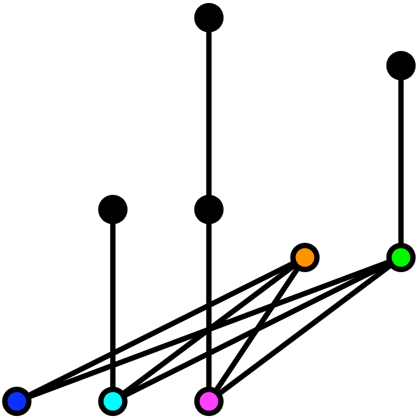

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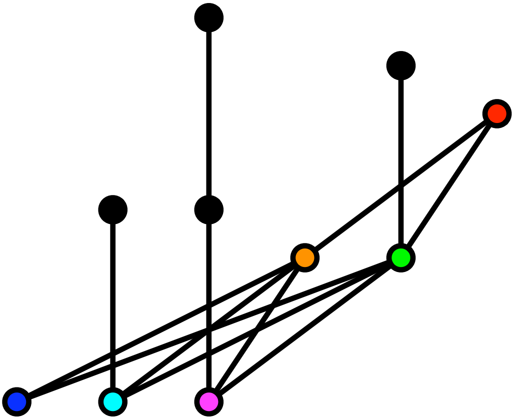

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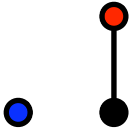
Remark

Every tree T is the comparability graph with $\omega(T) \leq 2$.

On-line Trees

$$\omega = 2$$



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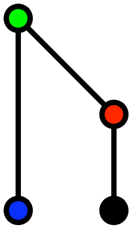

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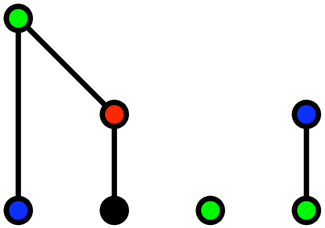

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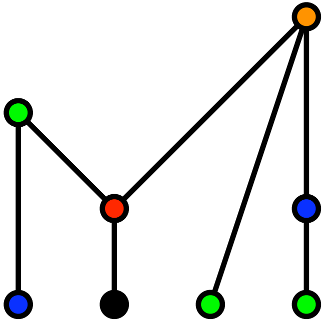

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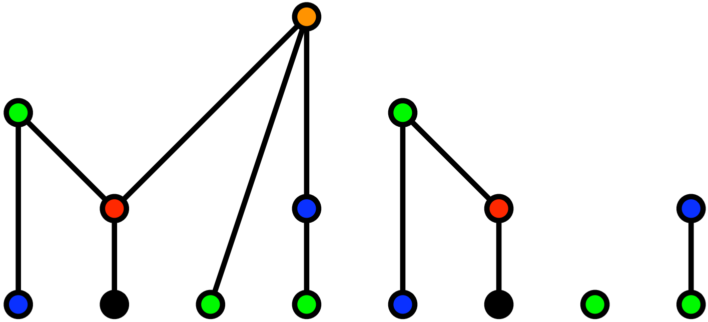

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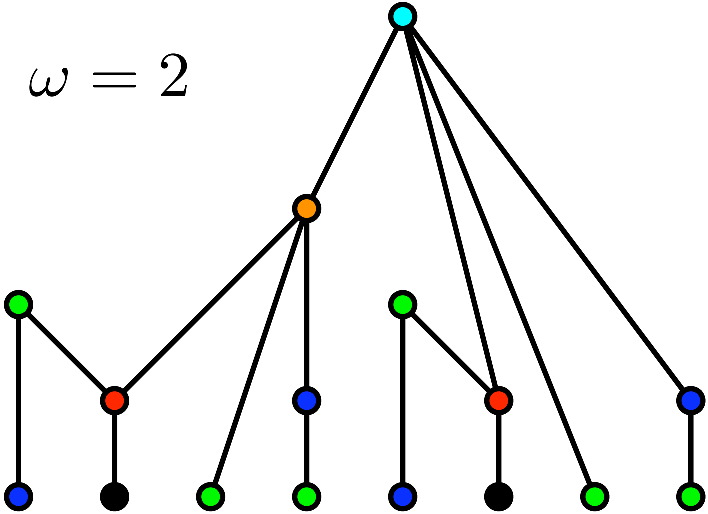

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- *(HK & Kolossa 1994)* $n^{\frac{10k}{\log\log n}}$ *colors, provided* G *is perfect.*

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Theorem (Irani 1994)

First-Fit colors every on-line graph G with $\mathrm{col}(G) \leq k$ on n vertices with $O(d \log n)$ colors. Moreover no on-line algorithm does better than this.

Lower bounds

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Theorem (Halldórsson & Szegedy)

For every $k \in \mathbf{N}$ and every on-line algorithm $\mathcal{A}$ there exists an on-line graph G such that $\chi(G) = k$ and $\chi_{\mathcal{A}}(G) \geq 2^k - 1$ and $|G| \leq k2^k$.

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Theorem (Vishwanathan 1992)

For every on-line coloring algorithm $\mathcal{A}$ there exists an on-line k-colorable perfect graph G on n vertices with $\mathcal{A} = \Omega(\frac{log^{k-1}(n)}{k^k})$.

Chain Partitioning

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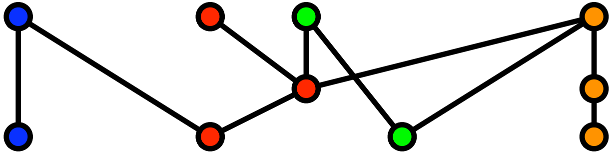
Theorem (Dilworth 1950)

Every poset P can be partitioned into $\text{width}(P)$ *chains.*



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Theorem (Szemerédi & Trotter 1981)

For every on-line algorithm $\mathcal{A}$ and positive integer w there exists an on-line poset P with $\mathrm{width}(P)=w$ such that

$$\mathcal{A}(P) \geq \binom{w+1}{2}.$$

Problems

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Problem (On-line Chain Partitioning)

Improve:

$$\binom{\mathrm{width}(P)+1}{2} \leq_{\forall \mathcal{A} \exists P} \mathcal{A}(P) \leq_{\exists \mathcal{A} \forall P} \frac{5^{\mathrm{width}(P)}-1}{4}$$

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Problem (On-line Coloring Cocomparability Graphs – Schmerl)

Does there exist an on-line coloring algorithm $\mathcal{A}$ and a function f such that for every on-line cocomparability graph G,

$$\mathcal{A}(G) \leq f(\omega(G))?$$

Special Posets: Interval Orders

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Theorem (HK & Trotter 1981)

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- The algorithm does not need an interval representation.

Special Posets: Bounded Dimension

Theorem (HK, McNulty & Trotter 1984)

There exists an on-line chain partitioning algorithm that covers the intersection P of d on-line linear orders with

$$\binom{\text{width}(P)+1}{2}^{d-1} \text{ chains.}$$

Graph Theory Detour: Forbidden Trees

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Definition

A graph class $\mathcal{C}$ is weakly perfect if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

Graph Theory Detour: Forbidden Trees

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A graph class $\mathcal{C}$ is weakly perfect if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

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For a graph H, let $\mathrm{Forb}(H)$ be the class of all graphs that do not induce H.

Conjecture (Gyárfás 1975; Sumner 1981)

$\mathrm{Forb}(T)$ *is weakly perfect for every tree* T*.*

Forbidden Trees

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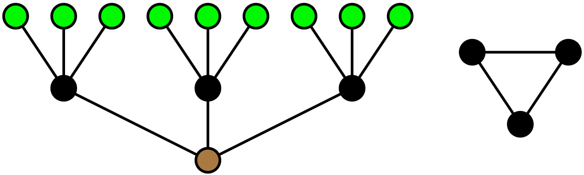
Theorem (Gyárfás, Szemerédi & Tuza 1980)

$\mathrm{Forb}(T, K_3)$ *is weakly perfect for every tree* T *with* $\mathrm{radius}(T) \leq 2$.



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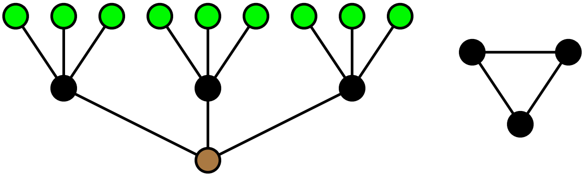

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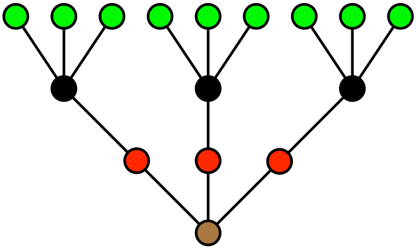
Theorem (HK & Y. Zhu 2004)

The Gyárfás-Sumner Conjecture is true if *T is obtained from a tree T' with* $\text{radius}(T') \leq 2$ *by subdividing every edge adjacent to the root exactly once.*



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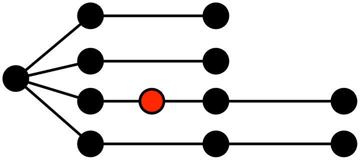
Corollary (Scott 1997)

The Gyárfás-Sumner Conjecture is true if T is a subdivision of a star.



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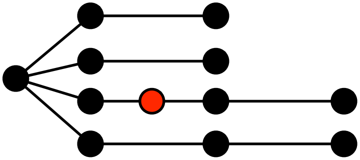

It is time to resolve the Gyárfás-Sumner Conjecture!

Schmerl's Problem

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Theorem (HK, Penrice & Trotter 1994)

For every tree T with $\mathrm{radius}(T) \leq 2$ *there exists an on-line coloring algorithm* $\mathcal{A}$ *and a function h such that for every on-line graph* $G \in \mathrm{Forb}(T)$

$$\mathcal{A}(G) \leq h(\omega(G)).$$

Moreover, if $\mathrm{radius}(T) > 2$ *then no such algorithm exists.*

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Corollary (Schmerl's Problem)

There exists an on-line coloring algorithm $\mathcal{A}$ *and a function* f *such that for every on-line cocomparability graph* G,

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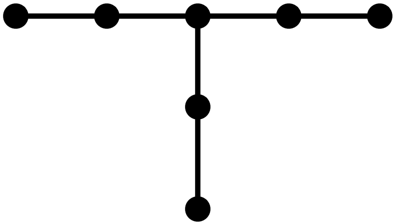
Proof.



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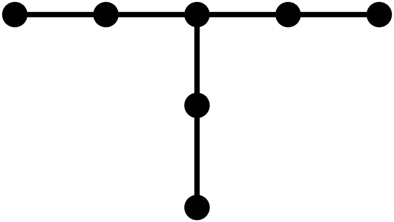

is not a cocomparability graph! □

First-Fit and Forbidden Trees

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There exists a function f *such that every on-line* $G \in \mathrm{Forb}(P_5)$ *satisfies* $FF(G) \leq f(\omega(G))$.

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Theorem (Gyárfás & Lehel + HK, Penrice & Trotter)

For all trees T, *if there exists a a function* f *such that every on-line* $G \in \mathrm{Forb}$ *satisfies* $FF(G) \leq f(\omega(G))$, *then* T *does not induce* $K_2 + 2K_1$.

First-Fit and Cocomparability Graphs

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Let G be the cocomparability graph of a poset that does not induce $K_{t,t}$. Then

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Remark

Interval graphs do not induce $K_{2,2}$.

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There exists an on-line interval graph G such that $\mathrm{FF}(G) \geq C\omega(G) - B(C)$, *where*

- *(Chrobak & Slusarek 1989)* $C = 4.4$

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- *The technique fails for* $C = 5$.

Tolerance Graphs

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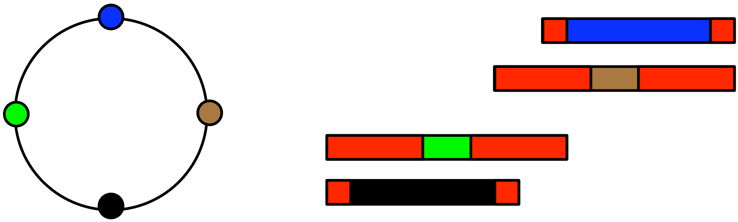

Definition

A graph $G = (V, E)$ is a tolerance graph if for each vertex v there is an interval I_v and a nonnegative real (tolerance) t_v such that

$$vw \in E \text{ iff } |I_v \cap I_w| \geq \min\{t_v, t_w\}.$$

Tolerance $\subseteq$ Perfect

Theorem (Golumbic, Monma &Trotter 1984)

All tolerance graphs graphs are perfect.

Classification of Tolerance Graphs

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Every 1-tolerance graph is the cocomparability graph of a poset with interval dimension 2.

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Theorem (Felsner 1998)

Every 1*-tolerance graph is the cocomparability graph of a poset with interval dimension* 2*.*

Corollary

Every $\frac{1}{2}$*-tolerance graph is the union of two interval graphs.*

Recent Results

Theorem (HK & Saoub 2008)

- *Every on-line* $\frac{1}{2}$*-tolerance graph* G *satisfies* $FF(G) \leq 16\omega(G)$.

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- *For every* k *there exists an on-line* 1*-tolerance graph* G *with* $\omega(G) = k$ *and* $FF(G) \geq 2^k$.
- *There is an on-line algorithm that colors every on-line low tolerance graph with* $\frac{9}{2}\omega^3(G)$ *colors, provided that the tolerance representation is also given on-line.*

Up-growing Posets

Definition (Felsner 1994)

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There exists an on-line algorithm $\mathcal{A}$ such that for any up-growing poset P

$$\mathcal{A}(P) \leq \binom{\text{width}(P)+1}{2}$$

Moreover this is best possible.

Up-growing Interval Orders

Theorem (Baier, Bosek & Micek 2008)

There exists an on-line algorithm $\mathcal{A}$ such that for any up-growing interval order P

$$\mathcal{A}(P) \leq 2\,\mathrm{width}(P) - 1$$

Moreover this is best possible.

Up-growing Semi-orders

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Theorem (Felsner, Kloch, Matecki & Micek 2008)

There exists an on-line algorithm $\mathcal{A}$ such that for any up-growing semi-order P

$$\mathcal{A}(P) \leq \lfloor \frac{1+\sqrt{5}}{2} \text{width}(P) \rfloor$$

Moreover this is best possible!

On-line Ramsey Theory

Theorem (HK & Konjevod 2008)

Fix integers u and t. For every on-line edge painting algorithm $\mathcal{A}$ there exists an on-line u-uniform hypergraph G with the same coloring number as K_u^t such that some copy of $K_u^t \subseteq G$ is monochromatic.