Connected-homogeneous graphs

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Homogeneous graphs

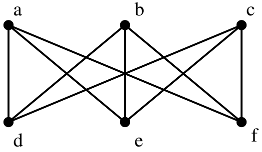
Definition

A graph Γ is called homogeneous if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

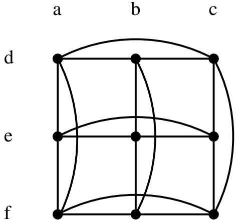
Homogeneity is the *strongest* possible symmetry condition we can impose on a graph.

Example

The line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$ is a finite homogeneous graph.



$K_{3,3}$



$L(K_{3,3})$

Classification of finite homogeneous graphs

Gardiner classified the finite homogeneous graphs.

Theorem (Gardiner (1976))

A finite graph is homogeneous if and only if it is isomorphic to one of the following:

1. *finitely many disjoint copies of a complete graph K_r (or its complement, complete multipartite graph)*
2. *the pentagon C_5*
3. *line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$.*

An infinite homogeneous graph

Definition (The random graph R)

Constructed by Rado in 1964. The vertex set is the natural numbers (including zero).

For $i, j \in \mathbb{N}$, $i < j$, then i and j are joined if and only if the ith digit in j (in base 2, reading right-to-left) is 1.

Example

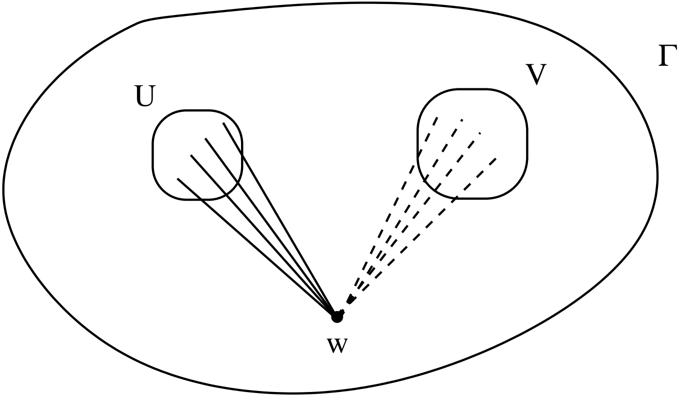
Since $88 = 8 + 16 + 64 = 2^3 + 2^4 + 2^6$ the numbers less that 88 that are adjacent to 88 are just $\{3, 4, 6\}$.

Of course, many numbers greater than 88 will also be adjacent to 88 (for example 2^{88}).

The random graph

Consider the following property of graphs:

(*) For any two finite disjoint sets U and V of vertices, there exists a vertex w adjacent to every vertex in U and to no vertex in V.



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Theorem

There exists a countably infinite graph R satisfying property (*), and it is unique up to isomorphism. The graph R is homogeneous.

Existence. The random graph R defined above satisfies property (*).

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Uniqueness and homogeneity. Both follow from a back-and-forth argument. Property (*) is used to extend the domain (or range) of any isomorphism between finite substructures one vertex at a time.

Building homogeneous graphs: Fraïssé's theorem

- The age of a graph Γ is the class of isomorphism types of its finite induced subgraphs.
- e.g. the age of the random graph R is the class of *all* finite graphs.

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Fraïssé (1953) - gives necessary and sufficient conditions for a class $\mathcal{C}$ of finite graphs to be the age of a countably infinite homogeneous graph M. The key condition is the amalgamation property.

If Fraïssé's conditions hold, then M is unique, $\mathcal{C}$ is called a Fraïssé class, and M is called the Fraïssé limit of the class $\mathcal{C}$.

Countable homogeneous graphs

Examples

- The class of all finite graphs is a Fraïssé class. Its Fraïssé limit is the random graph R.
- The class of all finite graphs not embedding K_n (for some fixed n) is a Fraïssé class. We call the Fraïssé limit the countable generic K_n-free graph.

Theorem (Lachlan and Woodrow (1980))

Let Γ be a countably infinite homogeneous graph. Then Γ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic K_n-free graph (or its complement).

Connected-homogeneous graphs

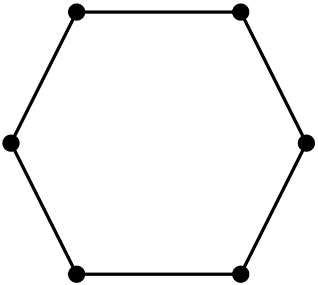
Definition

A graph Γ is connected-homogeneous if any isomorphism between *connected* finite induced subgraphs extends to an automorphism.

Example

The hexagon C_6 is connected-homogeneous

Use rotations and reflections



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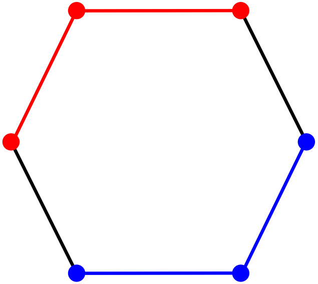

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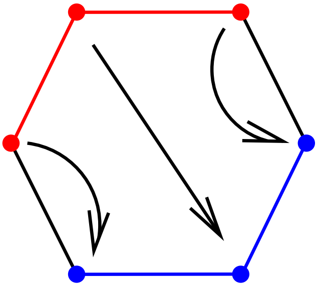

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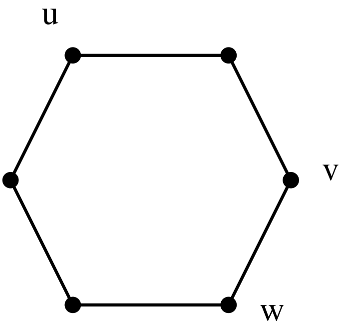
Example

On the other hand the hexagon is not homogeneous.

There is no automorphism α such that $(u, v)^{\alpha} = (u, w)$.



Connected-homogeneous graphs

Connected-homogeneity...

1. is a natural weakening of homogeneity;
2. gives a class of graphs that lie between the (already classified) homogeneous graphs and the (not yet classified) distance-transitive graphs.

$$\text{homogeneous} \Rightarrow \text{connected-homogeneous} \Rightarrow \text{distance-transitive}$$

(A graph is distance-transitive if for any two pairs (u, v) and (u', v') with $d(u, v) = d(u', v')$, where d denotes distance in the graph, there is an automorphism taking u to u' and v to v'.)

Finite connected-homogeneous graphs

Gardiner classified the finite connected-homogeneous graphs.

Theorem (Gardiner (1978))

A finite graph is connected-homogeneous if and only if it is isomorphic to a disjoint union of copies of one of the following:

1. *a finite homogeneous graph*
2. *bipartite "complement of a perfect matching"*
   (obtained by removing a perfect matching from a complete bipartite graph $K_{s,s}$)
3. *cycle C_n*
4. *the line graph $L(K_{s,s})$ of a complete bipartite graph $K_{s,s}$*
5. *Petersen's graph*
6. *the graph obtained by identifying antipodal vertices of the 5-dimensional cube Q_5*

Tree-like examples

Definition (Tree)

A tree is a connected graph without cycles. A tree is regular if all vertices have the same degree. We use T_r to denote a regular tree of valency r.

Fact. A regular tree T_r ($r \in \mathbb{N}$) is an example of an infinite locally-finite connected-homogeneous graph.

Definition (Semiregular tree)

$T_{a,b}$: A tree $T = X \cup Y$ where $X \cup Y$ is a bipartition, all vertices in X have degree a, and all in Y have degree b.

Locally finite infinite connected-homogeneous graphs

Let $r, l \in \mathbb{N}$ $(l \geq 2)$

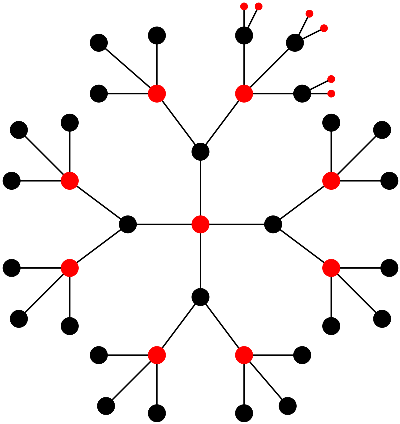
Take the bipartite semiregular tree $T_{r+1,l}$.

The graph $X_{r,l}$ is given by:

Vertices = bipartite block of $T_{r+1,l}$ of vertices of degree l.

Edges = adjacent in $X_{r,l}$ if their distance in the tree is 2.

(Macpherson (1982) proved that every connected infinite locally-finite distance transitive graph has this form)



$T_{3,4}$

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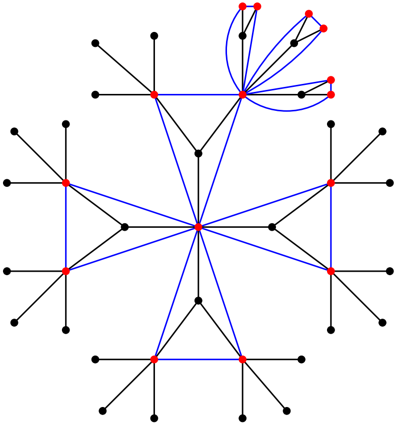

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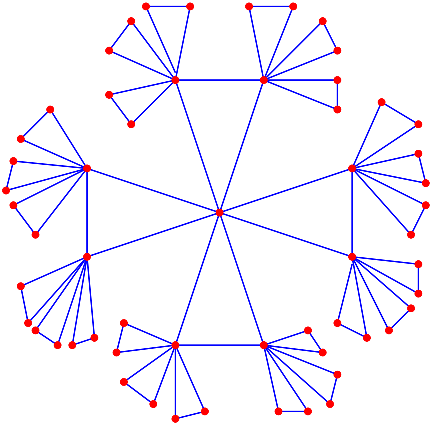

$X_{2,4}$

Infinite connected-homogeneous graphs

Theorem (RG, Macpherson (2007))

A countable graph is connected-homogeneous if and only if it is isomorphic to the disjoint union of a finite or countable number of copies of one of the following:

1. *a finite connected-homogeneous graph;*
2. *a homogeneous graph;*
3. *the random bipartite graph;*
4. *bipartite infinite complement of a perfect matching;*
5. *the line graph of the infinite complete bipartite graph $K_{\aleph_0,\aleph_0}$;*
6. *a treelike graph X_{κ_1,κ_2} with $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$.*

Future work

Digraphs

- Cherlin (1998) classified the countable homogeneous digraphs.
- There are $2^{\aleph_0}$ such graphs.

Problem 1. Classify the countably infinite connected-homogeneous digraphs.

Problem 2. Classify the locally-finite countably infinite connected-homogeneous digraphs.

Recent progress (with R. Möller).

In the case that the graph has more than one end we have:

1. a classification when the underlying graph embeds a triangle
2. underlying graph triangle-free $\Rightarrow$ digraph is highly-arc-transitive
   - can describe the descendants and the reachability graphs