Positive Polynomials and Optimization

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The following is the scientific report on our 5-days workshop "Positive Polynomials and Optimization". The aim of holding this workshop at this time was to continue a recent tradition of bringing researchers in optimization to gain access to the new mathematical tools related to positive polynomials. The conference was to give an opportunity for pure mathematicians to interact and exchange ideas/results with applied researchers in optimization. In a sense, the distinguishing property of the real numbers is positivity. There has been a recent explosion of interest in positive polynomials. This is due to the many interesting applications, the introduction of numerical algorithms for computing with sums of squares, as well as new theoretical results about sums of squares and representations of positive polynomials. Positive polynomials can be used to formulate problems in control theory, optimization, and other areas, and then these problems can be solved using the theory of positive polynomials coupled with numerical techniques from semidefinite programming.

This tradition goes back to a February 2002 meeting, "Positivität von Polynomen" held at Oberwolfach. This was the first time that people working on theoretical and algebraic geometrical aspects of positive polynomials had gotten together with people in optimization. Interest in this subject has grown through subsequent meetings; this was one of several topics at the MSRI Semester on Topological Aspects of Real Algebraic Geometry in Spring 2004, a meeting on optimization was held in Amsterdam in June 2004, a Luminy workshop on Positive Poynomials was held in March 2005, and Kuhlmann, Lall and Sottile have taught several short courses on real algebra and optimization at the "Trimestre Géométrie Algébrique Réelle" held at Institut Henri Poincaré, Paris September to December, 2005. These activities have featured particular aspects of these developments and have neither been comprehensive nor brought people together again for a period of sustained interaction. We felt that it was time for a comprehensive meeting on this subject to give the people working in different mathematical areas an overview of the developments since February 2002. The goal of this BIRS workshop was to deepen the connections between people working in different communities, as well as to disseminate these ideas and developments to students, post-doctoral fellows, and researchers working in nearby areas of mathematics. Special effort was made to invite "young researchers" to participate; several graduate students and post-doctoral fellows attended the meeting, and contributed actively to its success.

1 Overview of the Field

A multivariate polynomial is positive if it takes only non-negative values on its domain. For example, if a polynomial is a square, or a non-negative linear combination of squares (sos), then it is positive and this representation is an algebraic certificate of its positivity. Unfortunately, Hilbert showed that a polynomial which is positive on all of real affine n-space, where $n > 1$, is not necessarily sos. One may similarly ask about polynomials that are positive on some domain defined by polynomial inequalities (a semi-algebraic set). The solution of the moment problem by Schmüdgen, Putinar, and Jacobi and Prestel shows that polynomials positive on compact semi-algebraic sets have a similar algebraic certificates of positivity involving squares and the polynomials defining the semi-algebraic set.

Finding such a certificate having a particular form and involving squares of bounded degree d may be formulated and thus solved as a semidefinite program (SDP), for which there exist very efficient numerical tools. For example, Parrilo has developed software, SOStools, using SDPs to numerically solving problems involving sums of squares.

Lassere used this methodology to devise an algorithm to compute the minimum value of a polynomial f(x) on a compact semialgebraic set (an NP-complete problem, so it is hopeless to find an efficient algorithm). This algorithm finds the largest number m_d such that $f(x) - m_d$ has an algebraic certificate of positivity on the set, involving squares of degree at most d. This sequence of approximate minima m_d converges to the actual minimum, and thus we obtain a sequence of computationally feasible relaxations to this problem in class NP.

At the same time as these mathematical developments, there has been a new emphasis on control problems that are intrinsically computationally hard. There is a growing need for control techniques for problems where there is an explicit combinatorial structure, such as power control in wireless sensor networks with interference, combined task assignment and path planning for multiple vehicle systems, and multi-limbed robotic systems. There is also renewed interest and applications for control systems where the source of apparent intractability is not due to a combinatorial growth of discrete possibilities; instead the need for decentralization introduces complexity due to the structure of the desired control design. Examples of this kind of system include formation control of vehicles, and congestion control in wireline networks.

A certificate provides a mathematical proof which is easily verifiable automatically. In control theory, it is used to show that desirable outcomes will always occur, such as in stability analysis, where one shows that for all initial conditions states converge to the equilibrium. Equivalently, it is used to show that undesirable outcomes never occur, for example in reachability analysis, where one shows that two vehicles will never collide. The idea of certificates is a striking example of the common threads that interconnect the fields of optimization, algebraic geometry, control, and computational complexity. Another example of the applicability of these methods is found in recent work by K. Gatermann and P. Parrilo which looks at the global optimization problem for polynomials invariant under certain small finite groups. Using invariant theory, they show that the SDPs involved decompose into a series of much smaller, hence more computationally tractable, SDPs. They have used this idea to solve problems arising in chemistry, where the polynomials involved much symmetry.

2 Recent Developments and Open Problems

Recent developments, presented at the workshop, include the following: Lasserre's SDP relaxations exploiting some structural properties of the positive polynomials under consideration. Lasserre considers "sparse polynomials", that is, polynomials in which the variables occuring in the monomials satisfy specific partition and overlap conditions. On the same topic, and exploiting these ideas, Nie lectured on optimization of such structured polynomials. Both talks were very well recieved by the audience, and produced many intense discussions among the participants. Several open problems emerged naturally: e.g. to provide other, algebraic or analytic proofs of Lasserre's Theorem on the representation of sparse positive polynomials by sparse sums of squares. See the section "outcome of the workshop"below.

Curto's talk highlighted new aspects of the moment problem, by presenting his ideas about truncated moment problems; that is, representation by linear functionals of truncated moment sequences. This approach calls for many natural questions, for example, to investigate the various approximation methods available to date (such as saturation and closure of the preorderings, Schmúdgen's positivstellensatz) to polynomials of bounded degrees only.

Laurent's talk presented very interesting ideas to develop an algorithmic approach to compute real radical ideals, in the spirit of Gröbner basis in algebraic geometry. Many participants were inspired by these new ideas. A Ph.D student from Regensburg (Doris Augustin) have been studying the membership problem for a preorder in the polynomial ring (that is, searching for conditions which are necessary and sufficient for a polynomial f to lie in the preorder generated by finitely many other polynomials). She found a lot of inspiration in Laurent's talk and is now analyzing the relationship between the two approaches.

Netzer (Ph.D student from Konstanz, Germany) presented a new, elementary proof of Schmüdgen's celebrated result that a finitely preordering satisfies the strong moment property if the preorderings associated to the fibers do (a finitely generated preordering in the real polynomial ring is said to have the Strong Moment Property, if its closure with respect to the finest locally convex topology equals its saturation). Netzer solved another open problem: In [8] and [9], a property which implies the Strong Moment Property, known as the "Double-Dagger Property", was found. Netzer shows that this property is strictly stronger than the Strong Moment Property.

Scheiderer's talk presented recent progress made in [2] concerning representation of invariant (under the action of a locally compact group) positive polynomials by sums of squares. Several exciting open problems emerge from this work, for example, to characterize invariant semi-algebraic sets for which the invariant moment problem is solvable (that is, invariant linear functionals are represented by an invariant measure with support in the given invariant semi-algebraic set) whereas the usual moment problem fails (that is, not every linear functional is representable by a measure). His talk was related in many aspects to Theobald's talk about exploiting invariance in SDP-relaxations for polynomial optimization.

3 Presentation Highlights

List of Scientific Presentations:

Sunday

Mihai Putinar, *The trigonometric moment problem in several variables and related SOS decompositions*
Pablo Parillo, *Exact semidefinite representations for genus zero curves*
Konrad Schmüdgen, *Positivity and Positivstellensätze for matrices of polynomials*
Bill Helton, *Real algebraic geometry in a free *-algebra*
16:30–17:00 Second Chances

Monday

Raul Curto, *Algebraic geometric techniques for the truncated moment problem*
Claus Scheiderer, *Sums of squares and moment problems with symmetries*
Tim Netzer, *An elementary proof of Schmüdgen's Theorem*
Luis Zuluaga, *Closed-form solutions to certain moment problems with applications to business*
17:00–17:30 Second Chances

Tuesday

M.-F. Roy, *Certificate of positiviy in the Bernstein basis*
11:30–12:00 Second Chances

Wednesday

Monique Laurent, *A numerical algorithm for the real radical ideal*
Thorsten Theobald, *Symmetries in SDP-based relaxations for constrained polynomial optimization*
Markus Schweighofer, *Global optimization of polynomials using gradient tentacles and sums of squares*
Jiawang Nie, *Sparse SOS Relaxation and Applications*
Murray Marshall, *Representations of non-negative polynomials and applications to optimization*
17:00–17:30 Second Chances

Thursday

Bruce Reznick, *Hilbert's construction of psd quartics and sextics that are not sos*
J.-B. Lasserre, *A Positivstellensatz which preserves the coupling of variables*
11:00–11:30 Last Chances

ABSTRACTS

Speaker: **Raúl Curto** (Iowa)
Title: *Algebraic Geometric techniques for the truncated moment problem*
Abstract: For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy rank$\mathcal{M}(n) \leq$ card$\mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$. In joint work with Lawrence Fialkow and Michael Möller, we employ tools and techniques from algebraic geometry (e.g., Hilbert polynomials, Gröbner and H-bases, representation of positive polynomials) to prove that for the *extremal* case (rank$\mathcal{M}(n) =$ card$\mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, rank$\mathcal{M}(n)$-atomic) representing measure.

Truncated moment problems (TMP) as above for which the support of a representing measure is required to lie inside a closed set K are called truncated K-moment problems (TKMP). In case K is a semi-algebraic set determined by polynomials $q_1, ..., q_m$, the study of TKMP is dual to determining whether a polynomial nonnegative on K belongs to the positive cone consisting of polynomials of degree at most $2n$ which can be expressed as sums of squares, and of squares multiplied by one or more distinct q_i's.

The extremal case, which we have now solved, is inherent in the TMP. A recent result of C. Bayer and J. Teichmann (extending a classical theorem of V. Tchakaloff and its successive generalizations given by I.P. Mysovskikh, M. Putinar, and L. Fialkow and the speaker) implies that if $\beta^{(2n)}$ has a representing measure, then it has a finitely atomic representing measure. Fialkow and the speaker had previously shown that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ admits an extension to a positive moment matrix $\mathcal{M}(n+k)$ (for some $k \geq 0$), which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $\mathcal{M}(n+k+1)$. Further, we proved that any flat extension $\mathcal{M}(n+k+1)$ is an extremal moment matrix for which there is a computable rank $\mathcal{M}(n+k)$-atomic representing measure μ. In this sense, the existence of a representing measure for $\beta^{(2n)}$ is intimately related to the solution of an extremal TMP.

Speaker: **Bill Helton** (UC San Diego)
Title: *Real Algebraic Geometry in a Free $*$-Algebra*
Abstract: The talk will describe recent results and focus on new directions.

Speaker: **J.-B. Lasserre** (Toulouse)
Title: *A Positivstellensatz which preserves the coupling of variables*
Abstract: We specialize Schmüdgen's Positivstellensatz and its Putinar and Jacobi–Prestel refinement, to the case of a polynomial $f^2\mathbb{R}[X,Y] + \mathbb{R}[Y,Z]$, positive on a compact basic semi-algebraic set K described by polynomials in $\mathbb{R}[X,Y]$ and $\mathbb{R}[Y,Z]$ only, or in $\mathbb{R}[X]$ and $\mathbb{R}[Y,Z]$ only (i.e. K is cartesian product). In particular, we show that the preordering $P(g,h)$ (resp. quadratic module $Q(g,h)$) generated by the polynomials $\{g_j\} \subset \mathbb{R}[X,Y]$ and and $\{h_k\} \subset \mathbb{R}[Y,Z]$ that describe K, is replaced with $P(g) + P(h)$ (resp. $Q(g) + Q(h)$), so that the absence of coupling between X and Z is also preserved in the representation. A similar result applies with Krivine's Positivstellensatz involving the cone generated by $\{g_j, h_k\}$.

Speaker: **Monique Laurent** (CWI, Amsterdam)
Title: *Semidefinite characterization and computation of real radical ideals*
Abstract: For an ideal $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ given by a set of generators $h_1, \dots, h_m$, we propose a semidefinite characterization and a numerical method for finding the real radical ideal $\sqrt[\mathbb{R}]{I} = I(V_\mathbb{R}(I))$, provided it is zero-dimensional (even if I is not). Our method relies on expressing $I(V_\mathbb{R}(I))$ as the kernel of a suitable positive semidefinite moment matrix and uses semidefinite optimization for finding such a matrix.

One of our results can be sketched as follows. Let $M_t(y)$ be a maximum rank feasible solution to the system:

$$M_t(y) \succeq 0,\ M_{t-d_j}(h_j y) = 0\ (j = 1, \dots, m),$$

where $d_j := \lceil \deg(h_j)/2 \rceil$ and $t \geq \max_j d_j$. Then, $I(V_\mathbb{R}(I)) = \langle \mathrm{Ker} M_t(y) \rangle$ if the rank condition: $\mathrm{rank} M_t(y) = \mathrm{rank} M_{t-d}(y)$ holds. A maximum rank solution $M_t(y)$ can be found with a semidefinite programming solver; if the rank condition holds we have found $I(V_\mathbb{R}(I))$, otherwise iterate replacing t by $t+1$. The algorithm is +guaranteed to terminate when $V_\mathbb{R}(I)$ is finite. With our method we can compute directly from the optimal matrix $M_t(y)$ the following objects: the set $V_{\mathbb{R}}(I)$ of real roots, a linear basis of the quotient vector space $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$, a border basis of $I(V_{\mathbb{R}}(I))$ as well as a Gröbner basis for a total-degree monomial ordering.

A feature of our method is that it exploits right from the beginning the real algebraic nature of the problem. In particular, it does not need the determination of a Gröbner basis of the ideal I and we do not compute (implicitly or +explicitly) the complex variety $V(I)$. The method also applies to finding $I(V_{\mathbb{R}}(I) \cap S)$ where S is basic closed semialgebraic set.

This is joint work with J.-B. Lasserre and P. Rostalski

Speaker: **Murray Marshall** (Saskatchewan)
Title: *Representations of non-negative polynomials, degree bounds and applications to optimization*
Abstract: Natural sufficient conditions for a polynomial to have a local minimum at a point are considered. These conditions tend to hold with probability 1. It is shown that polynomials satisfying these conditions at each minimum point have nice presentations in terms of sums of squares. Applications are given to optimization on a compact set and also to global optimization. In many cases, there are degree bounds for such presentations. These bounds are of theoretical interest, but they appear to be too large to be of much practical use at present. In the final section, other more concrete degree bounds are obtained which ensure at least that the feasible set of solutions is not empty.

Speaker: **Tim Netzer** (Konstanz)
Title: *An Elementary Proof of Schmüdgen's Theorem on the Moment Problem of Closed Semi-Algebraic Sets*
Abstract: We discuss a more elementary proof of the main result from Schmüdgen's 2003 article "On the moment problem of closed semi-algebraic sets". The result states, that the question whether a finitely generated preordering has the so called Strong Moment Property can be reduced to the same question for preorderings corresponding to fiber sets of bounded polynomials.

Speaker: **Jiawang Nie** (IMA, Minnesota)
Title: *Sparse SOS Relaxation and Applications*
Abstract: SOS relaxation provides very good approximation for finding global minimum and minimzer of polynomial functions. However, the size of the resulting SDP is often very large and makes it difficult to solve large scale problems. This talk will discuss the global optimisation of large polynomial functions that are given as the summation of small polynomials. The sparse SOS relaxations are proposed. We analyze the computational complexity and the quality of lower bounds. Some numerical implementations of randomly generated problems shows that this sparse SOS relaxation is very successful. This sparse SOS relaxation is very useful in solving large scale sparse polynomial systems, like the polynomial systems derived from nonlinear PDEs and distance geometry problems (e.g., sensor network localization).

Speaker: **Bruce Reznick** (Illinois)
Title: *Hilbert's construction of psd quartics and sextics that are not sos*
Abstract: We will discuss both of these constructions, which become almost intuitive when one "counts constants". New and simple examples will be derived. Speaker: **Pablo Parrilo** (MIT)

Title: *Exact semidefinite representations for genus zero curves*
Abstract: The characterization of sets that admit an exact representation in terms of semidefinite programming constraints (perhaps with additional variables) is one of great interest in optimization. There have been a few recents results in this direction, based mainly on the work of Helton and Vinnikov and the related Lax conjecture, that point out to the existence of specific obstructions for (the interior of) a plane curve to be semidefinite representable. In this talk we discuss a procedure to explicitly construct exact representations for convex hulls of arbitrary segments of genus zero plane curves. In particular, it is shown that the new method enables the computation of representation for particular curves, for which a generic SOS-based construction fails.

Speaker: **Mihai Putinar** (UC Santa Barbara)
Title: *The multivariate trigonometric moment problem and related sums of squares decompositions*

Abstract: In the case of the one dimensional torus, Riesz-Fejer factorization of a non-negative trigonometric polynomial as the modulus square of another polynomial provides the basis of all positivity results related to the unit disk: Riesz-Herglotz parametrization of all non-negative harmonic functions, the solution to the trigonometric moment problem, as proposed by Schur, and separately by Caratheodory-Fejer, the spectral theorem for unitary operators.

In several variables, on the torus in $\mathbb{C}^n$, we reverse the flow, and start with Bochner's characterization of Fourier transforms of positive measures. This provides a sum of squares decomposition for positive trigonometric polynomials. And the result can easily be adapted to compact, semi-algebraic supports on the torus. The most intriguing case is however the unit sphere in $\mathbb{R}^n$, where a decopmpsition into squares of spherical harmonics is available. For odd dimensional sphere, the complex structure induced from $\mathbb{C}^n$ provides a decomposition (of a positive polynomial) into squares of pluriharmonic polynomials. As a consequence I will indicate a novel proof of an old theorem of Quillen.

Finally, the non-commutative sphere, or torus, associated to the free-$*$ algebra reveals stronger SOS deecompositions.

Speaker: **M.-F. Roy** (Rennes)
Title: *Certificate of positiviy in the Bernstein basis*
Abstract: We prove the existence of a polynomial size (in the degree d and bitsize t of coefficients) certificate of positivity for a positive univariate polynomial on $[-1, 1]$ using Berstein basis of degree d on well-chosen subintervals. This improves by an exponential factor previsouly known results by Powers and Reznick.

Speaker: **Claus Scheiderer** (Konstanz)
Title: *Sums of squares and moment problems with symmetries*
Abstract: Let G be a real algebraic subgroup of $GL(V)$, the general linear group of a finite-dimensional real vector space V, and assume that G is (semi-algebraically) compact. We study the cones of sums of squares in $R[V]$ and in $R[V]^G$, the ring of G-invariants, and relate them through the operations of contraction and extension. More generally, we do the same for arbitrary quadratic modules. In doing this, we use (and partially re-prove, partially generalize) results of Procesi-Schwarz, Bröcker and Gatermann-Parrilo. We prove that the Reynolds operator maps the cone $\Sigma R[V]^2$ into itself, and that this property is characteristic of the case where G is compact.

Given a basic closed set K in V which is G-invariant, we ask for (finite) characterizations of the G-invariant K-moment functionals. We isolate two conditions under which such characterizations exist, and show by examples that this may happen at the same time when the usual (full) K-moment problem is not finitely solvable.

The talk will contain (plenty of) explicit examples and (a few) open problems. (Joint work with Salma Kuhlmann and Jaka Cimpric.)

Speaker: **Konrad Schmüdgen** (Leipzig)
Title: *Positivity and Positivstellensätze for Matrices of Polynomials*
Abstract: The notion of k-positivity ,$0 \leq k \leq n$, for (n, n)-matrices of polynomials is introduced and discussed. Generalizations of Stengle's Positivstellensatz to matrices are given.

Speaker: **Markus Schweighofer** (Konstanz)
Title: *Global optimization of polynomials using gradient tentacles and sums of squares*
Abstract: We combine the theory of generalized critical values with the theory of iterated rings of bounded elements (real holomorphy rings). We consider the problem of computing the global infimum of a real polynomial in several variables. Every global minimizer lies on the gradient variety. If the polynomial attains minimum, it is therefore equivalent to look for the greatest lower bound on its gradient variety. Nie, Demmel and Sturmfels proved recently a theorem about the existence of sums of squares certificatesfor such lower bounds. Based on these certificates, they find arbitrarily tight relaxations of the original problem that can be formulated as semidefinite programs and thus be solved efficiently. We deal here with the more general case when the polynomial is bounded from below but does not necessarily attain a minimum. In this case, the method of Nie, Demmel and Sturmfels might yield completely wrong results. In order to overcome this problem, we replace the gradient variety by larger semialgebraic sets which we call gradient tentacles. It now gets substantially harder to prove the existence of the necessary sums of squares certificates.

Speaker: **Thorsten Theobald** (Berlin)
Title: *Symmetries in SDP-based relaxations for constrained polynomial optimization*
Abstract: (joint work with L. Jansson, J.B. Lasserre and C. Riener) We study methods for exploiting symmetries within semidefinite programming-based relaxation schemes for constrained polynomial optimization. Our main focus is on problems where the symmetric group or the cyclic group is acting on the variables. From the exact point of view, we extend the representation-theoretical methods of Gatermann and Parrilo for the unconstrained case to the constrained case (i.e., to Lasserre's relaxation scheme). In contrast to the viewpoint merely from the resulting semidefinite programs, the symmetries on the original variables induce much additional symmetry structure on the moment matrices of the relaxation scheme. We characterize the combinatorics of the resulting block decompositions in terms of Kostka numbers. Moreover, we present methods to efficiently compute lower and upper bounds for a subclass of problems where the objective function and the constraints are given by power sums.

Speaker: **Luis Zuluaga** (New Brunswick)
Title: *Closed-form solutions to certain moment problems with Applications to Business*
Co-authors: Donglei Du, Javier Pena, and Juan Vera.

Abstract: We present new closed-form solutions to certain moment problems with applications in mathematical finance, inventory theory, supply chain management, and Actuarial Science. In particular, we extend Lo's classical semiparametric closed-form bound for European call options by considering third-order moment information, and by finding a related semiparametric bound on the option's risk. Furthermore, we present a closed-form solution for a class of arbitrage bounds. We show how the latter result can be used to obtain a novel model for portfolio allocation with desirable properties for today's investors.

4 Scientific Progress Made and Outcome of the Meeting

Intensive discussions amongst the participants resulted in several new collaborations. The " second chances" offered a wonderful opportunity for lively question-answer sessions. The feedback from the participants was in general very positive. Here is a list of some of the collaborations initiated during this meeting. Many of these have already resulted in preprints or submitted papers (see the preprints in the bibliography below dated October or November 2006).

Putinar is collaborating with Kuhlmann on: the decision problem for systems of polynomial inequalities in a free *-algebra. We also collaborated on sparse polynomials [10]. Putinar, Lasserre and Helton started a collaboration on: sparsity pattern of inverses of truncated moment matrices associated to the multivariate distribution of independent random variables. Putinar and Schmüdgen started a collaboration on: an extension of Marcel Riesz ρ function for moment data in two variables; this function being the classical and most refined object to study moment sequences in one dimension.

An old paper of Gondard and Ribenboim [4] - in french - was quoted in Schmüdgen's talk, moreover this finally resulted into a new, constructive proof found by Hillar and Nie [6] , of the main theorem of [4].

Laurent and Schweighofer started collaborating on the project of writing an extensive survey on the recent developments on the use of moment matrices and sums of squares of polynomials in optimization. Schweighofer remarked that this very nice workshop is perhaps the nicest he has ever attended.

Reznick was inspired by the company and the interest to discover sufficiently simple proofs of Hilbert's construction [16].

Theobald used the breaks and evenings to exchange further ideas with Lasserre continuing work on symmetries, and also had interesting discussions with other participants on this.

Cimpric, Marshall and Kuhlmann discussed some further planned joint work on Invariant saturated preorders.

Cimpric, Scheiderer and Kuhlmann used this opportunity to discuss final details of [2] before submitting it.

5 Concluding Remarks

This meeting in the Banff Centre has been very interesting and enjoyable. Talks of one hour give a much better understanding than shorter talks as in some others meetings. The combination of optimization and real algebraic geometry is currently extremely fruitful. The workshop was an excellent opportunity for researchers in optimization to gain access to the new mathematical tools related to positive polynomials. This was one of the few conferences where pure mathematicians were able to interact and exchange ideas/results with applied researchers in optimization.

It is of great benefit to mathematical research in Canada to hold such workshops in this country. The Banff centre attracts visitors from abroad, and this provides an opportunity for Canadian researchers to invite these visitors further to their own institutions. A mini-conference " Topics in Real Algebraic Geometry" (http://math.usask.ca/ skuhlman/confgond.htm) was held at the University of Saskatchewan immediately before the Banff Workshop , to give an opportunity to the "young researchers" among the BIRS workshop participants to give talks in a less formal context.

D. Augustin lectured on The membership problem for preorders. She showed that the membership problem for a preorder in the polynomial ring over a real closed field is solvable if the set of coefficients of the polynomials which belong to the preorder is weakly semialgebraic; this means: the intersection of this set of coefficients with every finite dimensional subspace of the polynomial ring is semialgebraic. In the talk she concentrated on finitely generated preorders in the polynomial ring over the field of real numbers in one variable. By describing the structure of preorders generated by one polynomial in the local power series ring at a point she derived conditions which are necessary and sufficient for a polynomial f to lie in the preorder generated by another polynomial g if the basic closed semialgebraic set generated by g is compact. These conditions imply that the membership problem is solvable for preorders generated by one single polynomial.

Netzer lectured on The doubledagger condition: Recall that a finitely generated preordering in the real polynomial ring is said to have the Strong Moment Property, if its closure with respect to the finest locally convex topology equals its saturation. In the literature, there has been introduced a property which implies the Strong Moment Property, known as the "Double-Dagger Property". He presented an example which shows that this property is strictly stronger than the Strong Moment Property. And unlike the Strong Moment Property, a preordering does not necessarily have it if all preorderings corresponding to the fibers of some bounded polynomials have it.

Plaumann (Ph.D in Konstanz, Germany) lectured on "Sums of squares on reducible real curves": Scheiderer has classified all irreducible real affine curves for which every non-negative regular function is a sum of squares in the coordinate ring. Plaumann showedhow to extend some of these results to reducible curves. He also discussed the moment problem for reducible curves and applications to the moment problem in dimension 2.

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