MULTI-AGENT OPTIMIZATION

Roger J-B Wets

Mathematics, University of California, Davis

Banff IRS - May 2007

Collaborators & Contributors

⋆ Alejandro Jofré, Universidad de Chile
⋆ R.T. Rockafellar, University of Washington
⋆ Michael Ferris, University of Wisconsin
⋆ Adib Bagh & Sergio Lucero, University California, Davis

- Hedy Attouch, Université de Montpellier
- Ruben López, Universidad Catolica de Conception

- Conversations: William Zame, Martine Quinzii,
  Jacques Drżee, Kenneth Arrow, Yves Balasko,
  Okie Nomia, Monique Florenzano, Jean-Pierre Aubin

- Indirectly: S. Robinson, J.-S. Pang, D. Ralph, C. Kanzow,
  T. Munson, S. Dirkse

Multi-Agent Optimization

- 0. Introduction
  - Roger Guesnerie & Adam lectures
  - flow control: transportation, communication (hot topic)
  - energy pricing: oligopoly, price setting
  - financial markets: introduction of new instruments
  - economic modeling
- 1. Variational Analysis Tools
- 2. Deterministic Problems
  - foundations & computational schemes
- 3. Stochastic Problems (Walras)
  - foundations & computational schemes

Multi-Agent Optimization

- 0. Introduction
  - Roger Guesnerie & Adam lectures
  - flow control: transportation, communication (hot topic)
  - energy pricing: oligopoly, price setting
  - financial markets: introduction of new instruments
  - economic modeling
- 1. Variational Analysis Tools
- 2. Deterministic Problems
  - foundations & computational schemes
- 3. Stochastic Problems (Walras)
  - foundations & computational schemes

I. Variational Analysis Tools

$$\mathcal{N}_\infty^\# = \left\{ N \subset \mathbb{N} \,\middle|\, \forall\, N' \in \mathcal{N}_\infty,\ N \cap N' \neq \emptyset \right\}$$
$$\mathcal{N}_\infty = \left\{ N \subset \mathbb{N} \,\middle|\, \forall\, N' \in \mathcal{N}_\infty^\#,\ N \cap N' \neq \emptyset \right\}$$

I. Variational Analysis Tools

$$\mathcal{N}_\infty^\# = \left\{ N \subset I\!\!N \,\middle|\, \forall\, N' \in \mathcal{N}_\infty,\; N \cap N' \neq \emptyset \right\}$$
$$\mathcal{N}_\infty = \left\{ N \subset I\!\!N \,\middle|\, \forall\, N' \in \mathcal{N}_\infty^\#,\; N \cap N' \neq \emptyset \right\}$$

Outline

Outline

Outline

Sets Limits

Definition

Given $\{C^\nu \subset \mathbb{R}^n\}_{\nu \in \mathbb{N}}$, the *outer limit* is the set

$$\operatorname*{Limsup}_{\nu\to\infty} C^\nu = \left\{ x \,\middle|\, \exists\, N \in \mathcal{N}_\infty^\#,\ \exists\, x^\nu \in C^\nu\ (\nu \in N) \text{ with } x^\nu \underset{N}{\to} x \right\}$$

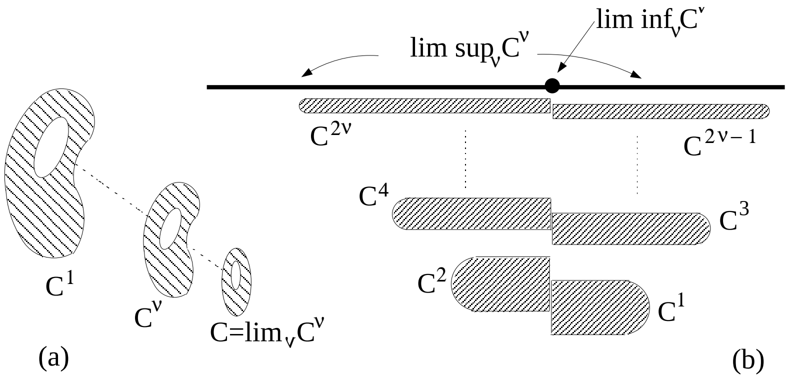
while the *inner limit* is the set

$$\operatorname*{Liminf}_{\nu\to\infty} C^\nu = \left\{ x \,\middle|\, \exists\, N \in \mathcal{N}_\infty,\ \exists\, x^\nu \in C^\nu\ (\nu \in N) \text{ with } x^\nu \underset{N}{\to} x \right\}$$

The *limit* of the sequence exists if the outer and inner limit sets are equal:

$$\operatorname*{Lim}_{\nu\to\infty} C^\nu = \operatorname*{Limsup}_{\nu\to\infty} C^\nu = \operatorname*{Liminf}_{\nu\to\infty} C^\nu,$$

then $C^\nu \to C$; Painlevé-Kuratowski *convergence*.


lim inf_ν C^ν
lim sup_ν C^ν
C^{2ν}
C^{2ν−1}
C^4
C^3
C^2
C^1
C^1
C^ν
C=lim_ν C^ν
(a)
(b)

Properties of Set Limits

- $\mathbb{B}(x^\nu, \rho^\nu) \to \mathbb{B}(x, \rho)$ when $x^\nu \to x$ and $\rho^\nu \to \rho$. When $\rho^\nu \to \infty$, $\mathbb{B}(x^\nu, \rho^\nu) \to \mathbb{R}^n$, their complements $\to \emptyset$.
- For set $D \subset \mathbb{R}^n$ with $\operatorname{cl} D = \mathbb{R}^n$ but $D \neq \mathbb{R}^n$ (e.g., $D =$ the rational vectors), $D \equiv C^\nu \to \mathbb{R}^n$, not to D.
- $\operatorname{Liminf}_\nu C^\nu$ and $\operatorname{Limsup}_\nu C^\nu$ (and $\operatorname{Lim}_\nu C^\nu$) are closed;
- $C^\nu \nearrow \quad \Longrightarrow \quad \operatorname{Lim}_\nu C^\nu = \operatorname{cl} \bigcup_\nu C^\nu$,
  $C^\nu \searrow \quad \Longrightarrow \quad \operatorname{Lim}_\nu C^\nu = \bigcap_\nu \operatorname{cl} C^\nu$
- $\emptyset \neq C^\nu$ and C closed, $\operatorname{Limsup}_\nu d_{C^\nu}(0) < \infty$,
  $C^\nu \to C \iff \forall\, x, \operatorname{Limsup}_\nu \operatorname{prj}_{C^\nu}(x) \subset \operatorname{prj}_C(x)$.
  also convex: $C^\nu \to C \iff \operatorname{prj}_{C^\nu}(x) \to \operatorname{prj}_C(x)\ \forall x$
- C^ν convex $\quad \Longrightarrow \quad \operatorname{Liminf}_\nu C^\nu$ (and $\operatorname{Lim} C^\nu$) convex,
  if $C = \operatorname{Liminf} C^\nu$, for any compact set $B \subset \operatorname{int} C$,
  then $B \subset \operatorname{int} C^\nu$ for ν large enough.

Convergence of solutions of convex systems

Theorem

$$C^\nu = \{x \in X^\nu \mid L^\nu(x) \in D^\nu\}, \qquad C = \{x \in X \mid L(x) \in D\},$$

$L^\nu, L : \mathbb{R}^n \to \mathbb{R}^m$ linear; $X^\nu, X \subset \mathbb{R}^n$ and $D^\nu, D \subset \mathbb{R}^m$ convex; and that $L(X)$ cannot be separated from D. If $L^\nu \to L$, $\mathrm{Liminf}_\nu X^\nu \supset X$ and $\mathrm{Liminf}_\nu D^\nu \supset D$, then $\mathrm{Liminf}_\nu C^\nu \supset C$.

$$L^\nu \to L,\ \ X^\nu \to X,\ \ D^\nu \to D \quad \Longrightarrow \quad C^\nu \to C.$$

$\star$ *for linear mappings $L^\nu \to L$ and convex sets $D^\nu \to D$, if D and* rge *L cannot be separated, then $(L^\nu)^{-1}(D^\nu) \to L^{-1}(D)$.*

$\star$ *$A^\nu \to A$, $b^\nu \to b$, A full rank, $\{x \mid A^\nu x = b^\nu\} \to \{x \mid Ax = b\}$.*

$\star$ $\mathrm{Liminf}_\nu(C_1^\nu \cap C_2^\nu) \supset \mathrm{Liminf}_\nu C_1^\nu \cap \mathrm{Liminf}_\nu C_2^\nu$ *holds when C_1^ν, C_2^ν are convex and $\mathrm{Liminf}_\nu C_1^\nu$, $\mathrm{Liminf}_\nu C_2^\nu$ cannot be separated. Indeed,*

$$C_1^\nu \to C_1,\ \ C_2^\nu \to C_2 \quad \Longrightarrow \quad C_1^\nu \cap C_2^\nu \to C_1 \cap C_2$$

as long as C_1 and C_2 cannot be separated.

The graph of a set-valued mapping

- $x \to \mathrm{sets}(U)$ collection of all subsets of U, or
- $\mathrm{gph}\, S = \{(x,u) \mid u \in S(x)\} \subset X \times U$

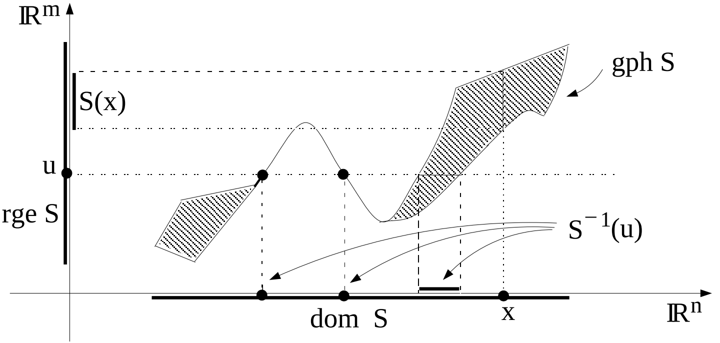
$$S : X \rightrightarrows U, \quad S(x) = \{u \mid (x,u) \in G\}, \quad G = \mathrm{gph}\, S.$$

$$\mathrm{dom}\, S = \{x \mid S(x) \neq \emptyset\}, \qquad \mathrm{rge}\, S = \{u \mid \exists\, x \text{ with } u \in S(x)\}$$

$$S(C) := \bigcup\nolimits_{x \in C} S(x) = \{u \mid S^{-1}(u) \cap C \neq \emptyset\},$$

while the *inverse image* of a set D is

$$S^{-1}(D) := \bigcup\nolimits_{u \in D} S^{-1}(u) = \{x \mid S(x) \cap D \neq \emptyset\}.$$


$\mathbb{R}^m$
S(x)
u
rge S
gph S
$S^{-1}(u)$
dom S
x
$\mathbb{R}^n$

Examples

- $F : X \to \mathbb{R}^m$, F^{-1} possibly set-valued, $\operatorname{rge} F^{-1} = X$.
- $F : X \to \mathbb{R}^m$, $F(x) = (f_1(x), \dots, f_m(x))$,
  for $u = (u_1, \dots, u_m) \in \mathbb{R}^m$,

$$F^{-1}(u) = \{x \in X \mid f_i(x_1, \dots, x_n) = u_i,\ i = 1, \dots, m$$

- Generalized equations and implicit mappings.

$$\text{find } \bar{x} \text{ such that } S(\bar{x}) \ni \bar{u}$$

  $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Emphasis on the behavior of $S^{-1}(u)$ near $\bar{u}$.
- Algorithmic mappings and fixed points. $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$:
  $\bar{x} \in S(\bar{x})$ is a *fixed point*
  Finding $\bar{x}$: from x^0 use the rule $x^\nu \in S(x^{\nu-1})$, i.e.,

$$x^1 \in S(x^0),\ x^2 \in (S \circ S)(x^0),\ \dots,\ x^\nu \in (S \circ \cdots \circ S)(x^0).$$

Semicontinuity

Definition

A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* (osc) at $\bar{x}$ if

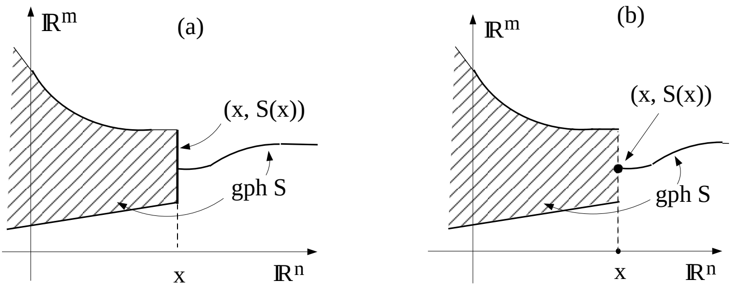
$$\operatorname*{Limsup}_{x \to \bar{x}} S(x) \subset S(\bar{x}),$$

or equivalently $\operatorname{Limsup}_{x \to \bar{x}} S(x) = S(\bar{x})$.
It's *inner semicontinuous* (isc) at $\bar{x}$ if

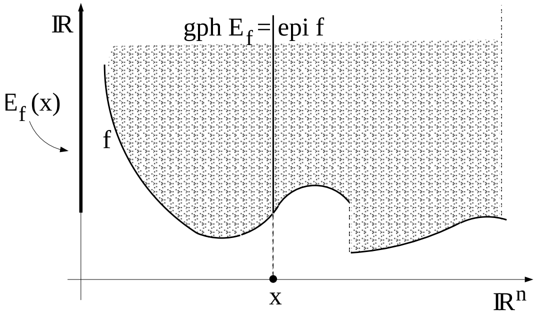
$$\operatorname*{Liminf}_{x \to \bar{x}} S(x) \supset S(\bar{x}),$$

equivalently, $\operatorname{Liminf}_{x \to \bar{x}} S(x) = S(\bar{x})$ when S is closed-valued.
It's *continuous* at $\bar{x}$ if it's osc and isc, i.e.,
if $S(x) \to S(\bar{x})$ as $x \to \bar{x}$.

Outer- and Inner-semicontinuity



Profile Mappings



Profile Mappings

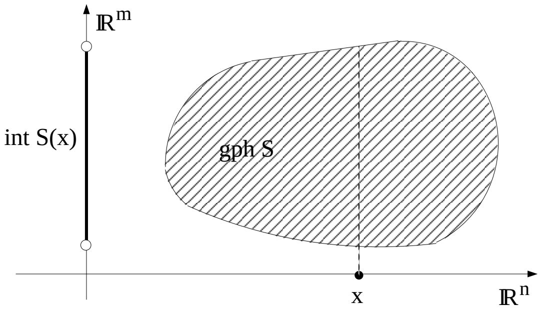
For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the *epigraphical profile* mapping $E_f : \mathbb{R}^n \rightrightarrows \mathbb{R}^1$

$$E_f(x) = \{\alpha \in \mathbb{R} \,|\, \alpha \geq f(x)\},$$

has $\operatorname{gph} E_f = \operatorname{epi} f$, $\operatorname{dom} E_f = \operatorname{dom} f$, and $E_f^{-1}(\alpha) = \operatorname{lev}_{\leq\alpha} f$.

- E_f is osc at $\bar{x} \iff f$ is lsc at $\bar{x}$
- it's isc at $\bar{x} \iff f$ is usc at $\bar{x}$.
- E_f continuous at $\bar{x} \iff f$ continuous at $\bar{x}$.
- $\alpha \mapsto \operatorname{lev}_{\leq\alpha} f$ osc $\iff f$ lsc
- the *hypographical profile* mapping $H_f : \mathbb{R}^n \rightrightarrows \mathbb{R}^1$ with $H_f(x) = \{\alpha \in \mathbb{R} \,|\, \alpha \leq f(x)\}$: analogous properties

Graph Convexity



Inner semicontinuity from convexity

Theorem

Consider a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and a point* $\bar{x} \in \mathbb{R}^n$.

(a) *If S is convex-valued and* $\operatorname{int} S(\bar{x}) \neq \emptyset$, *then a necessary and sufficient condition for S to be isc relative to* $\operatorname{dom} S$ *at* $\bar{x}$ *is that for all* $u \in \operatorname{int} S(\bar{x})$ *there exists* $W \in \mathcal{N}(\bar{x}, u)$ *such that* $W \cap (\operatorname{dom} S \times \mathbb{R}^m) \subset \operatorname{gph} S$*; in particular, S is isc at* $\bar{x}$ *if and only if* $(\bar{x}, u) \in \operatorname{int}(\operatorname{gph} S)$ *for every* $u \in \operatorname{int} S(\bar{x})$.

(b) *If S is graph-convex and* $\bar{x} \in \operatorname{int}(\operatorname{dom} S)$, *then S is isc at* $\bar{x}$.

(c) *If S is isc at* $\bar{x}$, *so is* $x \mapsto \operatorname{con} S(x)$.

Moreover: Let $T(w) = \{x \mid f_i(x, w) \leq 0, \ i = 1, \dots, m\}$
with f_i finite, continuous, $f_i(\cdot, w)$ convex in x.
If for $\bar{w}, \exists\, \bar{x}$ such that $f_i(\bar{x}, \bar{w}) < 0$ for all i,
then T is continuous on a neighborhood of $\bar{w}$.

Inner semicontinuity from convexity

Theorem

Consider a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and a point* $\bar{x} \in \mathbb{R}^n$.

(a) *If* S *is convex-valued and* $\operatorname{int} S(\bar{x}) \neq \emptyset$, *then a necessary and sufficient condition for* S *to be isc relative to* $\operatorname{dom} S$ *at* $\bar{x}$ *is that for all* $u \in \operatorname{int} S(\bar{x})$ *there exists* $W \in \mathcal{N}(\bar{x}, u)$ *such that* $W \cap (\operatorname{dom} S \times \mathbb{R}^m) \subset \operatorname{gph} S$; *in particular,* S *is isc at* $\bar{x}$ *if and only if* $(\bar{x}, u) \in \operatorname{int}(\operatorname{gph} S)$ *for every* $u \in \operatorname{int} S(\bar{x})$.

(b) *If* S *is graph-convex and* $\bar{x} \in \operatorname{int}(\operatorname{dom} S)$, *then* S *is isc at* $\bar{x}$.

(c) *If* S *is isc at* $\bar{x}$, *so is* $x \mapsto \operatorname{con} S(x)$.

Moreover: Let $T(w) = \{x \mid f_i(x, w) \leq 0,\ i = 1, \ldots, m\}$
with f_i finite, continuous, $f_i(\cdot, w)$ convex in x.
If for $\bar{w}$, $\exists\, \bar{x}$ such that $f_i(\bar{x}, \bar{w}) < 0$ for all i,
then T is continuous on a neighborhood of $\bar{w}$.

Pointwise and graphical limits

- $(\text{p-Limsup}_\nu S^\nu)(x) = \text{Limsup}_\nu S^\nu(x)$
- $(\text{p-Liminf}_\nu S^\nu)(x) = \text{Liminf}_\nu S^\nu(x)$
- when equal, the *pointwise limit* $\text{p-Lim}_\nu S^\nu$ exists
- *graphical outer limit*, $\text{g-Limsup}_\nu S^\nu$:

$$\text{gph}(\text{g-}\operatorname*{Limsup}_\nu S^\nu) = \operatorname*{Limsup}_\nu(\text{gph}\, S^\nu)$$

- *graphical inner limit*, $\text{g-Liminf}_\nu S^\nu$:

$$\text{gph}(\text{g-}\operatorname*{Liminf}_\nu S^\nu) = \operatorname*{Liminf}_\nu(\text{gph}\, S^\nu)$$

- they agree, the *graphical limit* $\text{g-Lim}_\nu S^\nu$ exists
- All these mappings are osc
- $\text{p-Lim}_\nu S^\nu = \text{g-Lim}_\nu S^\nu$, requires *equi-outer semicontinuity*

Approximation of generalized equations

Theorem

Consider the generalized equation $S^\nu(x) \ni \bar{u}^\nu$ *as an approximation to the generalized equation* $S(x) \ni \bar{u}$*, with solution sets* $(S^\nu)^{-1}(\bar{u}^\nu)$ *and* $S^{-1}(\bar{u})$*; assume the mappings* $S,\ S^\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *are closed-valued.*

(a) *When* $\text{g-Limsup}_\nu\, S^\nu \subset S$*, one has for every choice of* $\bar{u}^\nu \to \bar{u}$ *that* $\text{Limsup}_\nu (S^\nu)^{-1}(\bar{u}^\nu) \subset S^{-1}(\bar{u})$*. Thus, any cluster point of a sequence of approximate solutions is a true solution.*

(b) *When* $\text{g-Liminf}_\nu\, S^\nu \supset S$*,* $S^{-1}(\bar{u}) \subset \bigcap_{\varepsilon>0} \text{Liminf}_\nu\,(S^\nu)^{-1}\big(\mathbb{B}(\bar{u},\varepsilon)\big)$*. So, every true solution is the limit of approximate solutions for some* $\bar{u}^\nu \to \bar{u}$*.*

(c) *When* $S^\nu \xrightarrow{g} S$*, both conclusions hold.*

Framework

'Classical': $\mathrm{fcn}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \overline{\mathbb{R}}\}$
'New': $fv\text{-}\mathrm{fcn}(\mathbb{R}^n) = \{f : D \to \mathbb{R} \mid$ for some $\emptyset \neq D \subset \mathbb{R}^n\}$
Epigraph: $\mathrm{epi}\, f = \{(x, \alpha) \in D \times \mathbb{R} \mid \alpha \geq f(x)\} \subset \mathbb{R}^{n+1}$,
when $f \in \mathrm{fcn}(\mathbb{R}^n)$, $\mathrm{epi}\, f = \{(x, \alpha) \in \mathbb{R}^{n+1} \mid \alpha \geq f(x)\}$.

Definition

When $f \in fv\text{-}\mathrm{fcn}(\mathbb{R}^n)$ its lsc (lower semicontinuous) if $\liminf_\nu f(x^\nu) < \infty$, then for some subsequence

- if $x \in D$: $\liminf_\nu f(x^\nu) \geq f(x)$, and
- if $x \in \mathrm{cl}\, D \setminus D$: $f(x^\nu) \to \infty$.

Epi-limits

Definition

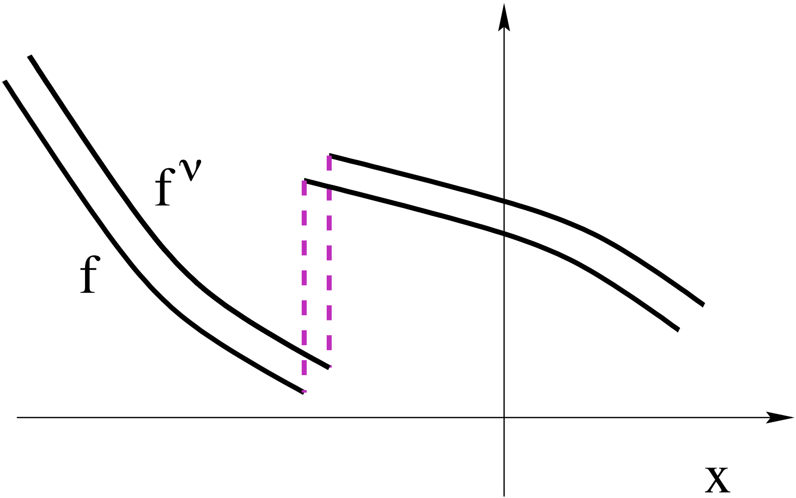
A sequence of functions $\{f^\nu, \nu \in \mathbb{N}\}$ *epi-converges* to f when $\operatorname{epi} f^\nu \to \operatorname{epi} f$ as subsets of $\mathbb{R}^{n+1}$; f belongs to $fv\text{-}\mathrm{fcn}(\mathbb{R}^n)$ or $\mathrm{fcn}(\mathbb{R}^n)$. One writes $f^\nu \xrightarrow{e} f$.

- $\operatorname{epi} f$ = outer limit of $\{\operatorname{epi} f^\nu\}$, then f is the *lower epi-limit*
- $\operatorname{epi} f$ = inner limit of $\{\operatorname{epi} f^\nu\}$, then f is the *upper epi-limit*

Theorem

Let $\{f^\nu\}_{\nu \in \mathbb{N}}$ be a sequence of functions with domains in $\mathbb{R}^n$. Then, the lower and upper epi-limits and the epi-limit, are all lsc.; the family of lsc functions is closed under epi-convergence. If the f^ν are convex, so is the upper epi-limit, and the epi-limit, if it exists.

Epi-topology



Analytic version

Example

Epi-limits are not necessarily in $fv\text{-fcn}(\mathbb{R}^n)$.

$$f^\nu(x) = \begin{cases} -\nu^2 x & \text{if } 0 \le x \le \nu^{-1}, \\ \nu^2 x - 2\nu & \text{if } \nu^{-1} \le x \le 2\nu^{-1}, \\ 0 & \text{for } x \ge 2\nu^{-1}, \end{cases}$$

converge to $f \equiv 0$ on $(0, \infty)$ and $f(0) = -\infty$

Theorem

Let $\{f : D \to \mathbb{R}, f^\nu : D^\nu \to \mathbb{R}\}$ *in* $fv\text{-fcn}(\mathbb{R}^n)$. *Then,* $f^\nu \xrightarrow{e} f \iff$

(a) $\forall\, x^\nu \in D^\nu \to x$ *in* D, $\liminf_\nu f^\nu(x^\nu) \ge f(x)$,

(a$^\infty$) *for all* $x^\nu \in D^\nu \to x \notin D$, $f^\nu(x^\nu) \nearrow \infty$,

(b) $\forall x \in D$, $\exists\, x^\nu \in D^\nu \to x$ *such that* $\limsup_\nu f^\nu(x^\nu) \le f(x)$.

Convergence of solutions

Theorem

Consider a sequence $\{f^\nu : D^\nu \to \mathbb{R},\ \nu \in \mathbb{N}\} \subset fv\text{-fcn}(\mathbb{R}^n)$ *epi-converging to* $f : D \to \mathbb{R}$*, also in* $fv\text{-fcn}(\mathbb{R}^n)$*. Then*

$$\limsup_{\nu\to\infty} (\inf f^\nu) \ \leq\ \inf f.$$

Moreover,

- *if* $x^k \in \mathrm{argmin}_{D^{\nu_k}} f^{\nu_k}$ *for* $\{\nu_k\}$ *and* $x^k \to \bar{x}$*, then* $\bar{x} \in \mathrm{argmin}_D f$ *and* $\min_{D^{\nu_k}} f^{\nu_k} \to \min_D f$*.*
- *If* $\mathrm{argmin}_D f$ *is a singleton, then every convergent subsequence of minimizers converges to* $\mathrm{argmin}_D f$*.*

Tight epi-convergence

Definition

$\{f^\nu : D^\nu \to I\!R\} \subset fv\text{-fcn}(I\!R^n)$ *epi-converges tightly* to $f : D \to I\!R$, when $f^\nu \xrightarrow{e} f$ and for all $\varepsilon > 0$, there exist a compact set B_ε and an index ν_ε such that

$$\forall\, \nu \geq \nu_\varepsilon : \quad \inf_{B_\varepsilon \cap D^\nu} f^\nu \leq \inf_{D^\nu} f^\nu + \varepsilon.$$

Theorem

$\{f^\nu : D^\nu \to I\!R\}_{\nu \in I\!N} \subset fv\text{-fcn}(I\!R^n)$ epi-converges to $f : D \to I\!R$ with $\inf_D f$ finite. Then, they epi-converge tightly

(a) $\iff \inf_{D^\nu} f^\nu \to \inf_D f$.

(b) $\iff \exists\, \varepsilon^\nu \searrow 0$: $\varepsilon^\nu\text{-argmin}\, f^\nu \to \text{argmin}\, f$.

Remark: no convergence of $\text{dom}\, f^\nu$ to $\text{dom}\, f$.

Reconciliation: $f \in \mathrm{fcn}(\mathbb{R}^n)$

Define

$$pr\text{-}\mathrm{fcn}(\mathbb{R}^n) := \left\{ f \in \mathrm{fcn}(\mathbb{R}^n) \,\middle|\, -\infty < f \not\equiv \infty \right\},$$

the proper functions in $\mathrm{fcn}(\mathbb{R}^n)$; f is *proper* if $f > -\infty$ and $f \not\equiv \infty$, i.e., finite on $\mathrm{dom}\, f$ (minimization context).

A *bijection* η between $fv\text{-}\mathrm{fcn}(\mathbb{R}^n)$ and $pr\text{-}\mathrm{fcn}(\mathbb{R}^n)$:

for $f : D \to \mathbb{R}$, set $\eta f = f$ on D and $\eta f \equiv \infty$ on $\mathbb{R}^n \setminus D$.

for $f \in pr\text{-}\mathrm{fcn}(\mathbb{R}^n)$, $\eta^{-1} f$ = restriction of f to $\mathrm{dom}\, f$.

Important: this bijection doesn't affect epigraphs. Thus, epi-conv. in $fv\text{-}\mathrm{fcn}(\mathbb{R}^n) \iff$ epi-conv. in $pr\text{-}\mathrm{fcn}(\mathbb{R}^n)$.

Hypo-convergence

Maximization setting: pass from f to $-f$.
Terminology: min to max (inf to sup), ∞ to $-\infty$, epi to hypo, $\leq$ to $\geq$ (and vice-versa), $\liminf$ to $\limsup$ (and vice-versa), and lsc to usc.

Definition

$f^\nu \xrightarrow{h} f$, when $-f^\nu \xrightarrow{e} -f$, or equivalently if $\mathrm{hypo}\, f^\nu \to \mathrm{hypo}\, f$.
Hypo-convergence tightly ... The *family of usc functions is closed under hypo-convergence*.