Hochschild Cohomology of Algebras: Structure and Applications

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Hochschild cohomology of associative algebras is important in many areas of mathematics, such as ring theory, commutative algebra, geometry, noncommutative geometry, representation theory, group theory, mathematical physics, homotopy theory and topology. Usually, associative algebras, occuring naturally, are not semisimple, and to understand their properties, homological methods are absolutely essential. Such algebras can be very different depending on the context, but fortunately, Hochschild cohomology is defined in complete generality, as it is constructed in terms of very basic linear algebra. In spite of this simplicity, it is a unifying concept for different areas. It has important invariance properties, for example it is invariant under derived, stable, and Morita equivalences.

The workshop concentrated on several algebraic aspects of Hochschild cohomology, reflecting connections among homological algebra, commutative algebra, representation theory, group theory and mathematical physics. It brought together experts from several of these fields, promoting an exchange of ideas and potentially new collaborations. Mathematicians using Hochschild cohomology presented recent results, techniques and open problems from their fields. This allowed us to take stock of progress achieved and led to discussions of possible solutions to open problems. A few mathematicians presented applications of Hochschild cohomology in fields outside of algebra proper, encouraging potentially useful interaction within a wider group of users of Hochschild cohomology, and broadening our understanding of the context to which the theory applies and what it all means.

During the week there were 21 talks. There were 41 participants in total, 5 of these Canadians.

1 Overview of the Field

Let A be an associative algebra over a field k, and M an A-bimodule (equivalently an $A\otimes_k A^{op}$-module where A^{op} is the algebra A with the opposite multiplication). The Hochschild cohomology of A, with coefficients in M, is the graded vector space $\mathrm{HH}^*(A,M) = \mathrm{Ext}^*_{A\otimes_k A^{op}}(A,M)$, where A is considered an $A\otimes_k A^{op}$-module under left and right multiplication. Equivalently, $\mathrm{HH}^*(A,M)$ is given in terms of the bar resolution of A by free $A\otimes_k A^{op}$-modules:

$$\cdots \xrightarrow{\delta_2} A\otimes_k A\otimes_k A \xrightarrow{\delta_1} A\otimes_k A \xrightarrow{\delta_0} A \to 0$$

where $\delta_i(a_0\otimes\cdots\otimes a_{i+1}) = \sum_{j=0}^{i}(-1)^j a_0\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_{i+1}$. Apply the functor $\mathrm{Hom}_{A\otimes_k A^{op}}(-,M)$ to the above sequence and let δ_i^* $(i\geq 0)$ denote the induced map. Then $\mathrm{HH}^i(A,M) = \ker(\delta_i^*)/\mathrm{im}(\delta_{i-1}^*)$

and $\mathrm{HH}^*(A, M) = \oplus_{i \geq 0} \mathrm{HH}^i(A, M)$.

The cohomology space $\mathrm{HH}^*(A, A)$ has a cup product, under which it is an associative algebra, and a graded Lie bracket. Further, $\mathrm{HH}^*(A, M)$ is a module for $\mathrm{HH}^*(A, A)$. One wants to understand the structure of $\mathrm{HH}^*(A, A)$ as a graded vector space, as a ring, and as a graded Lie algebra, and similarly to understand its modules $\mathrm{HH}^*(A, M)$. In turn such information about cohomology sheds light on the structure of the algebra A itself and on its bimodules M.

Many of the deep results on the structure of Hochschild homology and cohomology involve methods and results particular to a given field. It is one of the constant challenges to understand the unifying principles or possibilities of transfer from one area to another. We mention as one example the Hodge decomposition of Hochschild (co)homology for commutative algebras in characteristic zero (Quillen, Gerstenhaber-Schack, Loday in the algebraic framework; Kontsevich, Swan in geometry), that has caused new insights in combinatorics, such as the study of Eulerian idempotents, and in Lie theory, where it helps to understand better the nature of the Poincaré-Birkhoff-Witt isomorphism (Reutenauer, Loday, Bergeron-Wolfgang). This decomposition has as well a counterpart for the Ext algebra of the residue field of a commutative ring in positive characteristic (Tate, Assmus, Gulliksen-Levin, Quillen, Gerstenhaber and Schack), the decomposition holds universally for complete intersections (Wolffhardt), and there are decomposition theorems whenever a group action is present. To what extent the decomposition theorem is compatible with the higher multiplicative structure on Hochschild cohomology is an important problem, where a lot of research is currently under way. There are many questions open here, of interest in a broad range of mathematics.

2 Recent Developments and Open Problems

The vanishing of Hochschild cohomology in high degrees has implications for the structure of algebras; for example for finitely generated commutative algebras it is equivalent to the geometrically motivated notion of smoothness. There is evidence that such implications may play an important role also for noncommutative algebras, but they are not well understood and so bear further investigation. Computations of the ring structure of Hochschild cohomology of various types of algebras have been completed in the past decade by researchers from different fields using specialized techniques. When the Hochschild cohomology ring is finitely generated modulo nilpotents, support varieties of modules can be defined, analogous to those from group cohomology, and this is currently an active area of research. At present it is not clear how to produce systematically algebras with this property, as proofs of finite generation typically depend on difficult techniques and deep results from some specific area. The workshop was an ideal venue for sharing ideas and potentially making further progress.

2.1 Vanishing of Hochschild (co)homology in high degrees

For projective commutative algebras of finite type, eventual vanishing of either series of invariants is equivalent to the geometric notion of smoothness, and to finite projective dimension over the enveloping algebra (Avramov, Iyengar, Rodicio, Vigué-Poirrier). However, vanishing of cohomology does not imply vanishing of homology (Han) nor finite global dimension (Buchweitz, Green, Madsen, Solberg). The discovery of very small self-injective algebras with finite-dimensional Hochschild cohomology ring was a great surprise, and clearly, this is a very interesting avenue for further research.

The condition of finite projective dimension has been proposed as a definition for smoothness of noncommutative algebras (van den Bergh), but its properties are largely unknown and its relation to other generalizations of smoothness are not yet completely understood. The joint expertise of the participants of the workshop provided a good setting for discussing smoothness and other geometrically motivated properties of commutative algebras, such as being locally complete intersection, in the context of more general associative algebras.

2.2 The ring and Lie algebra structures of Hochschild cohomology

Ideally one might aim for a presentation by generators and relations, though this appears to be a very difficult problem in general. There is recent progress, and various groups of researchers have developed completely different methods. For example, the ring structures of Hochschild cohomology of certain path algebras and related algebras were determined using quiver techniques (Erdmann, Holm, and Snashall). A computation of the Hochschild cohomology ring of a finite group algebra used group cohomology (Siegel and Witherspoon proved a conjecture originally due to Cibils). The Hochschild cohomology rings of invariant subalgebras of Weyl algebras and generalizations were found by adapting techniques from commutative homological algebra (Alev, Farinati, Lambre, Solotar, and Suarez-Alvarez). These are just a few examples of computations of the ring structure of Hochschild cohomology completed within the past ten years.

2.3 Varieties for modules

In group representation theory, the support variety of a module is a powerful invariant, for example its dimension is equal to the complexity of the module, and it encapsulates many of the module's homological properties. It has found many applications, and has subsequently been extended, first to restricted enveloping algebras and more recently to finite group schemes and to quantum groups. The very definition of support varieties for these algebras depends on the difficult results that their cohomology algebras are finitely generated.

One would like to have some analogue for more general associative algebras. Recently much work has been done, showing that Hochschild cohomology, which is graded commutative, can serve as a substitute for group cohomology. It was shown that at least for self-injective algebras, many properties of support varieties for group representations have analogues in this more general setting (work by Snashall, Solberg, joined by Erdmann, Holloway, Taillefer). This however needs suitable finite generation properties. The required condition that the quotient of Hochschild cohomology by a nilpotent ideal be finitely generated is conjectured to be true for artin algebras (Snashall and Solberg), and known to be true in some cases. It would be very interesting to establish this conjecture for classes of algebras, especially finite-dimensional Hecke algebras. For these, some more information is available via rank varieties, and some analogue of 'Quillen stratification' might be true. In the context of more general associative algebras, such properties are even less well understood.

For some algebras, all representations or modules have periodic projective resolutions; these include modular group algebras of finite groups with cyclic p-Sylow or quaternion Sylow subgroups on one end of the spectrum, but also, apart from self-injective algebras of finite type, all finite-dimensional preprojective algebras that are of interest for quiver varieties and quantizations of singularities. In all cases known, this phenomenon is explained by periodicity of the Hochschild complex—and then one has in particular finite generation of cohomology.

2.4 Hochschild cohomology in related areas

To conclude we mention that in addition to the above algebraic directions of research, many questions arise from the use of Hochschild cohomology in fields less related to algebra that accordingly were not central to this meeting.

Geometric applications of Hochschild homology and cohomology play a significant role in work of Barannikov, Kontsevich, Markarian and Tsygan. One instance would be the connection with graph-complexes à la Kontsevich that was reported on by Burgunder, see below.

There are as well attempts to define more generally Hochschild theory for abelian, exact, or triangulated categories, see for example Lowen's contribution below.

As a specific example of a geometric application, mathematicians studying orbifolds have looked at Hochschild cohomology of the bounded derived category of equivariant sheaves (Baranovsky, Căldăraru, Kaledin) or their quantizations (Etingof, Ginzburg, Kaledin); in case of C*-algebras, not touched upon here, this work centers on the celebrated Baum-Connes conjecture. In the case of an affine space, this Hochschild cohomology ring is simply the Hochschild cohomology of an extension of a function algebra by the group, and thus it may be studied algebraically.

3 Presentation Highlights

The first day of the workshop consisted of five talks by younger mathematicians who introduced and represented the broad topics into which the remaining talks were organized. We include the abstracts of the talks submitted by the presenters.

3.1 Hochschild cohomology and support varieties

Nicole Snashall introduced this topic on the first day, and other talks were given by Petter Andreas Bergh and Henning Krause.

Nicole Snashall (University of Leicester)
The Hochschild cohomology ring modulo nilpotence
For a finite-dimensional algebra Λ, it was conjectured by Snashall and Solberg that the Hochschild cohomology ring of Λ modulo nilpotence is itself a finitely generated algebra. This talk described the current position of this conjecture and its connection to support varieties of modules. Reference was also made as to whether or not it is known that the Hochschild cohomology ring itself is finitely generated in certain cases, as this plays an important role in support varieties for self-injective algebras.

In particular, Snashall gave an overview of what is known for finite-dimensional self-injective algebras, then discussed the cases where Λ is a monomial algebra or is in a class of special biserial algebras which arise from the representation theory of $U_q(\mathfrak{sl}_2)$ (Erdmann, Snashall, Taillefer).

Petter Andreas Bergh (NTNU Trondheim)
Hochschild (co)homology of quantum complete intersections
This is joint work with Karin Erdmann. We construct a minimal projective bimodule resolution for finite dimensional quantum complete intersections of codimension 2. Then we use this resolution to compute the Hochschild homology and cohomology for such an algebra. In particular, we show that when the commutator element is not a root of unity, then the cohomology vanishes in high degrees, while the homology is always nonzero. Thus these algebras provide further counterexamples to "Happel's question", a question for which the first counterexample was given by Buchweitz, Green, Madsen and Solberg. On the other hand, the homology of the quantum complete intersections behave in accordance with Han's conjecture, i.e. the homology version of Happel's question.

Henning Krause (Universität Paderborn)
Localising subcategories of the stable module category of a finite group
This is a report on recent joint work with Srikanth Iyengar and Dave Benson. We classify the localising subcategories of the stable module category for a finite group. This enables us to prove the telescope conjecture in this context, as well as give a new proof of the tensor product theorem for support varieties.

In my talk I explain the history of this classification problem as well as the strategy of the proof. The challenge is basically to make proper use of the group cohomology ring which acts as a ring of cohomological operators. Thus we reduce this classification to a problem from commutative algebra. The main tools are support varieties and local cohomology. Then we use a similar classification of localising subcategories for the derived category of a commutative noetherian ring, which Neeman obtained some 15 years ago.

3.2 Structure of Hochschild cohomology

Srikanth Iyengar introduced this topic, and other talks were given by Thorsten Holm, Maria Julia Redondo, Marco Farinati, Mariano Suarez-Alvarez, and James Zhang.

Srikanth Iyengar (University of Nebraska, Lincoln)
Gorenstein algebras and Hochschild cohomology
This is joint work with L. L. Avramov. A classical result of Hochschild, Kostant, and Rosenberg characterizes smoothness of commutative algebras essentially of finite type over a field in terms of its Hochschild cohomology. I will discuss a similar characterization of the Gorenstein property.

Thorsten Holm (Universität Magdeburg)
Bilinear forms, Hochschild (co-)homology and invariants of derived module categories
This is joint work with C. Bessenrodt and A. Zimmermann. Let A be a symmetric algebra over a perfect field k of positive characteristic p. For such algebras, B. Külshammer introduced, for any n, spaces $T_n(A)$ as those elements of A whose p^n-th power lies in the commutator subspace $K(A)$. He then considered the orthogonal spaces with respect to the symmetrizing bilinear form on the symmetric algebra A. These $T_n(A)^\perp$ are ideals of the center of A, i.e. of the degree 0 Hochschild cohomology of A.

It has been shown by A. Zimmermann that the sequence of these ideals is invariant under derived equivalences of the symmetric k-algebras.

In the talk we first briefly discuss Zimmermann's results and then explain how to extend this theory to arbitrary finite-dimensional, not necessarily symmetric, k-algebras. The way to achieve this is by passing to the trivial extension algebras. In this way we obtain new invariants of the derived module categories of finite-dimensional k-algebras. This can be seen as an extension of the well-known fact that the degree 0 Hochschild homology $A/K(A)$ is invariant under derived equivalence.

We also present recent applications of the above results, e.g. to blocks of group algebras.

María Julia Redondo (Universidad Nacional del Sur)
Hochschild cohomology via incidence algebras
Let A be an associative, finite dimensional algebra over an algebraically closed field k. It is well known that if A is basic then there exists a unique finite quiver Q and a surjective morphism of k-algebras $\nu : kQ \to A$, which is not unique in general, with $I_\nu = \mathrm{Ker}\ \nu$ admissible. The pair (Q, I_ν) is called a *presentation* of A. Given a presentation (Q, I) of A we associate an incidence algebra $A(\Sigma_\nu)$ and study the connection between their Hochschild cohomology groups. If the chosen presentation is homotopy coherent we define a morphism between the complexes computing these cohomology groups, which induces morphisms $\mathrm{HH}(\Phi^n) : \mathrm{HH}^n(A(\Sigma_\nu)) \to \mathrm{HH}^n(A)$. Finally we find conditions for these morphisms to be injective.

Marco Farinati (Universidad de Buenos Aires)
The cohomology of monogenic extensions in the noncommutative setting
We extend the notion of monogenic extension to the noncommutative setting, and we study the Hochschild cohomology ring of such an extension. As an application we complete the computation of the cohomology ring of the rank one Hopf algebras begun by S. M. Burciu and S. J. Witherspoon.

Mariano Suarez-Alvarez (Universidad de Buenos Aires)
Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology
We consider the change-of-rings spectral sequence as it applies to Hochschild cohomology, obtaining a description of the differentials on the first page which relates it to the multiplicative structure on cohomology. Using this information, we are able to completely describe the cohomology structure of monogenic algebras as well as some information on the structure of the cohomology in more general situations.

We also show how to use the spectral sequence to reprove and generalize results of M. Auslander et al. about homological epimorphisms. We derive from this a rather general version of the long exact sequence due to D. Happel for a one-point (co)-extension of a finite dimensional algebra and show how it can be put to use in concrete examples.

James Zhang (University of Washington)
Twisted Hochschild (co)homology for Hopf algebras
This is joint work with K.A. Brown. The Hochschild homology and cohomology groups with coefficients in a suitably twisted free bimodule are shown to be non-zero in the top dimension d, when A is an Artin-Schelter regular noetherian Hopf algebra of global dimension d. (Twisted) Poincaré duality holds in this setting, as is deduced from a theorem of Van den Bergh.

3.3 Relations with algebraic geometry

Andrei Căldăraru introduced this topic, and other talks were given by Hubert Flenner, Joseph Lipman, and Amnon Yekutieli.

Andrei Căldăraru (University of Wisconsin, Madison)
Non-flat base change and the orbifold HKR isomorphism
I shall discuss and prove a base change theorem for derived categories of smooth schemes, in the absence of flatness assumptions. As an application I shall present a Hochschild-Kostant-Rosenberg isomorphism for smooth, global quotient orbifolds.

Hubert Flenner (University of Bochum)
Hochschild cohomology of singular spaces
This is a report on joint work with R.-O. Buchweitz.

In analogy with the cotangent complex we introduce the so called (derived) Hochschild complex of a morphism of analytic spaces or schemes; the Hochschild cohomology and homology groups are then the Ext and Tor groups of that complex. We prove that these objects are well defined, extend the known cases, and have the expected functorial and homological properties such as graded commutativity of Hochschild cohomology and existence of the characteristic homomorphism from Hochschild cohomology to the (graded) centre of the derived category.

We further generalize the HKR-decomposition theorem to Hochschild (co-)homology of arbitrary morphisms between complex spaces or schemes over a field of characteristic zero. To be precise, we show that for each such morphism $X \to Y$, the Hochschild complex decomposes naturally in the derived category $D(X)$ into $\bigoplus_{p\geq 0} \mathbb{S}^p(\mathbb{L}_{X/Y}[1])$, the direct sum of the derived symmetric powers of the shifted cotangent complex, a result due to Quillen in the affine case. The proof shows that the decomposition is given explicitly and naturally by the *universal Atiyah–Chern character*, the exponential of the universal Atiyah class. We also give further applications, in particular to the semiregularity map.

Joseph Lipman (Purdue University)
Hochschild homology, the fundamental class of a scheme-morphism, and residues
Let S be a noetherian scheme, $g : X \to Y$ a flat finite-type separated S-morphism, $\delta : X \to X \times_S X$ and $\gamma : Y \to Y \times_S Y$ the diagonal maps. We define a natural functorial map $\mathbf{L}\delta^*\delta_*\mathbf{g}^* \to \mathbf{g}^!\mathbf{L}\gamma^*\gamma_*$, the "relative fundamental class of g" (where $g^!$ is the twisted inverse image functor from Grothendieck duality). This can be interpreted as an orientation in a bivariant theory involving Hochschild complexes. It globalizes interesting commutative-algebra maps, like residues (which can also be described via Hochschild homology), and traces of differential forms. The talk will be about this theory, which, though at least 25 years old, still needs to be properly exposed.

Amnon Yekutieli (Ben Gurion University)
Twisted Deformation Quantization of Algebraic Varieties
This is joint work with F. Leitner (BGU). Let X be a smooth algebraic variety over a field of characteristic 0, endowed with a Poisson bracket. A quantization of this Poisson bracket is a formal associative deformation of the structure sheaf O_X, which realizes the Poisson bracket as its first order commutator. More generally one can consider Poisson deformations of O_X and their quantizations.

I will explain what these deformations are. Then I'll state a theorem which says that under certain cohomological conditions on X, there is a canonical quantization map (up to gauge equivalence). This is an algebro-geometric analogue of the celebrated result of Kontsevich (which talks about differentiable manifolds).

It appears that in general, without these cohomological conditions, the quantization will not be a sheaf of algebras, but rather a stack of algebroids, otherwise called a twisted associative deformation of O_X.

In the second half of the talk I'll talk about twisted deformations and twisted quantization, finishing with a conjecture.

3.4 Generalized Hochschild cohomology and category theory

Wendy Lowen introduced this topic, and other talks were given by Amnon Neeman, Emily Burgunder, and Claude Cibils.

Wendy Lowen (Vrije Universiteit Brussel/Université Denis Diderot Paris 7)
Hochschild cohomology of abelian categories

In contrast to the well known connection between Hochschild cohomology and deformation theory of associative algebras, Hochschild cohomology of schemes is harder to interpret in terms of deformations. We explain how it describes the deformation theory of suitable abelian categories over the sheme, like the categories of (quasi-coherent) sheaves, and we generalize both Hochschild cohomology and deformation theory to arbitrary abelian categories. There is a characteristic morphism from the Hochschild cohomology of an abelian category into the graded centre of its derived category. This morphism encodes the obstructions to deforming single objects of the (abelian or derived) category, and describes which part of the (enhanced) derived category is deformable as a differential graded category.

Amnon Neeman (Australian National University)
Brown representability via Rosicky
For some years we have known that well generated triangulated categories satisfy Brown representability; there are three proofs in the literature. The corresponding statement for the dual has stumped us all; we had no idea how to proceed. Then came a remarkable result of Rosicky's.

Emily Burgunder (Université de Montpellier II)
Leibniz homology and Kontsevich's graph complexes
The homology of the Lie algebra of matrices $gl(A)$ over an associative algebra A can be computed thanks to the cyclic homology of A:

$$H(gl(A)) = S(HC(A))$$

This theorem is known as the Loday-Quillen-Tsygan theorem. Another well-known theorem in homology is due to Kontsevich which says that the homology of the symplectic Lie algebra $K[p_1, ...p_n, q_1, ...q_n]$ can be explicited thanks to the homology of a certain "graph complex" G:

$$H(K[p_1, ..., p_n, q_1, ...q_n]) = S(H(G))$$

These two theorems are, in fact, examples of a more general theorem in the operadic setting, that we will present. If we replace the Lie homology by the Leibniz homology, then the cyclic homology has to be replaced by the Hochschild homology (Cuvier-Loday theorem). We show that, in Kontsevich case, there exists a "nonsymmetric graph complex" which computes the Leibniz homology of the symplectic Lie algebra.

Claude Cibils (Université de Montpellier II)
The Intrinsic Fundamental Group of a Linear Category
Joint work with Maria Julia Redondo and Andrea Solotar.

The main purpose is to provide a positive answer to the question of the existence of an intrinsic and canonical fundamental group associated to a linear category. The fundamental group we introduce takes into account the linear structure of the category, it differs from the fundamental group of the underlying category obtained as the classifying space of its nerve (see for instance G. Segal [1968] or D. Quillen [1973]).

We provide an intrinsic definition of the fundamental group as the automorphism group of the fibre functor on Galois coverings. We prove that this group is isomorphic to the inverse limit of the Galois groups associated to Galois coverings. Moreover, the graduation deduced from a Galois covering enables us to describe in a conceptual way the canonical monomorphism from its automorphism group to the first Hochschild-Mitchell cohomology.

The fundamental group that we define is intrinsic in the sense that it does not depend on the presentation of the linear category by generators and relations. In case a universal covering exists, we obtain that the fundamental groups constructed by R. Martínez-Villa and J. A. de la Peña depending on a presentation of the category by a quiver and relations are in fact quotients of the intrinsic fundamental group that we introduce.

If a universal covering exists, the fundamental group that we define is isomorphic to its automorphism group. Otherwise we show that the fundamental group is isomorphic to the inverse limit of the automorphisms groups of the Galois coverings of B. In case each connected component of the category of the Galois coverings admits an initial object – in other words if "locally" universal coverings exist – the intrinsic group that we define is isomorphic to the direct product of the corresponding automorphism groups.

The methods we use are inspired from the topological case as presented for instance in R. Douady and A. Douady's book. They are closely related to the way the fundamental group is considered in algebraic geometry after A. Grothendieck and C. Chevalley.

This work is very much indebted to the pioneer work of P. Le Meur [2006].

3.5 Relations with representation theory

Travis Schedler introduced this topic, and other talks were given by Andrea Solotar, Silvia Montarani, and Ching-Hwa Eu.

Travis Schedler (University of Chicago)
Calabi-Yau Frobenius algebras, stable Hochschild cohomology, and preprojective algebras
We will define and study the notion of Calabi-Yau Frobenius algebras over arbitrary base commutative rings k (especially the integers). This includes preprojective algebras of ADE quivers and finite group algebras. Such algebras have Calabi-Yau stable module categories and have a duality between "stable" Hochschild cohomology and homology—concepts which we define for any Frobenius algebra and interpret using the (unbounded) derived category of k-modules. We show that the stable Hochschild cohomology of "periodic" CY Frobenius algebras has a BV Frobenius algebra structure, which is closely related to the BV string topology algebra for compact spherical manifolds. We show that this includes the preprojective algebras above and group algebras for finite groups that act freely on a sphere. As a consequence, we also give a new explanation why any symmetric algebra has a BV algebra structure on ordinary Hochschild cohomology. If time permits, we will also compare the Dynkin preprojective algebra results with the non-Dynkin case (which is usual Calabi-Yau, and infinite-dimensional).

Andrea Solotar (Universidad de Buenos Aires)
Representations of Yang-Mills algebras
Joint work with Estanislao Herscovich.

Given $n \in \mathbb{N}$ and a field k, let $\mathfrak{f}(n)$ be the k-free Lie algebra with n generators $x_1, ..., x_n$. Consider the k-Lie algebra

$$\mathfrak{ym}(n) := \mathfrak{f}(n)/\langle \sum_{i=1}^{n} [x_i, [x_i, x_j]] : 1 \leq j \leq n \rangle,$$

which has been called the Yang-Mills algebra with n generators. It is a $\mathbb{N}$-graded Lie algebra, locally finite dimensional.

We denote $\mathrm{YM}(n)$ its enveloping algebra $\mathcal{U}(\mathfrak{ym}(n))$. Also,

$$\mathrm{YM}(n) \simeq TV(n)/\langle \sum_{i=1}^{n} [x_i, [x_i, x_j]] : 1 \leq j \leq n \rangle,$$

where $V(n)$ is the k-vector space with basis $\{x_1, ..., x_n\}$. As a consequence $\mathrm{YM}(n)$ is an homogeneous cubic algebra. As it has been noticed by Connes and Dubois-Violette, $\mathrm{YM}(n)$ is 3-Koszul. They also have computed some of the Hochschild cohomology k-vector spaces of this algebra.

However, $\mathrm{YM}(n)$ is not easy to handle: while for $n = 2$, $\mathfrak{ym}(2)$ is isomorphic to the Heisenberg algebra with generators x, y, z subject to relations $[x, y] = z, [x, z] = [y, z] = 0$, and so $\mathrm{YM}(n)$ is isomorphic to the down-up algebra $A(2, -1, 0)$, for $n > 2$ the associative algebra $\mathrm{YM}(n)$ is non-noetherian. So, the representation theory of $\mathrm{YM}(n)$ for $n > 2$ is highly non-trivial.

Our main result is then, given $n > 2$, to find families of representations of $\mathrm{YM}(n)$ big enough to separate points of the algebra. We manage to do so by showing that some well-known algebras, such as all the Weyl algebras and enveloping algebras of nilpotent finite-dimensional Lie algebras are quotients of $\mathrm{YM}(n)$. A mix of both methods allows us to describe several families of representations of $\mathrm{YM}(n)$, including infinite dimensional ones—via the Weyl algebras—and also finite dimensional representations of $\mathrm{YM}(n)$. The tools we use include A_∞-algebras and results of Bavula and Bekkert on representations of generalized Weyl algebras.

The interest on the representations of this family of algebras is mainly motivated by its physical applications, related to classical field theory and also to the study of D-branes.

Silvia Montarani (Massachusetts Institute of Technology)
Finite dimensional representations of symplectic reflection algebras associated with wreath products
Symplectic reflection algebras were introduced by Etingof and Ginzburg. They arise from the action of a finite group G of automorphisms on a symplectic vector space V, and are a multi-parameter deformation of the algebra $S(V) \rtimes G$, smash product of G with the symmetric algebra of V. A series of interesting examples

is provided by the wreath product symplectic reflection algebras, when G is the semidirect product of the symmetric group S_n of rank n with G'^n, where G' is a finite subgroup of $SL(2, C)$.

In this talk we will explain how to produce finite dimensional representations of these algebras. Our method is deformation theoretic and uses some properties of the Hochschild cohomology for this kind of algebras.

Time permitting, we will illustrate how Wee Liang Gan was able to recover the same representations by defining "reflection functors" between the categories of modules over wreath product symplectic reflection algebras corresponding to different values of the deformation parameters.

Ching-Hwa Eu (Massachusetts Institute of Technology)
Hochschild cohomology of preprojective algebras of ADE quivers
Preprojective algebras of ADE quivers are Calabi-Yau and Frobenius. We use these properties to compute the structure of the Hochschild cohomology and its product.

4 Workshop Participants

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