Indecomposable infinite graphs

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Interval

Definition

Given a graph $G = (V(G), E(G))$*, a subset* I *of* $V(G)$ *is an interval of* G *if for each* $x \in V(G) \setminus I$*, we have either* x — I *or* x I

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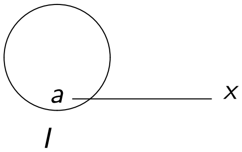
Remark 1

Given a tree G, if I is a non trivial interval of G, then all the elements of I are leaves of G.

Proof

G is connected $\Longrightarrow \exists\ a \in I$ and $\exists\ x \in V(G) \setminus I$ such that $\{a, x\} \in E(G)$
I is an interval of $G \Longrightarrow \forall\ i \in I\ \{i, x\} \in E(G) \Longrightarrow I$ is stable
By Contradiction : $\exists\ b \in I$ such that $\deg(b) \geq 2$
Consider $y \neq z \in V(G)$ such that $\{b, y\}, \{b, z\} \in E(G)$
I is stable $\Longrightarrow y, z \notin I$
I is an interval of $G \Longrightarrow$ for $c \in I$ with $c \neq b$: $\{c, y\}, \{c, z\} \in E(G)$
$\Longrightarrow (y, b, z)$ or (y, b, z, c) is a cycle



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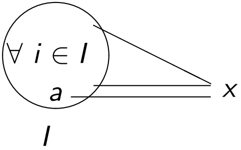

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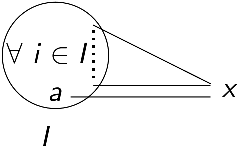

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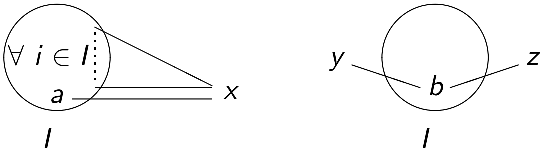

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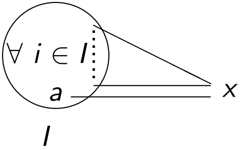

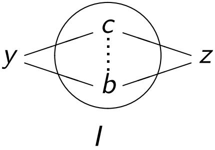

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Given a digraph $D = (V(D), A(D))$*, a subset* I *of* $V(D)$ *is an interval of* D *if for each* $x \in V(D) \setminus I$*, we have*
either $x \longleftrightarrow I$ *or* $x \longrightarrow I$ *or* $x \longleftarrow I$ *or* x I

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Given a tournament $T = (V(T), A(T))$*,*
a subset I *of* $V(T)$ *is an interval of* T
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Definition

For a total order $O = (V(O), A(O))$, a subset I of $V(O)$ is an interval of O if for each $x \in V(O) \setminus I$, we have $x > I$ or $x < I$
The usual notion of interval

Indecomposable graphs, digraphs,...

Definition

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Indecomposable : see also prime, primitive.

Remark 2

If $n \geq 4$, then P_n is indecomposable.

$$1 —— 2 —— \cdots\cdots —— n-1 —— n$$

The path P_n on $n \geq 2$ vertices

$$1 \text{———} 2 \text{———} \cdots\cdots\cdots \text{———} n-1 \text{———} n$$

The path P_n on $n \geq 2$ vertices

Proof

By contradiction :
if P_n admits a non trivial interval I, then $I = \{1, n\}$ by Remark 1.
But 1 — 2 .... n because $n \geq 4$.

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Theorem 1 (Sumner 1973)

Given an infinite or finite graph G, with $|V(G)| \geq 4$, if G is indecomposable, then G contains P_4 as an induced subgraph.

Indecomposable and infinite graphs, digraphs,...

Theorem 2 (Ille 1994)

Given an infinite graph G*,* G *is indecomposable iff for every finite subset* X *of* $V(G)$*, there exists a finite subset* Y *of* $V(G)$ *such that* $X \subseteq Y$ *and* $G[Y]$ *is indecomposable.*

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Given an infinite graph G, G is indecomposable iff for every finite subset X of $V(G)$, there exists a finite subset Y of $V(G)$ such that $X \subseteq Y$ and $G[Y]$ is indecomposable.

Hint of proof

$*$ $\Longleftarrow$ Use the following lemma for the family $\mathcal{F}$ of the finite subsets X of $V(G)$ such that $G[X]$ is indecomposable.

Lemma 1

Given a graph G, consider a family $\mathcal{F}$ of subsets of $V(G)$ satifying

- $\forall\ X \in \mathcal{F} \quad G[X]$ *is indecomposable;*
- $\forall\ X \neq Y \in \mathcal{F} \quad \exists\ Z \in \mathcal{F} \quad X \cup Y \subseteq Z.$

Then $G[\cup\mathcal{F}]$ is indecomposable.

$\ast \Longrightarrow$ Use Theorem 1 and the following proposition

Proposition 1

Given an indecomposable (and infinite) graph G, consider a finite subset X of $V(G)$ such that $G[X]$ is indecomposable (and $|X| \geq 3$). For every $x \in V(G) \setminus X$, there exists a finite subset Y of $V(G)$ such that $x \in Y$, $X \subset Y$ and $G[Y]$ is indecomposable.

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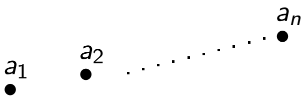

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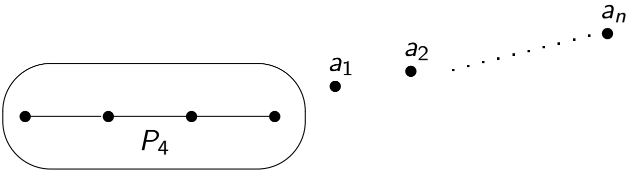

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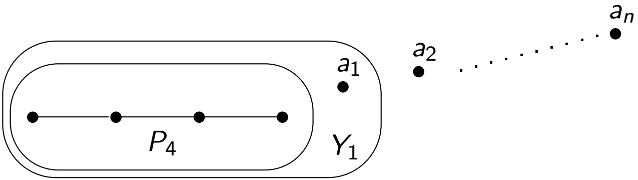

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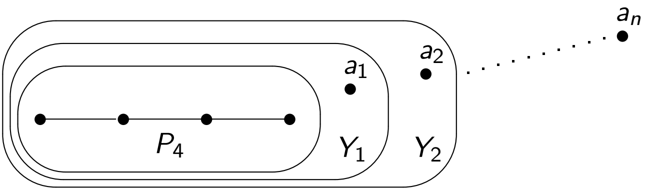

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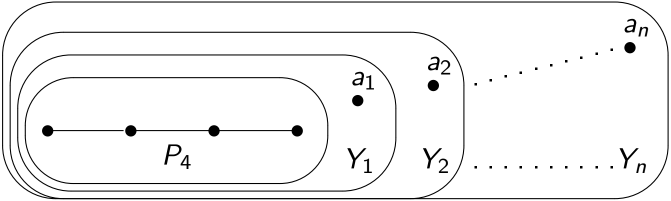

Application : duality theorems

Theorem 3 (Gallai 1967)

Given finite posets P and Q, with $V(P) = V(Q)$, if P and Q share the same comparability graph and if P is indecomposable, then $Q = P$ or $P^\star$.

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Theorem 4 (Boussaïri, Ille, Lopez, Thomassé 2004)

Given finite tournaments T and T', with $V(T) = V(T')$, if T and T' share the same C_3-structure and if T is indecomposable, then $T' = T$ or $T^\star$.

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$*$ $T[\{a, b\}] = T'[\{a, b\}] \Longrightarrow T'[Y] = T[Y] \Longrightarrow T[\{x, y\}] = T'[\{x, y\}]$

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$\Longrightarrow$ partition in intervals $I, J, K, \dots$ to define a new graph on $I, J, K, \dots$ by contraction.

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Given a graph G, a partition P of $V(G)$ is an interval partition of G if all the elements of P are intervals of G.

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Given a graph G, with each interval partition of G associate the quotient of G by P defined on $V(G/P) = P$ as follows. For any $I \neq J \in P$, I — J in G/P if I — J in G.

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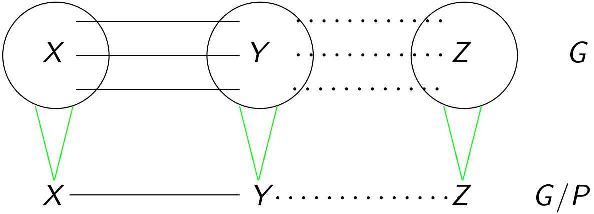

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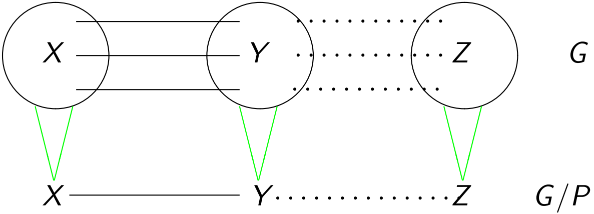

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Remark 6

The inverse operation is the lexicographic sum.

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- *The only non trivial intervals of G are $\{0, \ldots, n\}$ (for $n \neq 0$).*
- *For every interval partition P of G, $G/P \cong G$ or $\overline{G}$.*