Mathematics and the Environment: Public Decision Making

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Introduction

Evaluation of public policies

Introduction

- The budget of the Nation is finite >< our needs are infinite.
- Optimise/rank the use of our means.
- Remember that citizens are, in fine, the beneficiaries and the payers.
- Analyse their preferences to guide the State towards public policies that maximize social welfare.

The Copenhagen Consensus (2004)

Challenge	Opportunity
Diseases	Control of HIV/AIDS
Malnutrition	Providing micro nutrients
Subsidies and Trade	Trade liberalisation
Diseases	Control of malaria
Malnutrition	Development of new agricultural technologies
Sanitation & Water	Small-scale water technology for livelihoods
Sanitation & Water	Community-managed water supply and sanitation
Sanitation & Water	Research on water productivity in food production
Government	Lowering the cost of starting a new business
Migration	Lowering barriers to migration for skilled workers
Malnutrition	Improving infant and child nutrition
Malnutrition	Reducing the prevalence of low birth weight
Diseases	Scaled-up basic health services
Migration	Guest worker programmes for the unskilled
Climate	Optimal carbon tax
Climate	The Kyoto Protocol
Climate	Value-at-risk carbon tax

A panel of economic experts, comprising eight of the world's most distinguished economists, was invited to consider these issues. The members were Jagdish Bhagwati of Columbia University, Robert Fogel of the University of Chicago (Nobel laureate), Bruno Frey of the University of Zurich, Justin Yifu Lin of Peking University, Douglass North of Washington University in St Louis (Nobel laureate), Thomas Schelling of the University of Maryland, Vernon Smith of George Mason University (Nobel laureate), and Nancy Stokey of the University of Chicago.

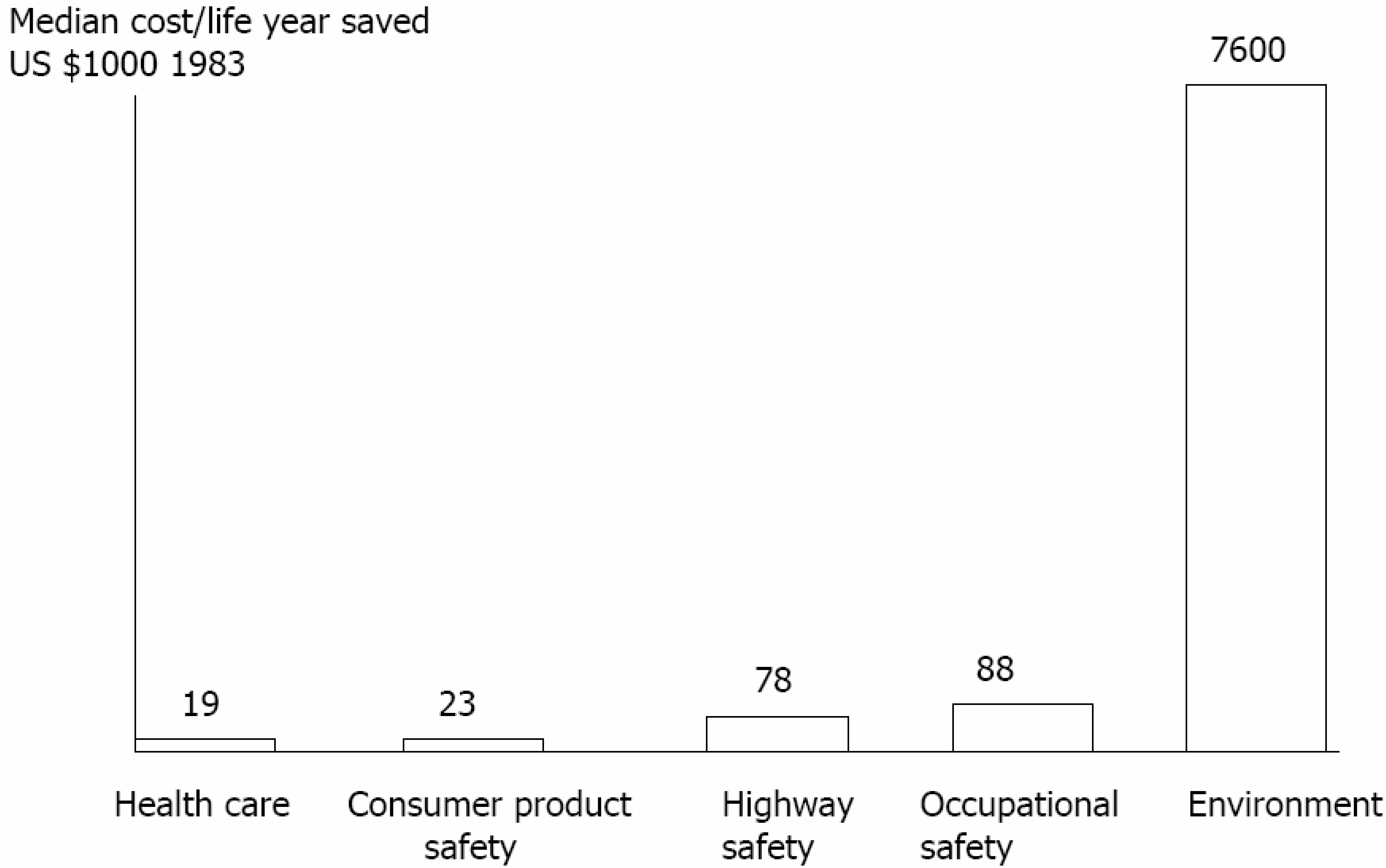
A simple tool: cost-benefit analysis

- Compare costs and benefits: this is what we all do in our day-to-day life!
- Two advantages of the method:
  - Provide a tool to help the decision maker;
  - Reveal the true Values that are at the heart of the public choice.
- Complex challenges:
  - Valuing non-monetary benefits;
  - Taking into account of uncertainty;
  - Taking into account of time.

The "cost of life"

Intervention	Cost/life-saved millions $1984
• Underground construction	0.3
• Crane suspended personnel platform	1.2
• Masonry construction	1.4
• Hazard communication	1.8
• Benzene exposure in rubber and tire industry	2.8
• Radionuclides/uranium mines	6.9
• Ethylene oxide	25.6
• Uranium mill tailings	53.0
• Abestos	104.2
• Arsenic glass plant	142.0
• Strenghen buildings in earthquake-prone areas	980.0
• Radiation emission standard for nuclear power plants	1,800.0

Source: Viscusi (1998)



Source: Tengs and Graham (1996) et Lomborg (2001)

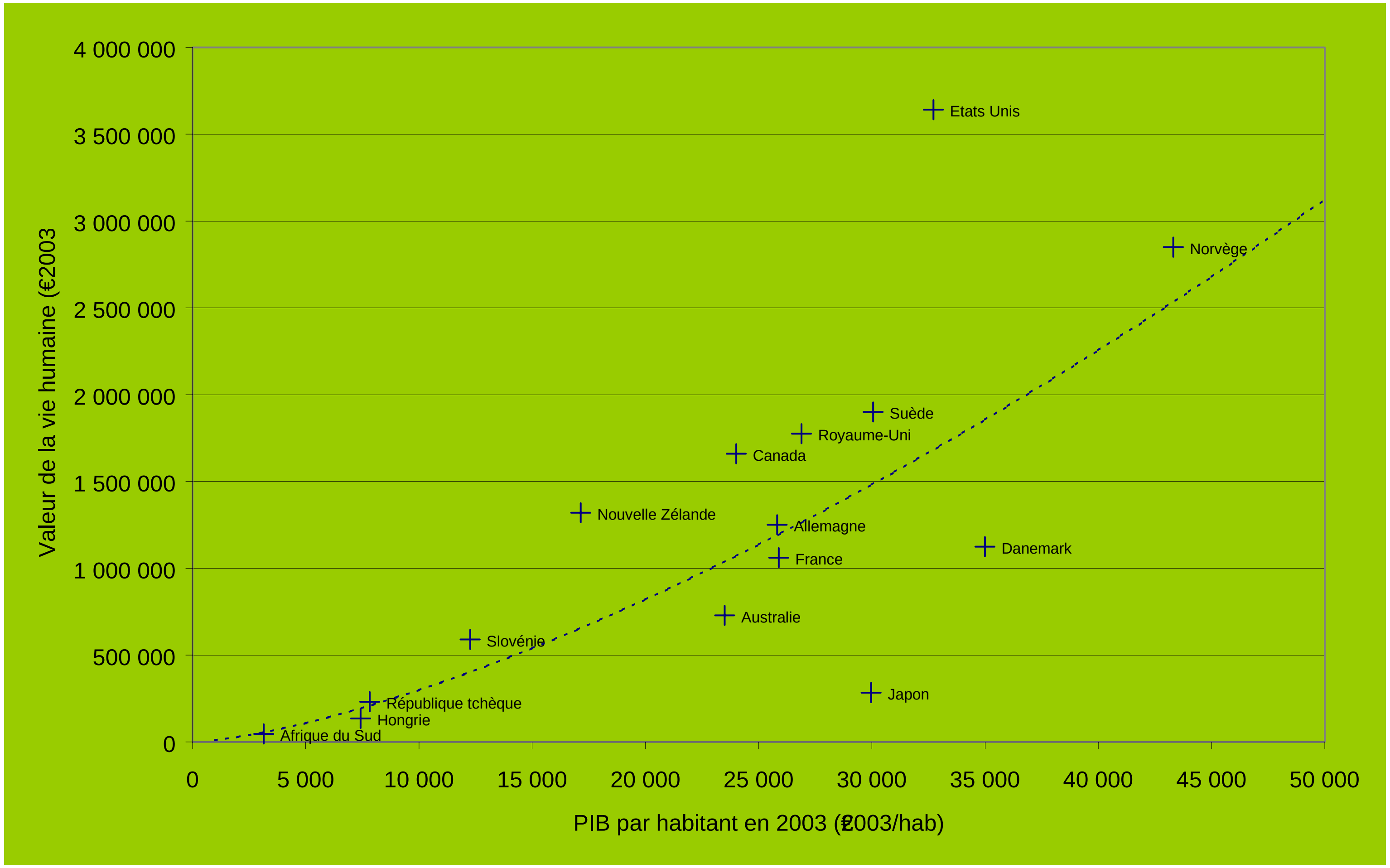
Estimate the value of your own life!

- Contingent valuation: You must play to the "russian roulette". There are 5 bullets in 10 000 chambers. How much are you ready to pay to remove one bullet?

  Value of life = 10 000 x p.

- Hedonic price:
  - Δ value of house in healthier places;
  - Δ level of salaries in safer jobs;
  - Δ price of safer cars…

The "value of life"



Risk estimation

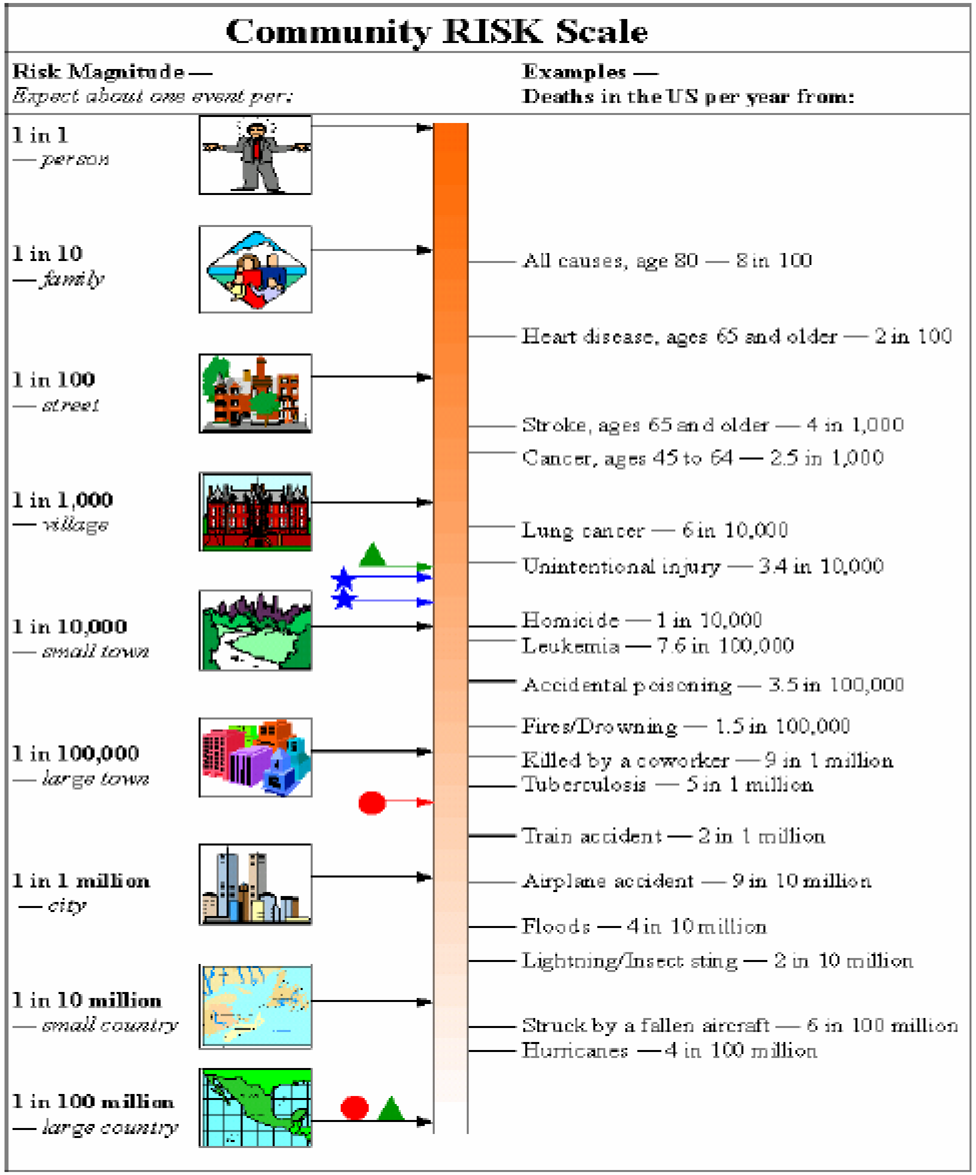
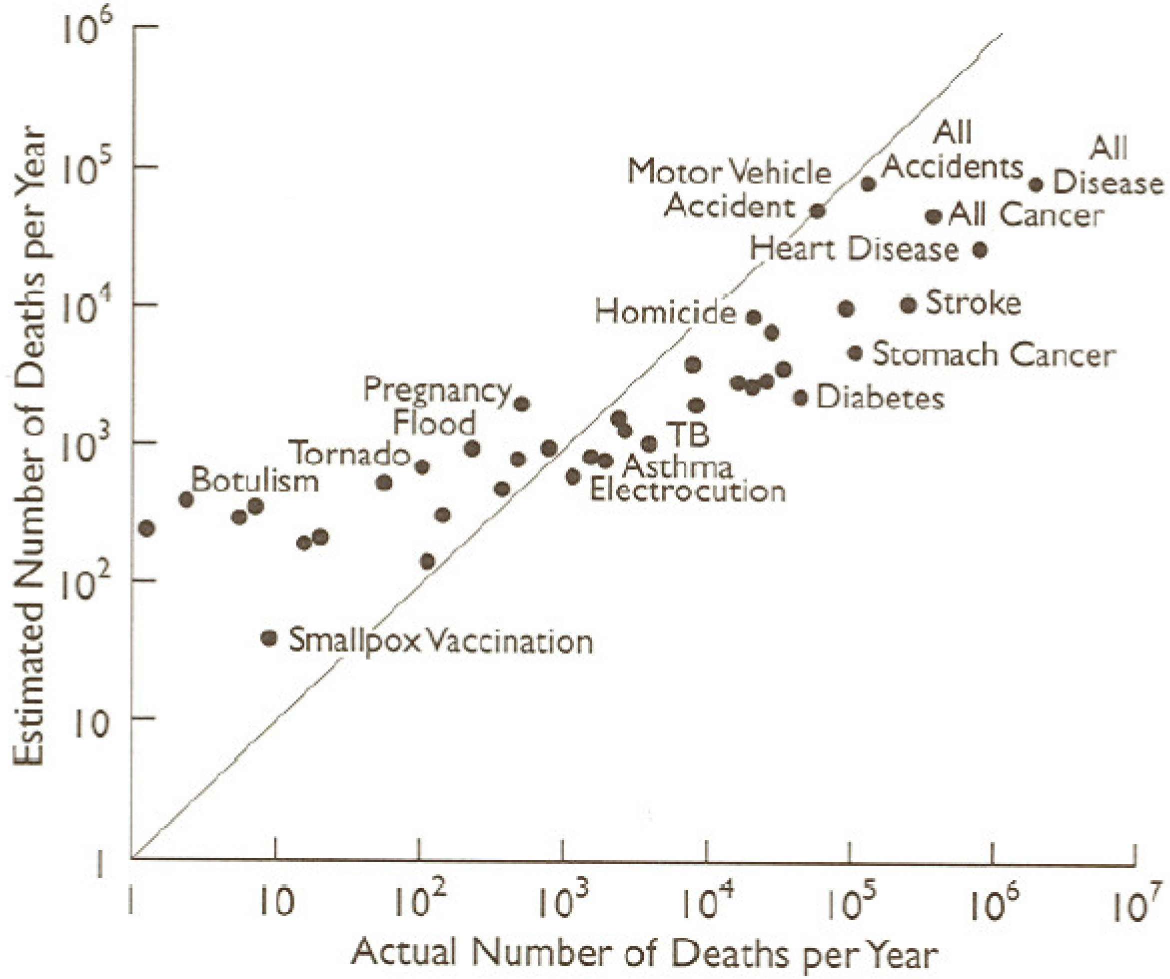


FIGURE 4-2

Relationship Between Judged Frequency and Actual Number of Fatalities per Year for 41 Causes of Death[a]



a. Respondents were informed of the actual number of motor vehicle accidents.

Source: Lichtenstein et al. (1978).

The Stern Report

- Global warming has an effect on intergenerational welfare that is equivalent to a permanent reduction of the world GDP by 10%.
- We should be willing to pay now and forever as much as 10% of GDP to eliminate the consequences of global warming.
- Copenhagen consensus?

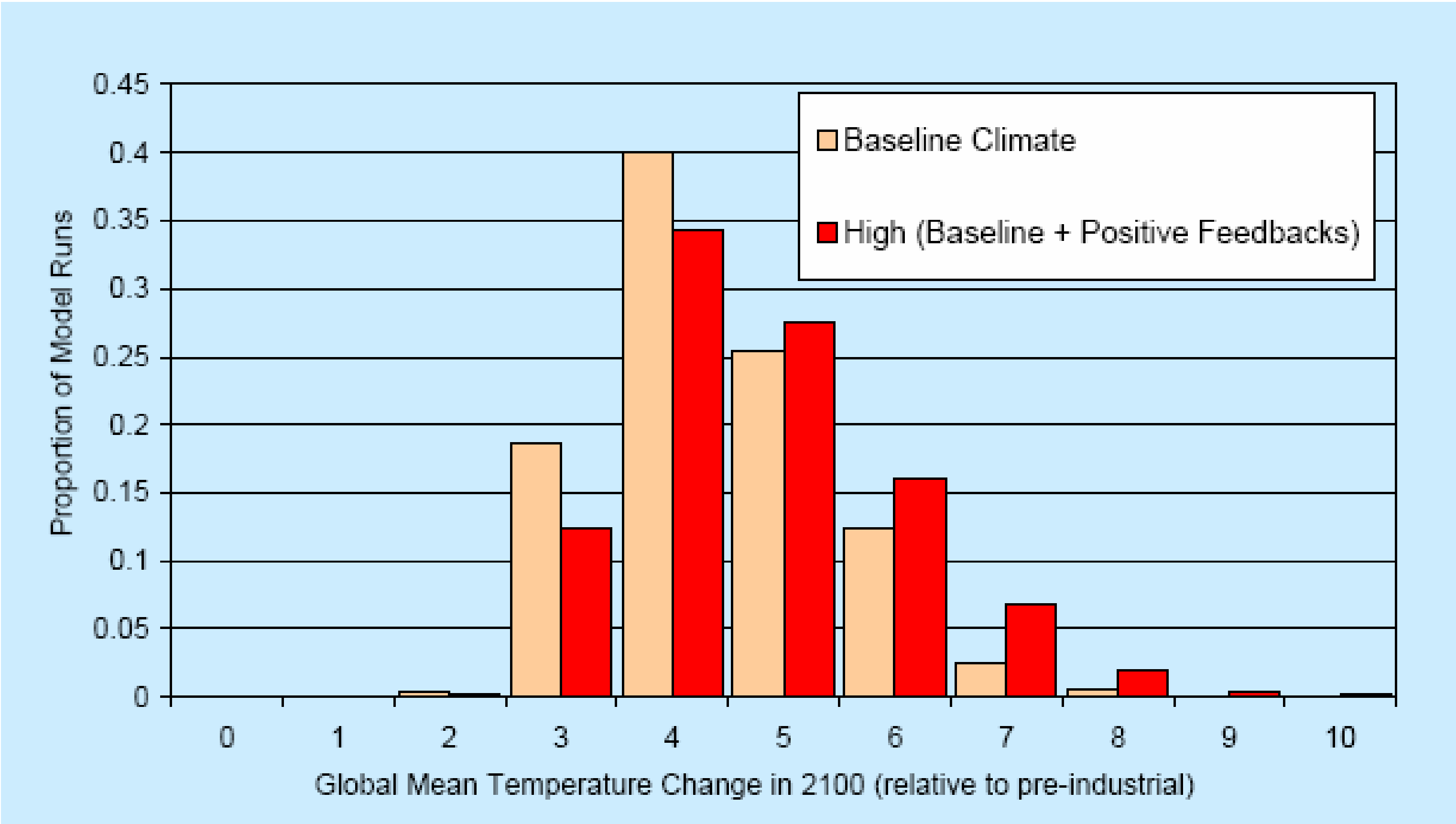
The costs in Stern

- Reduction of the average agricultural productivity due to reduced rainfall.
- Loss of human lives and real estate due to increased climatic extreme events.
- Increased energy consumption (air conditioning).
- Loss of environmental assets (biodiversity, lost species,...).
- Possibility of a bifurcation in climate above a certain threshold for ΔT.

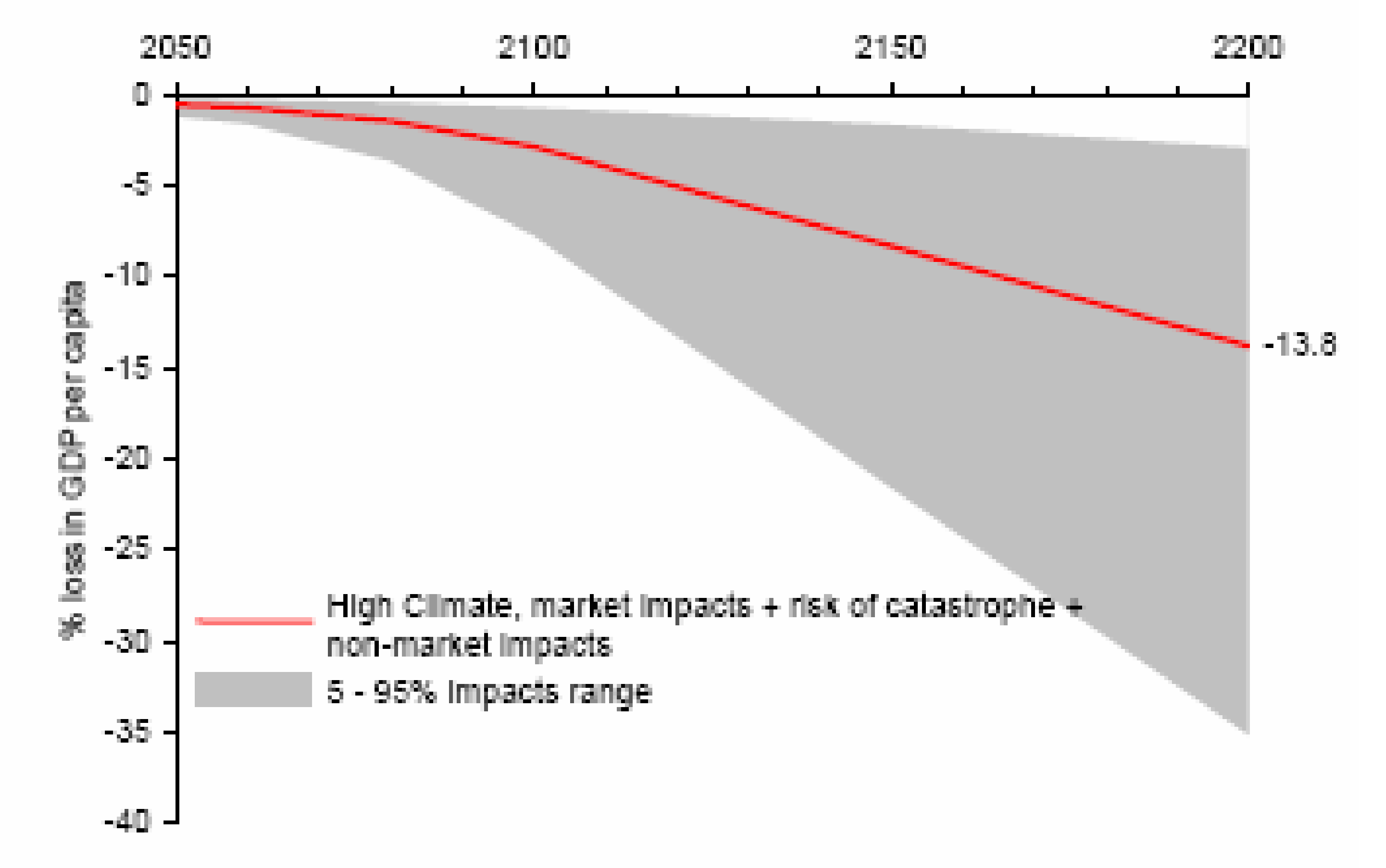
The Nordhaus-Stern methodology

- Coupling of a physico-climatic model and of an economic model, with feedbacks.
- A real effort to move from qualitative arguments to numbers.
- A clever and courageous use of the tools of CBA.

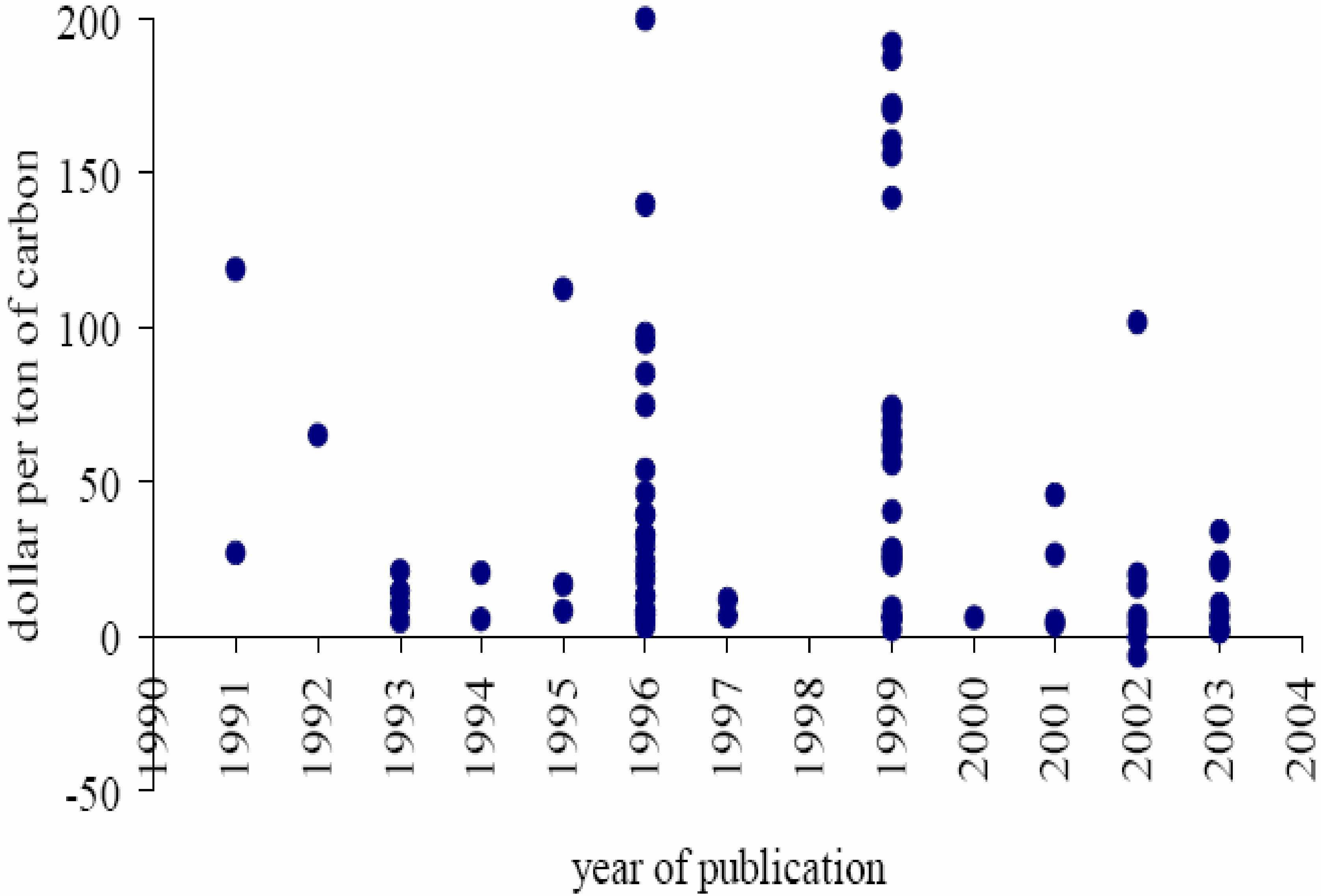
Uncertain ΔT



Uncertain dommages



Marginal social cost of greenhouse gas emissions from 28 studies



Source: Tol (2004). Note, one study from the early 1990s is excluded which has very high values (1800$/tC)

Decision Theory Under Uncertainty

A short overview

A brief history of some influential works on risk

- 1654 : Pascal, Fermat: ‘discovery’ of the theory of probabilities
- 1703: Jacob Bernouilli’s law of large numbers
- 1730: De Moine’s bell curve
- 1738: Dan Bernouilli’s decreasing marginal utility
- 1750: Bayes’ law
- 1875: Galton’s regression to the mean
- 1921: Keynes’ and Knight’s books on uncertainty
- 1944: von Neumann and Morgenstern’s seminal book
- 1952: Markowitz’s portfolio approach
- 1954: Savage’s foundations of statistics
- 1964: Pratt’s coefficient of risk-aversion
- 1979: Tversky and Kahneman’s prospect theory

Nobel-awarded economists for their contributions on risk-related issues:

- Allais (1988)
- Markowitz, Miller and Sharpe (1990)
- Merton and Sholes (1997)
- Kahneman and Smith (2002)

Contributors: Samuelson (1970), Arrow (1972), Friedman (1976), Simon (1978), Tobin (1981), Modigliani (1985), Stiglitz (2001), Schelling (2005)…

Introduction

- There will be a terrorist attack this year, whose intensity is unknown. The 90% interval of confidence for the loss of GDP is [3%,37%], with a mean equaling 13.8%. What % of GDP are you ready to pay to eliminate the threat?

- You have a lottery ticket that yields a payoff of 100 euros with probability ½. You get the possibility to exchange this ticket with a sure payoff CE. What is the minimum value of CE that makes you to accept the deal?

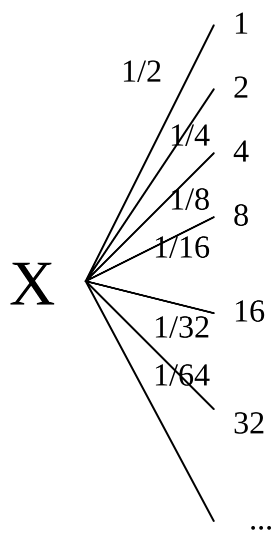
St Petersburg game

- Daniel Bernoulli (1700-1782) in St Petersburg:

  « Peter draw a coin as many time as is necessary to get "Head" for the first time. Peter accepts to give to Paul 1 ducat if only one draw is necessary, 2 ducats if two draws are necessary, 4 if Head appears only after 3 draws, and so on."

- "Let us determine what Paul is ready to pay to play this game".

St Petersburg game



EX= ((1/2)*1) +((1/4)*2) +((1/8)*4) +((1/16)*8) +((1/32)*16)+...

= (1/2) +(1/2) +(1/2) +(1/2) +(1/2) +...

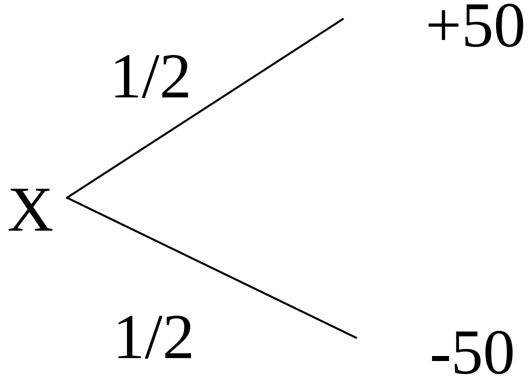
An explanation by D. Bernoulli

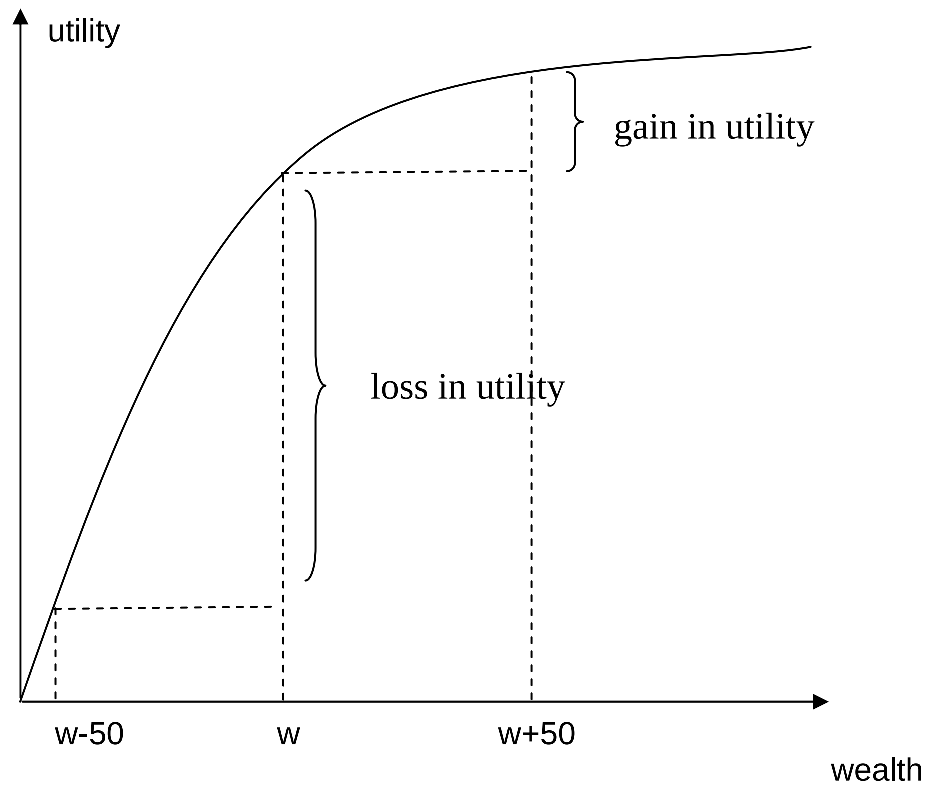
- Paradox: few people are ready to pay more than 10 ducats to play the game.
- Obviously, something else than the expected payoff matters!
- Bernoulli: This is because "the value of a good is not based on its price, but on the utility that it generates".

Decreasing marginal utility of wealth

- An increase in wealth yields an increase in utility which is inversely related to wealth.
- This explains risk aversion.
- An agent is risk-averse if he dislikes any *zero-mean* risk.

Risk aversion





Risk aversion

- Risk aversion on wealth.
- Risk aversion on time lost in traffic congestion, in holiday time,….
- Risk aversion on longevity.
- Risk aversion of bees and rats…

Expected utility theory

- Ex-ante, we measure welfare by the expected utility of final consumption:

$$Eu(X)$$

- In monetary terms, it is measured by the certainty equivalent consumption CE, defined by

$$u(CE) = Eu(X)$$

- Because u is concave, $CE<E(X)$.
- Risk premium: $\pi=E(X)-CE$.

Arrow-Pratt approximation

- Consider an agent with initial wealth w_0 and zero-mean risk tX.
- Arrow and Pratt obtained

$$\pi \cong 0.5At^2\sigma^2 \text{ with } A = -u''(w_0)/u'(w_0)$$

- Almost risk-neutral towards small risks. Focus on large risks!

A special case: Constant relative risk aversion

- One-parameter specification: $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$
- If $\gamma > 0$, u is increasing and concave.
- A larger γ implies a more concave u, a smaller CE, and a larger π.
- Parameter γ is called "relative risk aversion": $\gamma = -\frac{xu''(x)}{u'(x)}$

Estimate your own γ!

- Suppose that your wealth is subject to a 50-50 chance of being increased or reduced by α%.
- What percentage of your wealth are you ready to pay to eliminate this risk?

RRA	α=10%	α=30%
γ=0.5	π=0.3%	π=2.3%
γ=1	π=0.5%	π=4.6%
γ=4	π=2.0%	π=16.0%
γ=10	π=4.4%	π=24.4%
γ=40	π=8.4%	π=28.7%

Risk aversion in real life

- People are ready to pay more than the actuarial value of a risk to insure it.
- People are willing to invest in diversified portfolio of stocks only if they yield an expected return that is larger than the riskfree rate.
- Degrees of risk aversion can be infered from observed behaviors on these markets.

Allais Paradox

	0	1-10	11-99
A	100.000 euros	100.000 euros	100.000 euros
B	0 euro	500.000 euros	100.000 euros

	0	1-10	11-99
C	100.000 euros	100.000 euros	0 euro
D	0 euro	500.000 euros	0 euro

	0	1-10
a	100.000 euros	100.000 euros
b	0 euro	500.000 euros

Valuation of risk

A quick overview of asset pricing theory

Valuing a better world

- Uncertain GDP per capita next year: *c*.
- One wants to value a small investment that would increase *c* by *X* .

$$Eu(c+CE(\varepsilon)) = Eu(c+\varepsilon X)$$

$$\left.\frac{\partial CE(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{EXu'(c)}{Eu'(c)}$$

$$Asset\ value = \frac{EXu'(c)}{Eu'(c)}$$

Consumption-based asset pricing formula

Arrow-Lind

$$Asset\ value = \frac{EXu'(c)}{Eu'(c)}$$

- Arrow-Lind: If *X* and *c* are independent, then the *value* of the asset equals its *expected payoff*.
- The independent small risk on X has no effect on the investor/citizen's welfare: Remind Arrow-Pratt!
- Global warming: damages have
  - a diversifiable component that is independent of GDP;
  - a component proportional to GDP (remind Stern!).

The CAPM

$$Asset\ value = \frac{EXu'(c)}{Eu'(c)}$$

- If c and X are not independent:

$$\text{value} = EX + \text{cov}\left(X, \frac{u'(c)}{Eu'(c)} \right).$$

$$\begin{cases} u'(c) = u'(\mu) + \dfrac{c-\mu}{\mu}\mu u''(\mu) \\ Eu'(c) = u'(\mu) \end{cases}$$

$$\Rightarrow \frac{u'(c)}{Eu'(c)} = 1 - \gamma \frac{c-\mu}{\mu}$$

$$\text{value} \simeq EX - \gamma \frac{\text{cov}(X,c)}{\mu_c}$$

- Suppose that $X = \alpha + \beta c + \varepsilon$:

$$\text{value} \simeq EX - \beta\pi \quad \text{where} \quad \pi = \gamma \frac{\sigma_c^2}{\mu_c}$$

The CAPM for returns

- Similar formulas when X and c represent respectively the real return of the asset and the growth rate of real GDP.
- Estimate (α,β) in $X=\alpha+\beta c+\varepsilon$.
- β is the expected increase in the asset return when the growth rate of GDP is 1%.

$$\text{expected return} \simeq R_f + \beta\pi \quad \text{where} \quad \pi = \gamma\sigma_c^2$$

- π is the equity premium.
- If σ_c=2.5% and γ=2, Then, π=2 (2.5%)^2=0.1%.

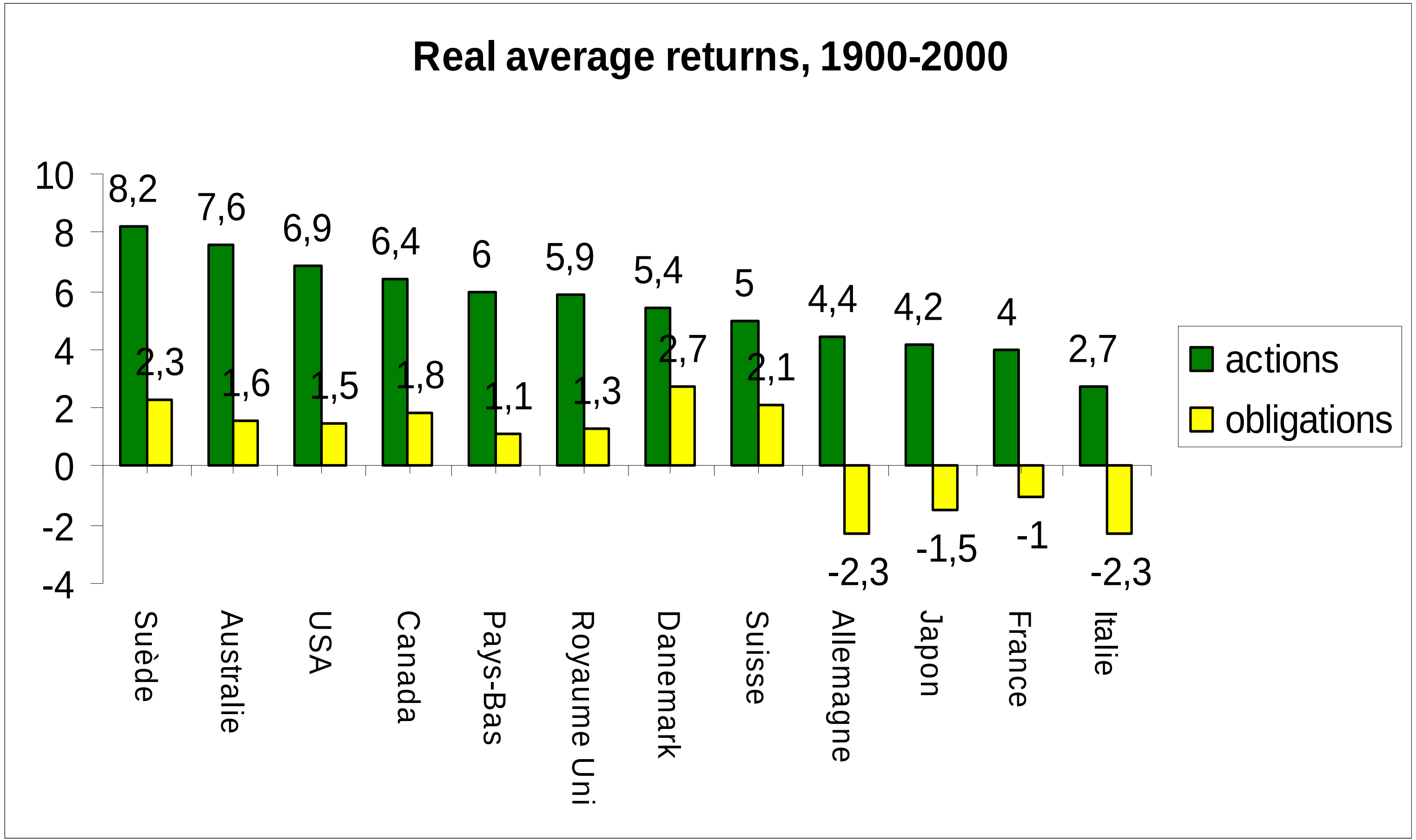
Estimated Beta

$$\text{expected return} \simeq R_f + \beta\pi$$

Air transport	1.80	Energy, raw materials	1.22
Real Property	1.70	Tires, rubber goods	1.21
Travel, outdoor rec.	1.66	Railroads, shipping	1.19
Electronics	1.60	Forest products, paper	1.16
Misc. Finance	1.60	Miscellaneous, conglom	1.14
Nondurables, entertain	1.47	Drugs Medicine	1.14
Consumer durables	1.44	Domestic oil	1.12
Business machines	1.43	Soaps, cosmetics	1.09
Retail, general	1.43	Steel	1.02
Media	1.39	Containers	1.01
Insurance	1.34	Nonferrous metals	0.99
Trucking, freight	1.31	Agriculture	0.99
Producer goods	1.30	Liquor	0.89
Aerospace	1.30	International oil	0.85
Business services	1.28	Banks	0.81
Apparel	1.27	Tobacco	0.80
Construction	1.27	Telephone	0.75
Motor vehicles	1.27	Energy, utilities	0.60
Photographic, optical	1.24	Gold	0.36
Chemicals	1.22		

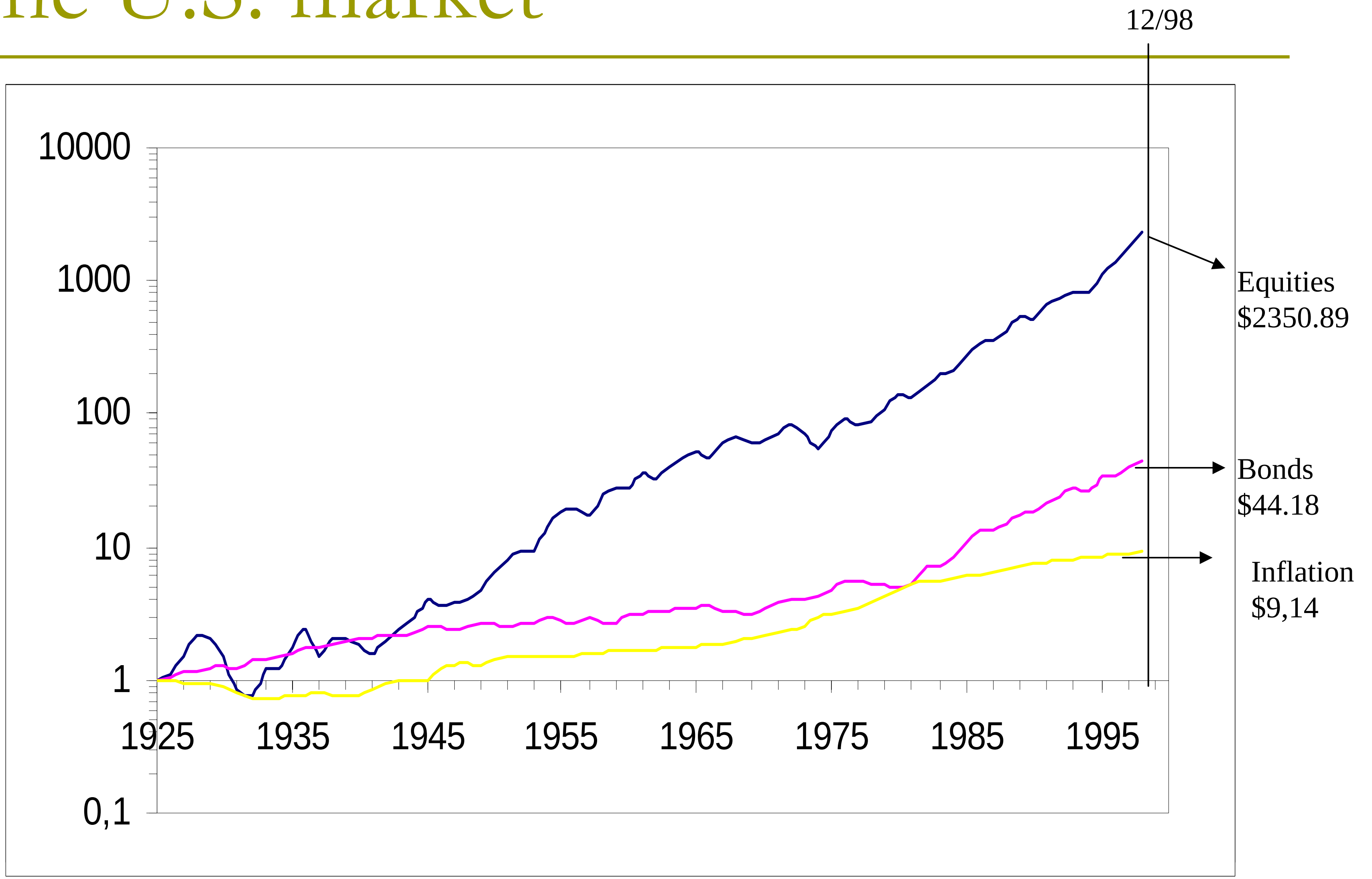
Risky assets

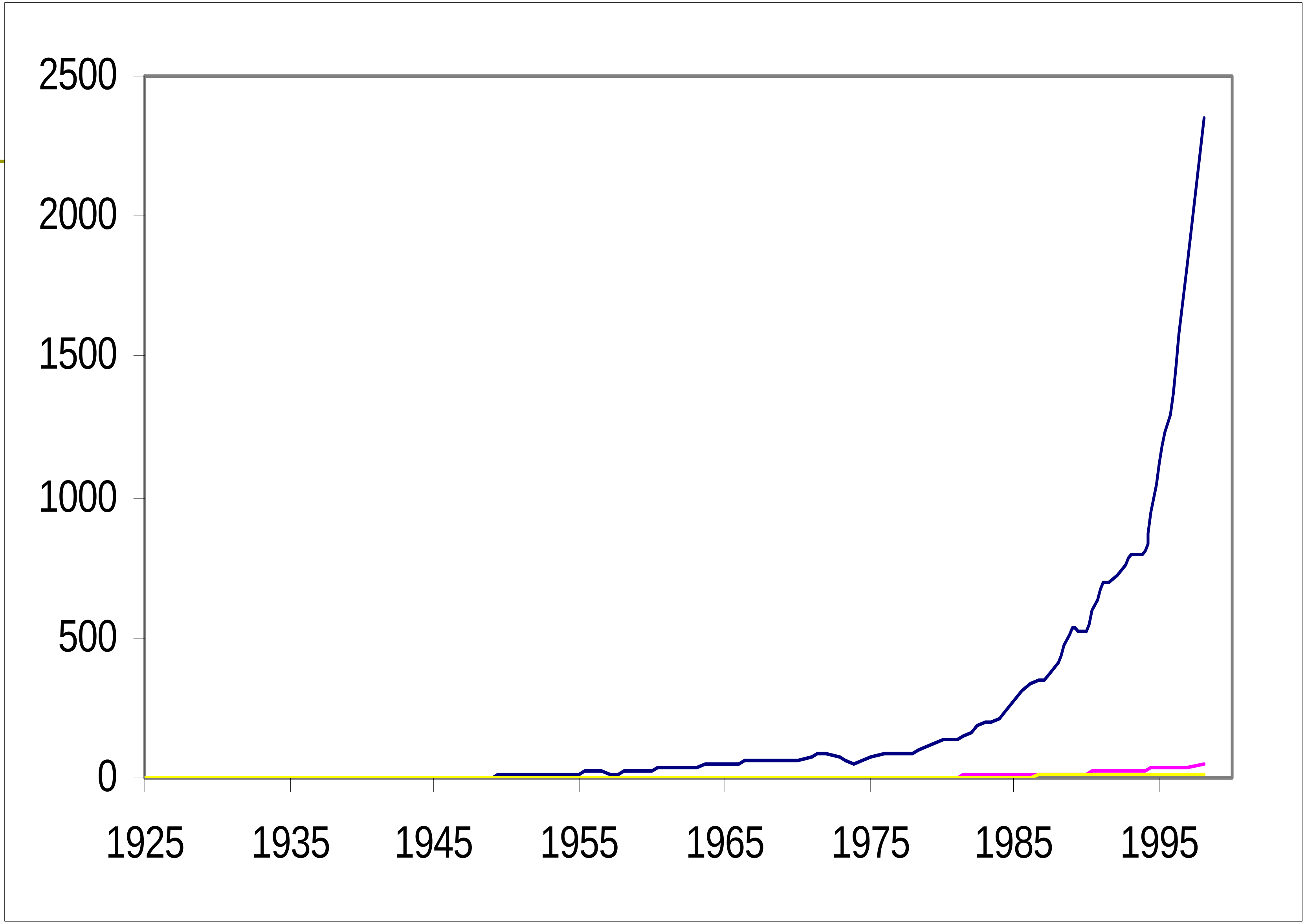
expected return (diversified port) $\simeq R_f + \pi$



π=4% iff γ=40!

The U.S. market




2500
2000
1500
1000
500
0
1925
1935
1945
1955
1965
1975
1985
1995

Conclusion

- People are risk-averse: γ=2...40!
- The presence of undiversifiable risks has a deep impact on their welfare: they are willing to take risk only if the expected reward is large.
- Reciprocally, they are ready to pay much to reduce undiversifiable risk (global warming!)
- Stern report takes into account of this aspect, but in a terribly conservative way (γ=1!)

Discounting

Investing for the future

- The State invests a lot for the future: infrastructure, education, sanitation, Apollo, Colbert's forests,...
- But there also exist many non-realised programs: Copenhagen projects, reducing pollutions, planned infrastructure, limiting extraction of non-renewable resources, ...
- Immediate costs, very distant benefits!
- By how much should we sacrifice the present for the future?
- Which future?

Improving the future?

- Consider a project that brings for sure 2 units of consumption in 20 years per unit of consumption invested today. It is financed through public debt.
- Is this project socially desirable?
- Reasoning by arbitrage
  - Suppose that the real interest rate is 4%.
  - One euro invested today at 4% yields 2,19 € in 20 years.
  - Investing in the project is welfare-deteriorating for the future generation.

NPV

- In fact, the NPV of the project is negative:

$$NPV = -1 + \frac{2}{1.04^{20}} = -0.09\ €$$

- Computing the NPV is equivalent to recognizing that any real investment project competes with investing at the interest rate!
- Key parameter: the discount rate.
- Reducing the discount rate means investing more for the future.

A technocratic approach?

- Behind the discount rate, there are real people!
- They accept to sacrifice their present by investing at 4%,...
- ... in spite of the fact that most of them expect larger incomes in the future.
- This rate balances the social cost of the present sacrifices with the social benefit coming from the positive performance of the investments.

Discounting the distant future

- A project yields 2000 € in 200 years per € invested today.

$$NPV = -1 + \frac{2000}{1.04^{200}} = -0.22\ €$$

$$-1 + \frac{2550}{1.04^{200}} = 0$$

- No observable interest rate for such distant future.
- Could/should we use a smaller discount rate for a more distant future?
- Build a normative model!

	Discount rate	Time horizon (years)
Afrique du Sud	8%	20-40
Allemagne	3%	variable
Australie	6 - 7%	20-30
Canada	5 -10%	20-50
Danemark	6 -7%	30
Etats-Unis	3 - 7%	variable
France	4%	30
Hongrie	6%	30
Italie	5%-8%	?
Japon	4%	40
Mexico	12%	30
Norvège	5%	25
Nouvelle Zélande	10%	25
Pays Bas	4%	30
Portugal	3%	20-30
République tchèque	7%	20-30
Royaume-Uni	6%	30
Suède	4%	15-60
Union Européenne	5%	?
BEI	5% en Europe, 10% ailleurs	?
Banque mondiale	10-12%	?

The three determinants of the discount rate

- Pure preference for the present/ethical attitude towards future generations (+)
- Preference for consumption smoothing over time + positive growth of GDP per capita (+): the marginal utility of 1 unit of consumption next period is smaller than the marginal utility of 1 unit of consumption now.
- Prudence + uncertain growth (-)

The slope of the yield curve

- Is it socially efficient to reduce the discount rate for longer time horizons?
- A potential argument:
  - more distant futures are more uncertain.
  - Under prudence, it has a negative effect on the discount rate.
  - But this is potentially counterbalanced by the fact that more distant generations are also wealthier on average.
- Comparing the degrees of riskiness of GDP per capita for different horizons.
- Serial correlations in growth rates are important.

Some preliminary remarks

- This paper is about the pricing of a safe marginal increase in the single consumption good in the future.
- Not in this paper:
  - evolution of relative prices;
  - riskiness of the project;
  - option values of flexible decisions;....
- Not a general equilibrium model.

The pricing formula

Notation

- The representative agent with a single consumption good:
  - Utility function *u* is increasing and concave;
  - Rate of PPP δ is constant;
  - Uncertain consumption flow *c(t)*.
- Social welfare function: $SWF = E\left[\int e^{-\delta t} u(c(t))dt\right]$
- Can be justified by Rawls' Veil of Ignorance, plus expected utility.
- Investment project:
  - It costs 1 unit of consumption today and yields $e^{r_t t}$ units of consumption in *t* years.
  - A zero-coupon bond with maturity *t*.

The pricing formula

$$SWF = E\left[\int e^{-\delta t} u(c(t))dt\right]$$

- The discount rate is the value of r_t such that the investment does not affect social welfare:

$$e^{r_t t} e^{-\delta t} Eu'(c_t) = u'(c_0)$$

$$r_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)}$$

$$r_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)}$$

The wealth effect

- Marginal utility is decreasing.
- Growth is positive.

$$c_t > c_0 \Rightarrow u'(c_t) < u'(c_0)$$

$$r_t = \delta - \frac{1}{t} \ln \frac{u'(c_t)}{u'(c_0)} > \delta$$

The precautionary effect

- Marginal utility is convex (prudence).
- Growth is uncertain.

$$Eu'(c_t) > u'(Ec_t)$$

$$r_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)} < \delta - \frac{1}{t} \ln \frac{u'(Ec_t)}{u'(c_0)}$$

An exact solution I

- $$\ln c_{t+1} = \ln c_t + x_t; \quad x_t \sim N(\mu, \sigma^2)$$
- $$d \ln c = \mu dt + \sigma dw$$
- $$E\left[\frac{c_{t+1}}{c_t}\right] = E \exp\left(\ln\left(\frac{c_{t+1}}{c_t}\right)\right) = E \exp x_t = \exp\left(\mu + 0.5\sigma^2\right) = \exp(\overline{\mu})$$
- $$E\left[\frac{c_t}{c_0}\right] = \exp(\overline{\mu} t)$$
- $$\frac{dc_t}{c_t} = \overline{\mu} dt + \sigma dw$$

A technical point

$$u(z) = -\exp(-Az) \text{ and } \tilde{z} \sim N(\mu, \sigma^2),$$

$$\begin{aligned}
Eu(\tilde{z}) &= \frac{-1}{\sigma\sqrt{2\pi}} \int \exp(-Az)\exp(-\frac{(z-\mu)^2}{2\sigma^2})dz \\
&= -\exp(-A(\mu - \frac{A\sigma^2}{2}))\frac{1}{\sigma\sqrt{2\pi}} \int \exp(-\frac{(z-(\mu- A\sigma^2))^2}{2\sigma^2})dz \\
&= -\exp(-A(\mu - \frac{A\sigma^2}{2})).
\end{aligned}$$

The certainty equivalent of z is μ - 0.5 A σ^2.

The "extended Ramsey rule"

$$r_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)}$$

$$u(c) = c^{1-\gamma} / (1-\gamma)$$

$$\frac{Eu'(c_t)}{u'(c_0)} = E \exp\left[-\gamma(\ln c_t - \ln c_0)\right] = \exp\left[-\gamma(\mu t - 0.5\gamma\sigma^2 t)\right]$$

$$r_t = \delta + \gamma\left[\mu - 0.5\gamma\sigma^2\right]$$

$$r_t = \delta + \gamma\left[\bar{\mu} - 0.5(\gamma+1)\sigma^2\right]$$

Growth of GDP

Year	GDP per capita[*]	Annualized growth rate[**]
-5000	130	
-1000	160	0.005
1	135	-0.017
1000	165	0.020
1500	175	0.012
1800	250	0.119
1900	850	1.224
1950	2030	1.741
1975	4640	3.307
2000	8175	2.265

$$r_t = \delta + \gamma\bar{\mu} - 0.5\gamma(\gamma+1)\sigma^2$$

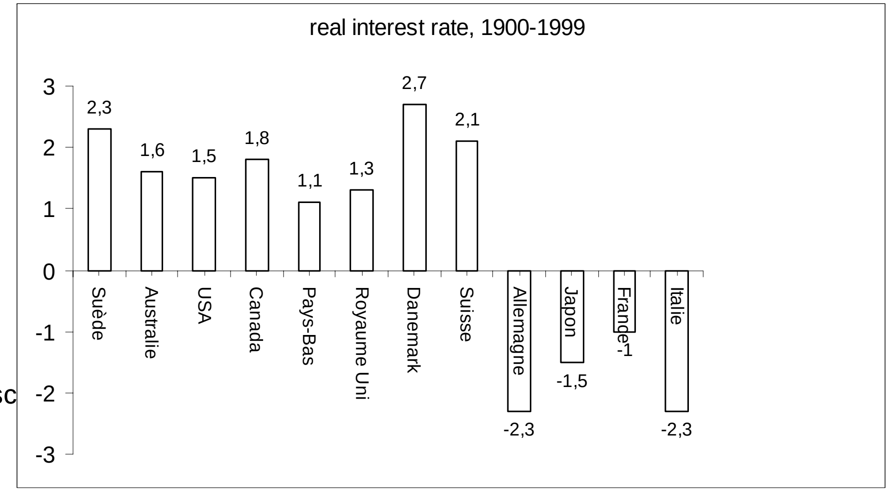
- The socially efficient discount rate is independent of the time horizon: Flat yield curve.
- Large wealth effect

$$\gamma = 2,\ \bar{\mu} = 2\% \quad \Rightarrow \quad \gamma\bar{\mu} = 4\%$$

- Small precautionary effect

$$\gamma = 2,\ \sigma = 2.5\% \quad \Rightarrow \quad \gamma(\gamma+1)\sigma^2 = 0.37\%$$

- The Rf puzzle!



Changing expectations

- The efficient term structure may not be flat if one expects that the growth rate will accelerate or decelerate in a deterministic way.
- $$d \ln c_t = \mu_t dt + \sigma dw_t$$
  $$d\mu_t = g(t)dt$$

$$r_t = \delta + \gamma m(t) - 0.5\gamma^2\sigma^2 \quad with \quad m(t) = \frac{1}{t}\int_0^t \mu_s ds$$

A model with parameter uncertainty

$$\ln c_{t+1} = \ln c_t + x_t(\theta)$$

$$\textit{Conditional to } \theta, x_1, x_2, \ldots \textit{ are i.i.d.}$$

$$\textit{Prior distribution on } \theta$$

- Learning: a larger growth in the first periods raises the expected growth for subsequent periods.
- Decreasing yield curve.

The case of learning with CRRA preferences

$$r_t = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)} \qquad \Rightarrow r_t = \delta - \frac{1}{t} \ln E\left(\exp\left[\sum_{\tau=1}^{t} x_\tau(\theta) \right] \right)^{-\gamma}$$

$$r_t = \delta - \frac{1}{t} \ln E_\theta E \prod_{\tau=1}^{t} \exp - \gamma x_\tau(\theta) = \delta - \frac{1}{t} \ln E_\theta \prod_{\tau=1}^{t} E \exp - \gamma x_\tau(\theta)$$

$$\alpha(\theta) = E e^{-\gamma x(\theta)}$$

$$r_t = \delta - \ln \sqrt[t]{E\left(\alpha(\theta)^t \right)}$$

Learning and CRRA: A simple result

$$r_t = \delta - \ln \sqrt[t]{E\left(\alpha(\theta)^t\right)}$$

- Proposition 5: *Under these assumptions, the efficient discount rate is decreasing with t. It tends to the smallest possible rate*

$$\delta - \max_\theta \ln \alpha(\theta)$$

when t tends to infinity.

Weitzman's calibration (2004)

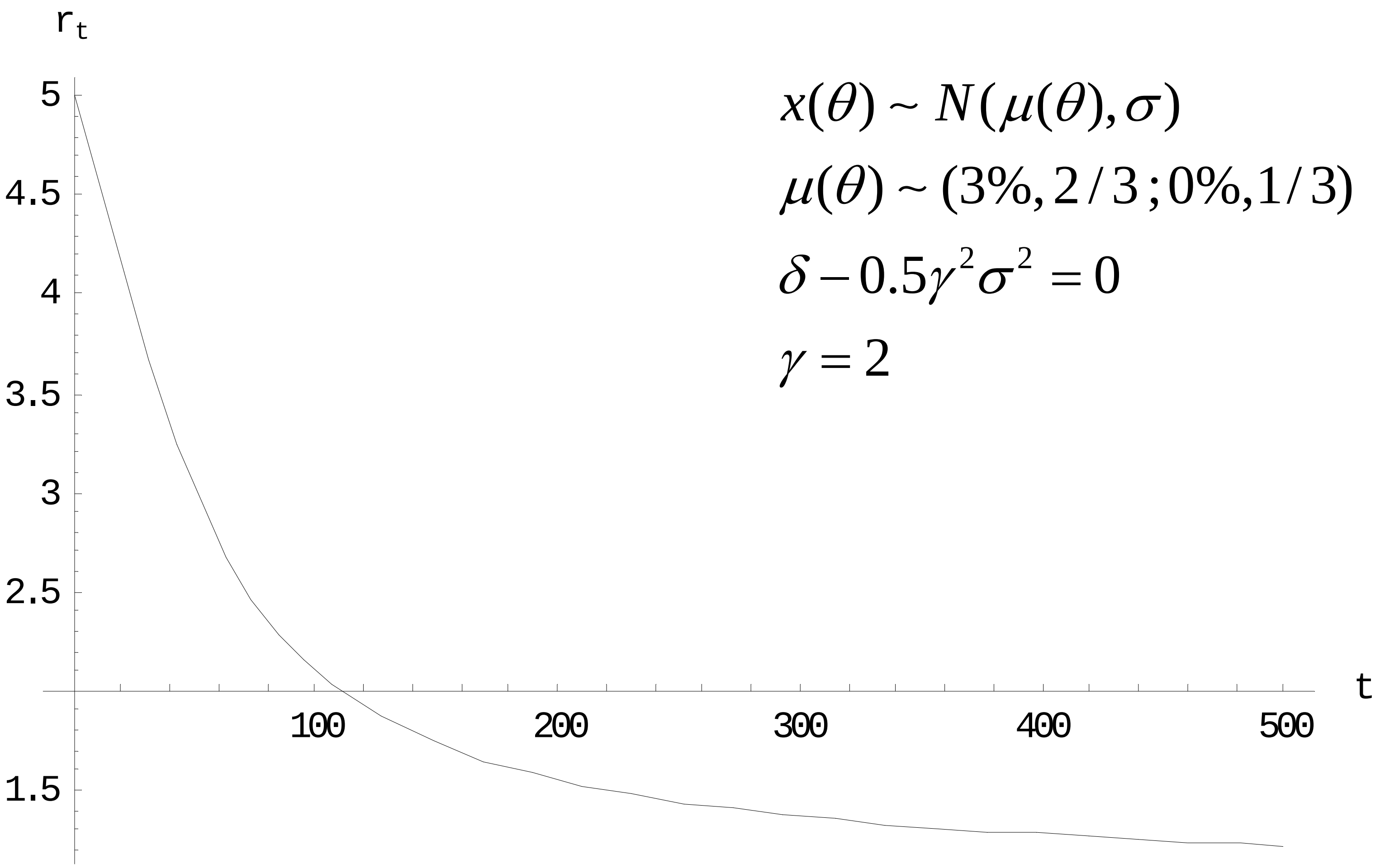
$$r_t = \delta - \frac{1}{t}\ln E\alpha(\theta)^t \quad \underset{t\to\infty}{\longrightarrow} \quad \delta - \max_\theta \ln \alpha(\theta)$$

- $x(\theta) \sim N(\mu(\theta),\sigma) \qquad \mu(\theta) \sim N(\mu,\sigma_\mu)$
- It implies that

$$\alpha(\theta) = \exp\left[-\gamma\mu(\theta) + 0.5\gamma^2\sigma^2\right]$$

is unbounded below. So is the efficient discount rate.

Calibration



$$x(\theta) \sim N(\mu(\theta), \sigma)$$

$$\mu(\theta) \sim (3\%, 2/3\, ; 0\%, 1/3)$$

$$\delta - 0.5\gamma^2\sigma^2 = 0$$

$$\gamma = 2$$

First-order stochastic dependence

Definitions

- $c_t = c_0 + x_1 + x_2.$
- $G(x) = \Pr\left[x_1 \le x\right]$ *and* $F(x \mid x_1) = \Pr\left[x_2 \le x \mid x_1\right].$
- (x_1, x_2) *are "positively first-order stochastically dependent" (FSD) if F is non-increasing in* x_1 *for all* x_2*.*
- Example: AR(1): $x_2 \mid x_1 = \phi x_1 + \varepsilon$ *with* $\phi > 0$*.*
- *Let* y_2 *denote the independent r.v. whose CDF is H, with* $H(y) = \int F(y \mid x_1)\, dx_1.$

Isolating the effect of FSC

- We compare

$$r_t = \delta - \frac{1}{t}\ln\frac{Eu'(c_0+x_1+x_2)}{u'(c_0)}$$

to

$$\widehat{r}_t = \delta - \frac{1}{t}\ln\frac{Eu'(c_0+x_1+y_2)}{u'(c_0)}.$$

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$$r_t \le \widehat{r}_t \quad \Leftrightarrow \quad Eu'(c_0+x_1+x_2) \ge Eu'(c_0+x_1+y_2)$$

Two results

$$r_t \leq \widehat{r}_t \quad \Leftrightarrow \quad Eu'(c_0 + x_1 + x_2) \geq Eu'(c_0 + x_1 + y_2)$$

- Lemma 1: Consider any positive FSD pair (x_1, x_2).

$$Eh(x_1, x_2) \geq Eh(x_1, y_2) \Leftrightarrow h \text{ is supermodular: } \frac{\partial^2 h}{\partial x_1 \partial x_2} \geq 0.$$

- Proposition 1: *The presence of a positive FSD dependent in changes in consumption reduces the long-term efficient discount rate if and only if the representative agent is prudent (u' convex).*

- *Corollary: CRRA+FSD implies decreasing term structure.*

Intuition

- The positive FSD dependence in Δc raises the risk of the distant future compared to the i.i.d. case.

$$E\left(c_0 + x_1 + x_2\right) = E\left(c_0 + x_1 + y_2\right)$$
$$E\left(c_0 + x_1 + x_2\right)^2 \geq E\left(c_0 + x_1 + y_2\right)^2$$

- Under prudence, it is efficient to make more efforts for that distant future. This is done by reducing the long-term discount rate.

Remark: growth *rates* independent

- We compare

$$r_t = \delta - \frac{1}{t}\ln\frac{Eu'(c_0 e^{x_1+x_2})}{u'(c_0)} \quad to \quad \hat{r}_t = \delta - \frac{1}{t}\ln\frac{Eu'(c_0 e^{x_1+y_2})}{u'(c_0)}$$

- Proposition 2: *The presence of a positive FSD dependence in changes in* <u>*log*</u> *consumption reduces the long-term efficient discount rate if and only if –cu'''/u''>1.*
- *Notice that*

$$Ec_0 e^{x_1+x_2} \geq Ec_0 e^{x_1+y_2}.$$

Second-order stochastic dependence: stochastic volatility

Definitions

- (x_1,x_2) are positive SSD if an increase in x_1 raises the riskiness of the $x_2|x_1$.

- Example: $x_2 \left| x_1 = \mu + \varepsilon \sqrt{x_1} \right.$

- Lemma 2: Consider any positive SSD pair (x_1,x_2).

$$Eh(x_1, x_2) \geq Eh(x_1, y_2) \Leftrightarrow -\frac{\partial h}{\partial x_2} \textit{ is supermodular: } \frac{\partial^3 h}{\partial x_1 \partial x_2^2} \leq 0.$$

Results

- Proposition 3: *The presence of a positive SSD correlation in changes in consumption raises the long-term efficient discount rate if and only if u''''<0.*

- Intuition: increased skewness reduces $Eu'(c_t)$ if *u''''<0.*

Results

- Proposition 4: *The presence of a positive SSD correlation in changes in log consumption raises the long-term efficient discount rate if and only if*

$$f(c) = u''(c) + 3cu'''(c) + c^2 u''''(c) \leq 0.$$