Valuation of a Power Plant Under Production Constraints and Market Incompleteness

Arnaud PORCHET
Joint work with Nizar Touzi and Xavier Warin

CREST - Laboratoire de Finance et Assurance
and
Électricité de France - R&D division

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Outline

Outline

1. Overview
2. Utility indifference value
3. Characterization by BSDE
   - BSDE
   - Main result
4. Numerical Application
   - Complete market
   - Incomplete market

Real option valuation with constraints

No constraints and complete market ⇒

$$p = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} q(S_t^e - HS_t^g)^+ dt\right]$$

In reality:

- Physical constraints: switching costs, minimal on/off times, ramp rates, outage uncertainties etc...
- Incompleteness of energy markets

Motivation: Extend the AAO value to take these features into account

Methodology

- Define the value by utility indifference
- Derive the associated optimal control problems
- Characterize the solution by means of Backward SDE and PDE
- Solve the equations numerically

$\Rightarrow$ Quantify the impact of production constraints and market incompleteness

Outline

Utility indifference pricing

Methodology for pricing a contingent claim ξ:

- define a utility function U
- compute

$$V(x) := \sup_{\pi} \mathbb{E}[U(X_T^{x,\pi} + \xi)]$$
$$v(x) := \sup_{\pi} \mathbb{E}[U(X_T^{x,\pi})]$$

- define

$$p(x) := \sup \{ p \in \mathbb{R} : \ V(x-p) \geq v(x) \}$$

$\longrightarrow$ Maximal amount of cash the agent is ready to pay to buy ξ

$\longrightarrow$ Reduces to the no arbitrage pricing formula in a complete market

Investment strategies

We suppose

- Exponential utility

$$U(x) \; := \; -e^{-\eta x} \, , \eta > 0$$

- Diffusion price process

$$\frac{dS_t^i}{S_t^i} \; = \; \mu_i(t, S_t)dt + \sigma_i(t, S_t) \cdot dW_t \; , \; 1 \leq i \leq n$$

- Wealth process

$$X_t^{x,\pi} \; := \; x + \int_0^t \pi_u \cdot \frac{dS_u}{S_u} \quad , \quad \pi_u \; \in \; K \subset \mathbb{R}^n$$

Management strategies - Case of 2 modes

We suppose

- Instantaneous rates of benefit ψ^0 (off) and ψ^1 (on)
  Ex: $\psi_t^1 = q\left(S_t^e - HS_t^g\right)$, $\psi_t^0 = 0$
- Shut-down and start-up costs C_0 and C_1
- Minimal on/off times δ_0 and δ_1

$\Rightarrow$ Management strategy $\theta := (\theta_n, n \geq 0)$ s.t.

$$\theta_{2n+i} + \delta_i \leq \theta_{2n+1+i}$$

$\theta_{2n+i} \leftrightarrow$ switch to mode i (we assume $\theta_0 = 0$)

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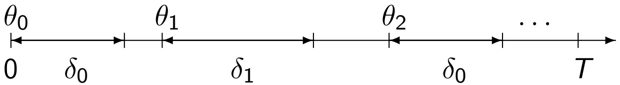

Cumulated benefit

$$B_t^\theta \;:=\; \overbrace{\int_0^t \psi(u,\theta)du}^{\text{Total Gains}} \;-\; \overbrace{\sum_{n\geq 1}\left(C_0 \mathbf{1}_{\{0<\theta_{2n}\leq t<T\}} + C_1 \mathbf{1}_{\{0<\theta_{2n-1}\leq t<T\}}\right)}^{\text{Sum of Switching Costs}}$$

where

$$\psi(u,\theta) \;:=\; \sum_{n\geq 0}\left(\underbrace{\psi_u^0 \mathbf{1}_{\{\theta_{2n}\leq u<\theta_{2n+1}\}}}_{\text{Off}} + \underbrace{\psi_u^1 \mathbf{1}_{\{\theta_{2n+1}\leq u<\theta_{2n+2}\}}}_{\text{On}}\right)$$

Management strategies - General Case

- Instantaneous rates of benefit ψ^i, $1 \leq i \leq M$
- Switching costs $C_{i,j}$
- Minimal on/off times $\delta_{i,j}$

$\Rightarrow$ Management strategy $\xi_t := \sum_{n \geq 0} \xi^n \mathbf{1}_{\{\theta_n \leq t < \theta_{n+1}\}}$ s.t.

$$\theta_n + \delta_{\xi^{n-1},\xi^n} \leq \theta_{n+1}$$

- Total Gains

$$B_t^{\xi} = \int_0^t \psi_u^{\xi_u} du + \sum_{n \geq 1, \theta_n \leq t} C_{\xi^{n-1},\xi^n}$$

Optimal control problems

- Maximal utility functions

$$V_0(x) := \sup_{\theta,\pi} \mathbb{E}\left[U\left(X_T^{x,\pi} + B_T^{\theta}\right)\right]$$
$$v_0(x) := \sup_{\pi} \mathbb{E}\left[U\left(X_T^{x,\pi}\right)\right]$$

- Price

$$p_0(x) := \sup\left\{p \in \mathbb{R},\ V_0(x-p) \geq v_0(x)\right\}$$

Outline

Backward Stochastic Differential Equation - BSDE

- Forward SDE - X unique $\mathcal{F}$-adapted process s.t.

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

given $x_0 \in \mathbb{R}$, μ and σ

- Backward SDE - (Y, Z) unique $\mathcal{F}$-adapted processes s.t.

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds + \int_t^T Z_s dW_s$$

given $\xi \in \mathcal{F}_T$ and f.

$\rightarrow$ if $f = 0$, Y and Z linked by representation thm $\Rightarrow$ not ind

BSDE: Applications to option pricing

- European option: Compute $\mathbb{E}^{\mathbb{Q}}[g(X_T)]$

$$Y_t = g(X_T) + \int_t^T \frac{\mu_s}{\sigma_s} Z_s ds + \int_t^T Z_s dW_s$$

$\Rightarrow Y_t = \mathbb{E}^{\mathbb{Q}}[g(X_T)|\mathcal{F}_t]$ is the option price at time t

- American option: Compute $\sup_{\tau \in \mathcal{T}(0,T)} \mathbb{E}^{\mathbb{Q}}[g(X_\tau)]$

$$Y_t = g(X_T) + \int_t^T \frac{\mu_s}{\sigma_s} Z_s ds + \int_t^T Z_s dW_s + (K_T - K_t)$$

$$Y_t \geq g(X_t) \, , \; 0 = \int_0^T (Y_t - g(X_t)) dK_t$$

with K increasing, continuous, $K_0 = 0$

$\Rightarrow$ Reflected BSDE - Y_t is the option price at time t

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BSDE: Applications to optimal control

Quasi Linear Hamilton-Jacobi-Bellman PDE:

$$0 = -v_t(t,x) - \mathcal{L}v(t,x) + \sup_{a \in A} f\left(t, x, v(t,x), v_x(t,x), a\right)$$

$\mathcal{L}v := \mu v_x + \frac{1}{2}\mathrm{Tr}(\sigma\sigma' v_{xx})$ and $v(T,.) = \phi(.)$

$\Longrightarrow$ $Y_t := v(t, X_t)$, $Z_t := \sigma(t, X_t) v_x(t, X_t)$ solve

$$dY_t = \sup_{a \in A} f\left(t, X_t, Y_t, \sigma^{-1}(t, X_t) Z_t, a\right) dt + Z_t dW_t$$

with $Y_T = \phi(X_T)$

$\longrightarrow$ Link between PDE and BSDE

$v_0(x)$ and BSDE

We define

$$g_t(z) \quad := \quad \frac{\eta}{2}\left|\Sigma_t z - \Pi_t(\Sigma_t z)\right|^2 + \Pi_t(\Sigma_t z)\cdot\Pi_t(\Sigma_t^{-1}\mu_t)$$

where $\Pi_t(x)$ is the orthogonal projection of x on $\Sigma_t K$. Consider

$$y_t \quad = \quad \frac{1}{2\eta}\int_t^T \left|\Pi_u(\Sigma_u^{-1}\mu_u)\right|^2 du - \int_t^T g(u, y_u)du - \int_t^T z_u \Sigma_u dW_u$$

Imkeller-Hu-Müller proved that

Proposition

$$v_0(x) := \sup_{\pi\in\mathcal{A}_0} \mathbb{E}\left[U\left(X_T^{x,\pi}\right)\right] = -e^{-\eta(x+y_0)}$$

$V_0(x)$ and RBSDE

We define

$$f_t^i(z) \;:=\; g_t(z) - \frac{1}{2\eta}\left|\Pi_t(\Sigma_t^{-1}\mu_t)\right|^2 - \psi_t^i$$

Consider

$$\begin{aligned}
Y_t^i &= -\int_t^T f_u^i\left(Z_u^i\right) du - \int_t^T Z_u^i \cdot \Sigma_u dW_u + \left(K_T^i - K_t^i\right) \\
Y_t^i &\geq \overline{Y}_t^{1-i} - C_{1-i} \\
K^i &\in \mathcal{J}(\mathbb{R}) \, , \int_0^T \left(Y_t^i - \overline{Y}_t^{1-i} + C_{1-i}\right) dK_t^i = 0 \\
\overline{Y}_t^i &= \mathcal{E}^g_{t,\overline{\delta}_i(t)}\left[Y^i_{\overline{\delta}_i(t)} + \int_t^{\overline{\delta}_i(t)} \left(\frac{1}{2\eta}\left|\Pi_u(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right) du\right]
\end{aligned}$$

$V_0(x)$ and RBSDE - 2

Then we shall prove that

Proposition

$$V_0(x) := \sup_{(\theta,\pi)\in\mathcal{T}_\infty\times\mathcal{A}_0} \mathbb{E}\left[U\left(X_T^{x,\pi} + B_T^{\theta}\right)\right] = -e^{-\eta(x+\overline{Y}_0^0)}$$

Optimal switching and trading

- No Delays - mode i at time t: switch if $Y_t^i < Y_t^{1-i} - C_{1-i}$
- Delays - introduce $\overline{Y}^i$ to anticipate over the no-switching period: switch if $Y_t^i < \overline{Y}_t^{1-i} - C_{1-i}$

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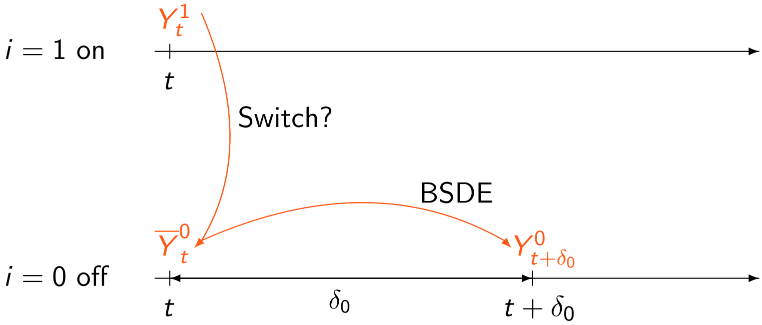


Properties

- Complete Market
  If $\delta_0 = \delta_1 = 0$, $C_0 = C_1 = C$ and $K = \mathbb{R}^n$, the value of the physical asset p_0^C converges as C tends to 0 to

$$p_0^C \underset{C \to 0}{\longrightarrow} \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \psi_t^0 \vee \psi_t^1 dt \right]$$

- Incomplete Market

$$\lim_{\eta \to 0} p_0^\eta = \sup_{\xi \in \mathcal{X}_0} \mathbb{E}^{\mathbb{Q}_e} [B_T^\xi] \;, \quad \lim_{\eta \to \infty} p_0^\eta = \sup_{\xi \in \mathcal{X}_0} \inf_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}} [B_T^\xi]$$

Outline

Link with a non-linear PDE

$$0 = \min\left\{u^i - \overline{u}^{1-i} + C_{1-i}, -\mathcal{L}u^i + f^i\left(.,.,\frac{\partial u^i}{\partial s}\right)\right\}, \text{ on } [0,T]\times\mathbb{R}^n$$
$$u^i(T,s) = \phi(s)\ ,$$

for $i \in \{0,1\}$, and for all $t_0 \in [0,T]$, $\overline{u}^i(t_0,s) = w^i(t_0,t_0,s)$, where $w^i(t_0,t,s)$ solves:

$$0 = -\mathcal{L}w^i + f^i\left(.,.,\frac{\partial w^i}{\partial s}\right), \text{ on } [t_0,\overline{\delta}_i(t_0)]\times\mathbb{R}^n$$
$$w^i(t_0,\overline{\delta}_i(t_0),s) = u^i(\overline{\delta}_i(t_0),s)\ ,$$

where

$$\mathcal{L}u(t,s) := \frac{\partial u}{\partial t}(t,s) + \mu(t,s)\frac{\partial u}{\partial s}(t,s) + \frac{1}{2}\mathrm{Tr}\left(\Sigma\Sigma^*(t,s)\frac{\partial^2 u}{\partial s^2}(t,s)\right)$$

Overview

2 methods

- BSDE $\Rightarrow$ Monte Carlo
- PDE

2 cases

- Complete Market $\Rightarrow$ linear equations (independent of η)

$$p = \sup_{\theta \in \mathcal{T}_\infty} \mathbb{E}^{\mathbb{Q}}[B_T^\theta]$$

- Incomplete Market $\Rightarrow$ quadratic equations (harder to solve)

Numerical computation of BSDE

Euler scheme on X and Y:

$$\begin{aligned} Y_{t_{k+1}} - Y_{t_k} &= -\int_{t_k}^{t_{k+1}} f(s, X_s, Y_s, Z_s)ds - \int_{t_k}^{t_{k+1}} Z_s dW_s \\ &\simeq -f(t_k, X_{t_k}, Y_{t_k}, Z_{t_k})\Delta t_{k+1} - Z_{t_k}\Delta W_{k+1} \end{aligned}$$

Conditional expectations:

$$\begin{aligned} \mathbb{E}_{t_k}[\Delta W_{k+1} \times .] \Rightarrow Z_{t_k} &= \frac{1}{\Delta t_{k+1}} \mathbb{E}_{t_k}[\Delta W_{k+1} Y_{t_{k+1}}] \\ \mathbb{E}_{t_k}[.] \Rightarrow Y_{t_k} &= \mathbb{E}_{t_k}[Y_{t_{k+1}}] + f(t_k, X_{t_k}, Y_{t_k}, Z_{t_k})\Delta t_{k+1} \end{aligned}$$

Coal-fired power plant

- Forward price processes of electricity and coal

$$\frac{dF(t,T)}{F(t,T)} = \mu_F e^{-a(T-t)}dt + \sigma_F e^{-a(T-t)}dW_t^1$$
$$\frac{dG(t,T)}{G(t,T)} = \mu_G e^{-b(T-t)}dt + \sigma_G e^{-b(T-t)}dW_t^2$$

- Payoffs

$$\psi_t^0 := 0,\ \psi_t^1 := q\left(F(t,t) - HG(t,t)\right),$$

$\Rightarrow$ Complete market, 2-D linear BSDE/PDE

Results in complete market

- Parameters

$\delta_0 = 24$ h	$\delta_1 = 8$ h	$T = 8760$ h
$C_0 = 0$ €	$C_1 = 35530$ €	$H = 0.36$ ton/MWh

- Low electricity prices

$$p_c = 15.91\ 10^6\ \text{€}\ ,\ p_{nc} = 21.17\ 10^6\ \text{€}$$

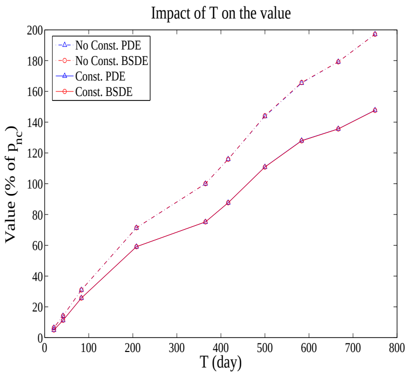
$\Rightarrow$ 25 % decrease

- High electricity prices

$$p_c = 119.3\ 10^6\ \text{€}\ ,\ p_{nc} = 125.9\ 10^6\ \text{€}$$

$\Rightarrow$ 5 % decrease

Results in complete market - 2



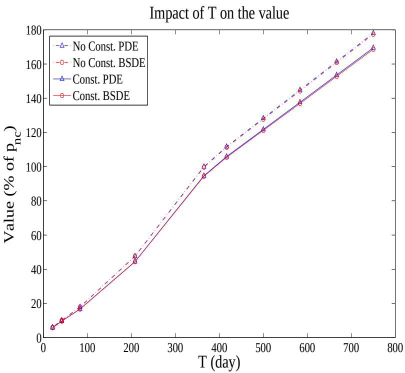


Figure: Impact of T: low (left) and high (right) electricity prices.

Results in complete market - 3

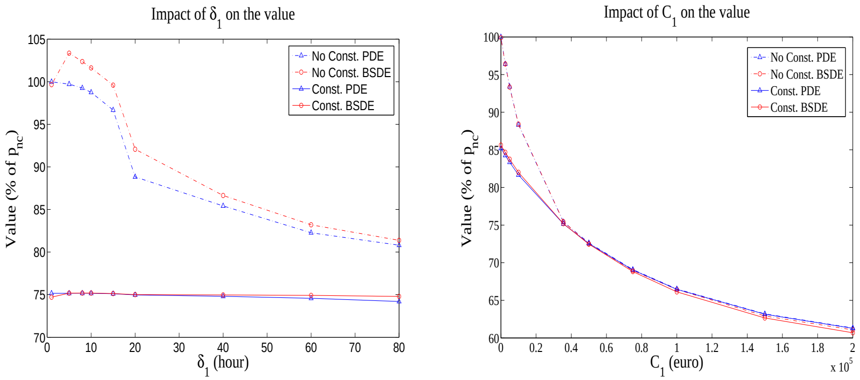


Figure: Impact of δ_1 (left) and C_1 (right).

Computational time

Horizon T (day)	83	208	365	500	750
PDE (CPU mn)	14	20	35	62	79
BSDE (CPU mn)	99	230	302	477	718

Table: Comparison of time performances between the PDE and BSDE algorithms.

BSDE: Lemor-Gobet-Warin with an 8×8 grid, linear approximation and 25600 simulations. $\Delta t = 1$ h.
PDE: Crank-Nicholson within a domain $[-5, 5] \times [-5, 5]$. In each direction, 100 step mesh. $\Delta t = 1$ h.

Incomplete market

1 source of incompleteness: exogenous shock

$$\psi_t^1 := q\left(F(t,t) + \varepsilon_t - HG(t,t)\right)$$
$$d\varepsilon_t = -\kappa\varepsilon_t dt + \gamma dW_t^3$$

$\Rightarrow$ 3-D quadratic BSDE/PDE

- BSDE algorithm does not converge easily
- PDE is more tractable but very high computational time ($T = 6$ months $\Rightarrow$ 1 week)
- Price now depends on η
- $\kappa = 0.02$, $\gamma = 0.01$, $\eta = 1 \Rightarrow$ 25 % decrease

Results in incomplete market

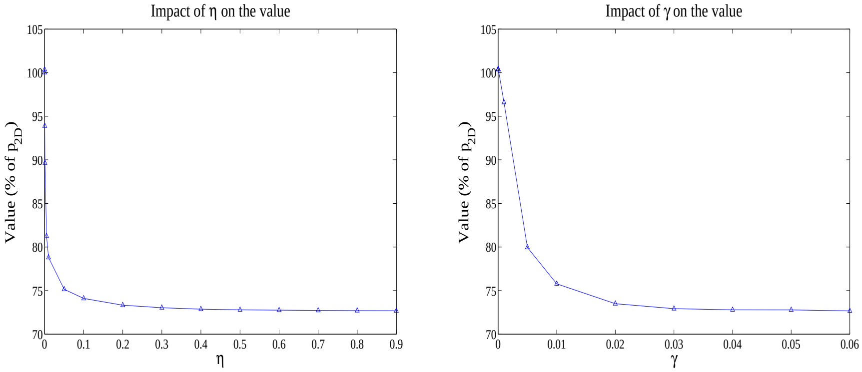


Figure: Impact of η and γ.

Conclusion

- Develop a utility indifference framework for valuing physical assets
- Numerically intensive method, esp. in incomplete market
- Upper bound given by the pricing under minimal entropy measure

Extensions:

- Improve numerical methods for quadratic BSDE
- Hydropower: BSDE methods for stochastic optimal constraint with state constraints?