Nonstandard Arithmetic, Reverse Mathematics, and Recursive Comprehension

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First order reasoning about hyperintegers proves things about sets of integers. The advantage is that hyperintegers have more structure than sets of integers.

1. The languages $L_1, L_2, {}^*L_1$
2. Basic Nonstandard Arithmetic (BNA)
3. The Standard Part Principle (STP)
4. Nonstandard Counterparts
5. Weak Koenig Lemma
6. Arithmetical Comprehension
7. Arithmetical Transfinite Recursion
8. Π^1_1-Comprehension
9. Recursive Comprehension

Reminder: The 5 basic theories in L_2

$\mathsf{RCA}_0 =$
$I\Sigma_1 + \Sigma^0_1$ Induction $+\Delta^0_1$ Comprehension.

$\mathsf{WKL}_0 = \mathsf{RCA}_0+$ the Weak Koenig Lemma
(every infinite binary tree has an infinite branch).

$\mathsf{ACA}_0 = \mathsf{RCA}_0+$ Arithmetical Comprehension
(for each k, every Σ^0_k formula defines a set).

$\mathsf{ATR}_0 = \mathsf{RCA}_0 + \Sigma^1_1$ Separation
(any two disjoint Σ^1_1 properties are separated by a set).

Π^1_1-$\mathsf{CA}_0 = \mathsf{RCA}_0 + \Pi^1_1$ Comprehension
(any Π^1_1 property defines a set).

Some References

A. Enayat. From bounded arithmetic to second order arithmetic via automorphisms. In Logic in Tehran, Lecture Notes in Logic 26 (2006).

C.W. Henson, M. Kaufmann, and H.J. Keisler. The Strength of Nonstandard Methods in Arithmetic. JSL 49 (1984), 1039–1058.

H.J. Keisler, Nonstandard Arithmetic and Reverse Mathematics. BSL 12 (2006), 100–125.

S.G. Simpson, Subsystems of Second Order Arithmetic. Springer-Verlag, 1999.

K. Tanaka, The self-embedding theorem of WKL_0 and a non-standard method. APAL 84 (1997), 41–49.

K. Yokoyama, Formalizing non-standard arguments in second order arithmetic. To appear.

1. The languages $L_1, L_2, {}^*L_1$

First Order Arithmetic L_1:
Sort N with variables $m, n, q, r, \ldots$
Vocabulary $0, 1, +, \dot{-}, \cdot, <$
(where $m \dot{-} n = \max(m-n, 0)$)
Terms $s(\vec{n})$ of sort N

2nd Order Arithmetic L_2:
$L_1 \subseteq L_2$. Models: $(\mathcal{N}, \mathcal{P})$, $\mathcal{P} \subseteq \mathcal{P}(\mathcal{N})$
Sort P with variables $X, Y, Z, \ldots$
Relation $\in$ of sort $N \times P$

Nonstandard Arithmetic ${}^*L_1$:
$L_1 \subseteq {}^*L_1$. Models $(\mathcal{N}, {}^*\mathcal{N})$, $\mathcal{N} \subseteq {}^*\mathcal{N}$
Sort ${}^*N$ with variables $x, y, z, \ldots$
Vocabulary $0, 1, +, \dot{-}, \cdot, <$ in both sorts
Terms $t(\vec{u})$ of sort ${}^*N$
where $\vec{u}$ has variables of both sorts.

Combined language: $L_2 \cup {}^*L_1$
Models: $(\mathcal{N}, \mathcal{P}, {}^*\mathcal{N})$

2. Basic Nonstandard Arithmetic (BNA)

A weak theory in ${}^*L_1$
which says $\mathcal{N} \equiv_{\forall} {}^*\mathcal{N}$ and $\mathcal{N} \subset_{end} {}^*\mathcal{N}$

Axioms of $I\Sigma_1$ in sort N:
Recursive rules for $0, 1, +, \dot{-}, \cdot, <$,
Σ^0_1 Induction in L_1

$\forall$-Transfer: $\forall \vec{m}\, \varphi(\vec{m}) \leftrightarrow \forall \vec{x}\, \varphi(\vec{x})$,
$\forall \vec{m}\, \varphi(\vec{m})$ a universal sentence in L_1

Proper Initial Segment Axioms:

$$\forall n \exists x (x = n)$$

$$\forall n \forall x [x < n \rightarrow \exists m\, x = m]$$

$$\exists y \forall n [n < y]$$

3. The Standard Part Principle (STP)

The bridge between ${}^*L_1$ and L_2.

STP is a sentence in $L_2 \cup {}^*L_1$ meaning:
"Every hyperinteger codes a set, and every set is coded by a hyperinteger"

$(p_n|x)$ means "The n-th prime divides x",

$st(x) = \varphi(\cdot, \vec{u})$ denotes $\forall m \, [(p_m|x) \leftrightarrow \varphi(m, \vec{u})]$
"x codes the class $\{m : \varphi(m, \vec{u})\}$"

$st(x) = X$ denotes $\forall m \, [(p_m|x) \leftrightarrow m \in X]$
"x codes X", "X is the standard part of x"

STP: $\forall x \exists X \, st(x) = X \wedge \forall X \exists x \, st(x) = X$

4. Nonstandard Counterparts

A theory T' in $L_2 \cup {}^*L_1$ is **conservative** over a theory T in L_2 if every sentence of L_2 provable from T' is provable from T.

T' is a **nonstandard counterpart** of T if T' implies and is conservative over T.

We will give nonstandard counterparts of each of the five basic theories RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0 of second order arithmetic.

Each of these counterparts will be of the form $U + \mathsf{STP}$ where U is a theory in ${}^*L_1$.

5. Weak Koenig Lemma

Theorem. *The theory*
$^*\mathsf{WKL}_0 = \mathsf{BNA} + \mathsf{Int}\text{-}\mathsf{IND} + \Sigma_1^S\text{-}\mathsf{IND} + \mathsf{STP}$
is a nonstandard counterpart of WKL_0*.*
So is $^*\mathsf{WKL}_0+$ *Transfer for* FO *sentences.*

Δ_0^S formula in $^*L_1$: Built from atomic formulas, connectives, and bdd quantifiers $(\forall m < s(\vec{n})), (\forall x < t(\vec{u}))$.
Σ_1^S means $\exists m\, \varphi$ where φ is Δ_0^S. And so on.

Internal Induction (Int-IND):

$$[\varphi(0, \vec{u}) \wedge \forall x[\varphi(x, \vec{u}) \rightarrow \varphi(x+1, \vec{u})]] \rightarrow \forall x\, \varphi(x, \vec{u})$$

where $\varphi(x, \vec{u})$ is Δ_0^S.

Σ_1^S Induction (Σ_1^S-IND):

$$[\psi(0, \vec{u}) \wedge \forall n[\psi(n, \vec{u}) \rightarrow \psi(n+1, \vec{u})]] \rightarrow \forall n\, \psi(n, \vec{u})$$

where $\psi(n, \vec{u})$ is Σ_1^S.

Transfer for FO sentences says $\mathcal{N} \equiv {}^*\mathcal{N}$.

5. Weak Koenig Lemma (Continued)

Proof that *WKL_0 implies WKL_0 uses the **Overspill Lemma**:
For each Δ^S_0 formula $\varphi(x,\vec{u})$,
$\forall n\, \varphi(n,\vec{u}) \rightarrow \exists x[\varphi(x,\vec{u}) \wedge \forall m\, m < x]$

Proof that *WKL_0 is conservative over WKL_0 uses Tanaka's result that
Every countable nonstandard model of WKL_0 has an isomorphic proper end extension.

6. Arithmetical Comprehension

Theorem. *The theory* ${}^*\mathsf{WKL}_0 + S\text{-}\mathsf{ACA}$ *is a nonstandard counterpart of* ACA_0*.*
So is ${}^*\mathsf{WKL}_0 + S\text{-}\mathsf{ACA} + \mathsf{FOT}$*.*

S-Arithmetical Comprehension (S-ACA):
(Each S-arithmetical class is coded by an x)

$$\exists x\, st(x) = \varphi(\cdot, \vec{u})$$

where $\varphi(m, \vec{u}) \in \bigcup_k \Sigma_k^S$.

FOT is Transfer for all first order formulas ($\mathcal{N} \prec {}^*\mathcal{N}$).

Using results of Enayat, one can get even stronger nonstandard counterparts of ACA_0.

7. Arithmetical Transfinite Recursion

Theorem. *The theory* $^*\mathsf{WKL}_0 + \Sigma_1^*\text{-}\mathsf{SEP}$ *is a nonstandard counterpart of* ATR_0*.*
So is $^*\mathsf{WKL}_0 + \Sigma_1^*\text{-}\mathsf{SEP} + \mathsf{FOT}$*.*

Σ_1^* formula: $\exists x\,\varphi(x,\vec{u})$ where $\varphi(m,\vec{u}) \in \bigcup_k \Sigma_k^S$.

Σ_1^*-Separation (Σ_1^*-SEP):
(Two disjoint Σ_1^* classes can be separated by an x)

$\forall m[\psi(m,\vec{u}) \to \neg\theta(m,\vec{u})] \to$
$\exists x[\psi(\cdot,\vec{u}) \subseteq st(x) \wedge st(x) \subseteq \neg\theta(\cdot,\vec{u})]$
where $\psi(m,\vec{u}), \theta(m,\vec{u})$ are Σ_1^*.

8. Π^1_1 Comprehension

Theorem. *The theory* $^*\mathsf{WKL}_0 + \Pi^*_1\text{-CA}$ *is a nonstandard counterpart to* $\Pi^1_1\text{-CA}_0$*.* *So is* $^*\mathsf{WKL}_0 + \Pi^*_1\text{-CA} + \mathsf{FOT}$*.*

Π^*_1 formula: $\forall x\,\varphi(x,\vec{u})$ where $\varphi(m,\vec{u}) \in \bigcup_k \Sigma^S_k$.

Π^*_1 Comprehension (Π^*_1-CA):
(Each Π^*_1 class is coded by an x)

$$\exists x\, st(x) = \varphi(\cdot,\vec{u})$$

where $\varphi(m,\vec{u})$ is Π^*_1.

The 1984 paper of Henson, Kaufmann, and Keisler gave nonstandard counterparts of some theories which are stronger than Π^1_1-CA.

9. Recursive Comprehension

The theory $^*\mathsf{RCA}_0 = {}^*\mathsf{WKL}_0 - \mathsf{STP} + \text{weak STP}$ is a nonstandard counterpart of RCA_0.
Here is another, with full STP:

Theorem. *The theory* $^*\mathsf{RCA}_0{}' =$ $\mathsf{BNA}+$*Special* Σ_1^S-$\mathsf{IND}+$*Special* Δ_1^S-$\mathsf{CA} + \mathsf{STP}$ *is a nonstandard counterpart of* RCA_0*.*

Special Δ_0^S formulas: Built from atomic formulas, connectives, $(s(\vec{n})|t(\vec{u}))$, bdd quantifiers $(\forall m < s(\vec{n}))$ of sort N.
Special Σ_1^S means $\exists m\,\varphi$, φ special Δ_0^S.

Special Δ_1^S Comprehension (Special Δ_1^S-CA):
(Each special Δ_1^S class is coded by an x)

$$\forall m[\varphi(m,\vec{u}) \leftrightarrow \neg\psi(m,\vec{u})] \rightarrow \exists x\, st(x) = \varphi(\cdot,\vec{u})$$

where $\varphi(m,\vec{u}), \psi(m,\vec{u})$ are special Σ_1^S.

9. Recursive Comprehension (Continued)

But each of the following is a nonstandard counterpart of WKL_0:

$^*\mathsf{RCA}_0{}' +$ Overspill Lemma

$^*\mathsf{RCA}_0{}' +$ Internal Induction

$^*\mathsf{RCA}_0{}' +$ Transfer for Π^0_1 sentences

$^*\mathsf{RCA}_0{}' + \Delta^S_0$ Comprehension

The analogue of $^*\mathsf{RCA}_0{}'$ with a symbol for each primitive recursive function.

9. Recursive Comprehension (Open Questions)

Is $\mathsf{BNA} + \Sigma_1^S\text{-}\mathsf{IND} + \mathsf{STP}$ conservative over RCA_0?

Is ${}^*\mathsf{RCA}_0{}' + $Transfer for universal formulas (rather than sentences) conservative over RCA_0?

Note: The above two theories do not imply the Weak Koenig Lemma.

Is the analogue of ${}^*\mathsf{RCA}_0{}'$ with an added symbol for exponentiation conservative over RCA_0?

What happens if one uses a different method of coding sets by hyperintegers?