08w2133: Singular phenomena in nonlinear optics, hydrodynamics and plasmas

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1 Overview of the Field

Nonlinear wave phenomena arising in the areas of optics, hydrodynamics and plasmas have led to fundamental discoveries in applied mathematics. This is mainly due to the universality of the governing equations such as the nonlinear Schrödinger equation (NLS),

$$i\psi_t + \triangle\psi + |\psi|^2\psi = 0, \tag{1}$$

which describes nonlinear interaction of waves. It is a common trend that solutions of the governing nonlinear equations result in the formation of singularities. This has been observed for example in the filamentation of laser beams in nonlinear media, wave breaking in hydrodynamics, collapse and Langmuir waves in plasmas. Analytical and numerical models continue to evolve to allow deeper understanding of these phenomena at fine spatial and temporal scales. The workshop brought some of the leading scientists in these fields combining theoretical, numerical and experimental studies.

2 Recent Developments

This field experiences rapid developments in analytical, numerical and experimental directions. It includes new results from the more theoretical side such as standing ring solutions to a super critical NLS, recent discovery of ground state selection and energy equipartition in NLS and Gross Pitaevskii Equation, lattice solitons, orbital instabilities, and the band-gap interface, as well as new bounds on Kolmogorov spectra for the Navier - Stokes equations. These results are complemented with growing achievements in numerical simulations of strongly nonlinear phenomena, examples include the modeling of freakons which is a new type of soliton, simulation of quasisolitonic turbulence in the Majda-McLaughlin-Tabak equation, modelling of weak electron-phase space holes, regularized point vortex simulations of vortex sheet roll-up. Another important development is the necessity of combining theoretical and numerical results to achieve better understanding of experiments. Experimental studies are now becoming much more elaborative; this allows to distinguish relative contributions from different types of strongly nonlinear processes. Examples are the use of femtosecond bullets for laser inscription of photonic structures, study of mixing transitions and oscillations in low-Reynolds number viscoelastic fluids.

3 Presentation Highlights

We had many excellent presentations at the workshop. Here we highlight some representative presentations:

Walter Craig found new bounds on Kolmogorov spectra for the Navier - Stokes equations. An argument involving scale invariance and dimensional analysis given by Kolmogorov in 1941, and subsequently refined by Obukov, predicts that in three dimensions, solutions of the Navier - Stokes equations at large Reynolds number and exhibiting fully developed turbulent behavior should obey *energy spectrum*

$$E(\kappa, t) \sim C\kappa^{-5/3},$$

at least in an average sense. A global estimate on weak solutions in the norm $|\mathcal{F}\partial_x u(\cdot, t)|_\infty$ was found which gives bounds on a solution's ability to satisfy the Kolmogorov law. The result gives rigorous upper and lower bounds on the inertial range, and in the unforced case an upper bound on the time of validity of the spectral regime.

Philippe Guyenne presented numerical simulations of 3D overturning water waves over bottom topography. The numerical model solves the fully nonlinear potential flow equations using a 3D high-order boundary element method combined with an explicit time integration scheme, expressed in a mixed Eulerian-Lagrangian formulation. Results on wave profiles and kinematics were presented as well as comparisons with 2D results as well as with theoretical predictions.

Mark Hoefer considered a dispersive regularization of degenerate rarefact ion wave Interactions. In an Eulerian fluid, shock waves do not propagate into a region of zero density. Instead, degenerate rarefaction or expansion waves describe the fluid behavior. The interaction of two such waves does however generate gradient singularities which must be regularized. A dissipative regularization for this problem leads to two counter-propagating shock waves moving away from the initial interaction point. A dispersive regularization for this problem was presented using the Whitham averaging method which gives rise to an oscillatory interaction region described by a modulated train of solitons decaying to small amplitude linear waves. Viewed in the context of dispersive shock waves (DSWs), the interaction region can be thought of as two expanding DSWs placed back-to-back. A comparison of the asymptotic results with numerical simulations and experiments demonstrates that this interaction region corresponds to the macroscopic, quantum mechanical interference of matter-waves in a Bose-Einstein condensate.

Boaz Ilan investigated positive soliton solutions of nonlinear Schrodinger equations with periodic and irregular-lattice potentials. Using rigorous, asymptotic and computational methods he showed that the solitons are (un)stable precisely whenever they (violate) satisfy the power-slope (Vakhitov-Kolokolov) and lesser-studied spectral condition. Violations of the power-slope and spectral conditions induce focusing and drift instabilities, respectively. This unified approach predicts the strength of the instabilities as well. These results are elucidated by computation of soliton dynamics with periodic, defect, and quasi-crystal lattice structures.

Robert Krasny considered vortex sheets as weak solutions of the incompressible fluid equations describing velocity fields with a tangential discontinuity. Physically, a vortex sheet represents a thin shear layer in slightly viscous flow. The initial value problem for vortex sheets is ill-posed due to the Kelvin-Helmholtz instability. Moore showed that a curvature singularity forms in finite time in a perturbed vortex sheet. To go past the critical time, it is necessary to regularize the principal value integral defining the sheet velocity. This talk presented regularized point vortex simulations of the problem. For a large value of the regularization parameter, the sheet rolls up into a smooth spiral past the critical time. As the regularization parameter is reduced, the onset of chaotic dynamics due to resonances in the vortex core was observed.

Brenton LeMesurier developed a new approach for generating time discretizations for a large class of Hamiltonian systems which exactly conserve the energy and other quadratic conserved quantities of the corresponding differential equations. An essential feature is a procedure for constructing discrete approximations of partial derivatives in a way that mimics essential properties of derivatives, in particular for the quadratic forms of most "momenta". The approach is applied to a class of systems which includes models of energetic pulse propagation in protein due to Davydov, Scott et al. These models have the integrable nonlinear Schrödinger equation as a continuum limit, with sech pulse solutions. Extensions will be shown to systems like coupled systems of discrete nonlinear Schrödinger equations and oscillators. The discrete models have self-focusing effects not seen in the 1D cubic NLS.

Pavel Lushnikov derived a regularization of collapse in cellular dynamics. In this case biological cells interact through chemotaxis when cells secret diffusing chemical (chemoattractant) and move towards the gradient of the chemoattractant creating effective nonlocal attraction between cells. Macroscopic description of cellular density dynamics through Keller-Segel model has striking qualitative similarities with the nonlinear Schrödinger equation including critical collapse in two dimensions and supercritical in three dimensions. Critical collapse has logarithmic corrections to $(t_0 - t)^{1/2}$ scaling law of self-similar solution. Microscopic motion of eucaryotic cells is accompanied by random fluctuations of their shapes. A nonlinear diffusion equation was derived coupled with chemoattractant from microscopic cellular dynamics in dimensions one and two using excluded volume approach. The nonlinear diffusion coefficient depends on cellular volume fraction and it provides regularization (prevention) of cellular density collapse. A very good agreement is shown between Monte Carlo simulations of the microscopic Cellular Potts Model and numerical solutions of the macroscopic equations for relatively large cellular volume fractions.

Vladimir Mezentsev presented a full vectorial modeling of femtosecond bullets for laser inscription of photonic structures. This means that the full set of Maxwell's equations coupled to standard Drude model of the generated plasma is modeled. The results are compared with orthodox models based on the paraxial envelope approximation.

Pierre Raphaël discussed the description of singularity formation for some focusing nonlinear Schrödinger equations $iu_t + \Delta u + u|u|^{p-1} = 0$ in N dimensions. He used the analysis of the stable "log-log" blow up regime for the L^2 critical case $p = 1 + \frac{4}{N}$ to prove the existence of standing ring blow up solutions in the super critical case. In particular, for $p = 5$ and *any* $N \geq 2$, he proved the existence of radially symmetric blow up solutions which concentrate their L^2 mass on the unit sphere N dimentions and the stability of this singularity formation in the radial class.

Benno Rumpf addressed important issues of strong turbulence by studying quasisolitonic turbulence in the Majda-McLaughlin-Tabak equation which is a simple one dimensional model system for turbulence. He discussed formation of collapses and of quasisolitons from a weakly turbulent background in this system.

Catherine Sulem discussed the problem of nonlinear wave motion of the free surface of a body of fluid over a variable bottom. The objective was to describe the character of wave propagation in a long wave asymptotic regime, under the assumption that the bottom of the fluid region is described by a stationary random process whose variations take place on short length scales. The principal result is the derivation of effective equations and a consistency analysis. The effects of random modulations on the solutions was computed giving an explicit expression for the scattered component of the solution due to waves interacting with the random bottom.

Becca Thomases studied mixing transitions and oscillations in low-Reynolds number viscoelastic fluids. In the past several years, it has come to be appreciated that in low Reynolds number, the flow the nonlinearities provided by non-Newtonian stresses of a complex fluid can provide a richness of dynamical behaviors more commonly associated with high Reynolds number Newtonian flow. For example, experiments by V. Steinberg and collaborators have shown that dilute polymer suspensions being sheared in simple flow geometries can exhibit highly time dependent dynamics and show efficient mixing. The corresponding experiments using Newtonian fluids do not, and indeed cannot, show such nontrivial dynamics. To better understand these phenomena, a computational study of the Stokes-Oldroyd-B viscoelastic model in 2D was performed. For low Weissenberg number, flows are "slaved" to the four-roll mill geometry of the fluid forcing. For sufficiently large Weissenberg number, such slaved solutions are unstable and under perturbation transit in time to a structurally dissimilar flow state dominated by a single large vortex, rather than four vortices of the four-roll mill state. The transition to this new state also leads to regions of well-mixed fluid and can show persistent oscillatory behavior with continued destruction and generation of smaller-scale vortices.

Natalia Vladimirova investigated the qualitative behavior of solutions of a Burgers-Boussinesq system – a reaction-diffusion equation coupled via force to a Burgers equation – by a combination of numerical and asymptotic techniques. When the force is small, the solutions decompose into a traveling wave and an accelerated shock wave moving in opposite directions. When the force exceeds some critical value, the solutions are composed of three elementary pieces: a wave fan, a reaction traveling wave, and an accelerating shock, with the whole structure traveling in the same direction. With further increase of the force, the wave fan catches up with the accelerating shock wave – the solution drops below reaction threshold and reaction is ceased. Extinction results irrespective of the size of initial data – a major difference with what happens in advection-reaction-diffusion equations where an incompressible flow is imposed.

Michael Weinstein discussed ground state selection and energy equipartition in the NLS and the Gross Pitaevskii Equation which are central equations to the mathematical description of nonlinear optical and macroscopic quantum systems. It was shown that NLS/Gross Pitaevskii systems support multiple families of nonlinear bound states (”solitons”) that (i) the generic evolution is towards a nonlinear ground state and (ii) in the weakly nonlinear regime an energy equipartition law holds.

4 Scientific Progress Made and Outcome of the Meeting

This workshop provided a unique opportunity for the advancement and understanding of strongly nonlinear phenomena in nonlinear optics, biology, hydrodynamics and plasma through the integration of theoretical, numerical and experimental studies. Cross-fertilization of these interdisciplinary approaches allowed achieving a new level of understanding of nonlinear phenomena. Because of the diversity of the underlying nonlinear phenomena, correct models require good understanding of the underlying physics and the recognition of characteristic dominant scales and effects so that asymptotic methods could lead to tracktable models. An important outcome of the workshop was the study of mechanisms of regularization of strongly nonlinear solutions which depend on the particular application, but show many universal features such as dissipative anomaly in Navier-Stokes turbulence when regularization does not depend on the exact value of the dissipative regularization provided viscosity of fluid is small enough. An example is the formation of freak waves and wave breaking which are described by nondissipative Euler’s equations up to point of formation of white caps which are responsible for dissipation.

The workshop brought together many leading researchers in the field of strongly nonlinear phenomena and we only regret that we applied for a 2-days workshop instead of a 5-days workshop, because in the shorter workshop was really a stretch to cover such a large number of important results. The workshop allowed active engagement of junior faculty and researchers to what is the happening in this interdisciplinary area. A total 8 junior researchers and 3 women participated and their contribution was crucial for the success of the workshop.