Random Matrices, multi-orthogonal Polynomials and Riemann-Hilbert Problems

Organizer: John Harnad (Department of mathematics and Statistics, Concordia University and Centre de recherches mathématiques, Université de Montréal),
Participants: Marco Bertola, (Department of mathematics and Statistics, Concordia University and Centre de recherches mathématiques, Université de Montréal)
Betrand Eynard (SPhT CEA Saclay)
Ken McLaughlin, (Department of Mathematics, University of Arizona)

30 April–14 May, 2005

The objective of this collaborative project was to further advance the computation of large N asymptotics in multimatrix models by extending the previously known methods used in 1-matrix models [8, 5] and applying them to the Riemannn-Hilbert formulation of multi-orthogonal polynomials developed in [2, 9].

The specific objectives were;

1. To relate the "dual" formulations of the Riemann-Hilbert problem characterizing biorthogonal polynomials obtained by the different members of this group [1, 2, 3, 9].

2. To extend the asymptotic analysis, based on the Riemannn-Hilbert method, and variational equations, to obtain rigorous large N asymptotics for the partition function in 2-matrix models [6, 7], the equilibrium distributions for the eigenvalues, and correlation functions in terms of asymptotics of the associated biorthogonal polynomials.

Considerable progress was already made on item 1 between the planning of this meeting and the actual event. The relation between the two different approaches to the Riemann-Hilbert problem for biorthogonal polynomials was in fact completely determined by M. Bertola and J. Harnad, in colllaboration with A. Its, in the months prior to the meeting, and these results were communicated to the other members of the group at the beginning of the meeting. The full details are currently being written up in final form, but a preliminary version is now available in the preprint [4].

The essential difference between the two approaches was that, whereas the large argument asymptotics in the formulation ref. [9] were fairly simple, involving only exponentials and power law dependence on the arguments, the jump discontinuities across the integration contours on which the biorthogonality is defined involves transcendental nonconstant dependence. In the approach of [2] however, the jump discontinuities are piecewise constant, but the large argument asymptotics involve fractional powers of the arguments and have sectorial behaviour, with Stokes matrices relating the different sectors. Moreover, only the "dual" fundamental systems were given an explicit integral representation in [2], with the asymptotics of the "direct" systems determined through the invariant bilinear pairing. The new approach, described in [4], gives an integral representation also for the "direct" fundamental systems, and these integral representations are used to deduce the sectorial large argument asymptotics and jump discontinuities explicitly, as well as the differential equations satisfied, without recourse to either the "folding" methods used previously, or further algebraic manipulations based on infinite recursions. Moreover, the integral representation in [4] is shown to factorize into a product of: 1) an explicitly known matrix factor, which is constant in the arguments of the system, though not in the degrees of the biorthogonal polynomials; 2) the integral representation of [9]; 3) a matrix factor that is independent of the polynomial degree N, consisting of the Wronskian matrix of an associated higher degree constant coefficient equation whose coefficients are determined by the polynomial potential. This factorization relation shows how the transcendental jump matrix in the system of [9] is transformed into a constant one, while at the same time introducing the N-independent sectorial behaviour that is required.

Since this point was already resolved by the start of the meeting, the remaining time could be devoted to addressing the set of problems listed under item 2. In fact, considerable progress in this direction had also already been made prior to the meeting, so the actual time spent at BIRS could, in part, be devoted to communicating this further progress, and to planning out the future steps needed for fully resolving these problems. The progress regarding large N asymptotics was made, partly on a heuristic, and partly a rigorous basis, by B. Eynard [6, 7].

He explained to the others in the group:

1. How the three different versions of the 2-matrix models, the "Normal" model, the "symmetry broken" normal model, and the "formal" model are related. The first of these, which is the one studied in [1, 2, 3] is based on integration on homology classes of contours; the second is based on grouping together multiple integrals by partitions of N in which the parts indicate the number of factors in the multiple integrals along a given contour and the third, the "formal" model, is based on a combinatorial definition of the partition function involving the multiplication of the weights of Feynmann graphs associated with a perturbative development about a Gaussian measure and evaluation of the integrands at the critical point contributions via gaussian integration.

2. How the existence of an "equilibrium" spectral curve may be deduced from a suitable definition of the free energy $\mathcal{F}_0$, which coincides with $\frac{1}{N^2}$ times the logarithm of the partition function in the "formal" model. This definition can be given on any "spectral curve" of the general form deduced from the "loop equation" [6] (which follows from the reparametrization invariance of the partition function),

$$E(x, y) = -(V_1'(x) - y)(V_2'(y) - x) + P(x, y),$$

   where $V_1(x)$ and $V_2(y)$ are the polynomial potentials, of degrees $d_1 + 1$ and $d_2 + 1$, respectively, defining the biorthogonality measure and $P(x, y)$ is a polynomial of degree $\leq d_1 - 1$ in x and $\leq d_2 - 1$ in y. The free energy is given by residue formulæ involving the meromorphic differential ydx, which determine $\mathcal{F}_0$ as a functional on the moduli space of algebraic curves of the above the form. Its real part may be shown to be a convex function. The extrema are therefore well-defined, and the variational equations for these imply the vanishing of the real parts of the cycles of the abelian integral $\int ydx$ on the curve around any cycles.

3. Explicit forms - partly conjectural, partly proved, expressing the asymptotic forms of the fundamental systems of refs. [2, 4] in terms of ratios of Riemann theta functions on the equilibrium curve. Since these formulæ were deduced assuming the applicability of saddle point and WKB techniques which require more rigorous justification, the Riemann-Hilbert method is required to complete the analysis.

During the remainder of the two week period of the meeting, preliminary calculations were undertaken with a view to determining the branch cut structure for the Riemann surface of the spectral curve in the case when the potentials are even quartic polynomials, and the genus of the curve is 0. The purpose was to determine a contour that is homologically equivalent to the contour of integration on which the various transformations may be applied to reduce the Riemann-Hilbert problem to factors that differ from the identity only by terms that are exponentially decreasing. This is accomplished through the introduction of a generalization of the g-function, as done for the 1-matrix case in ref. [5]. The definition of this g-function seems now to be clear: it is the multivalued function defined by the abelian integral $\int ydx$ on the spectral curve.

These preliminary calculations for the genus 0 case appear to lead to the correct cut structure that should arise. It also appears, from these preliminary discussions and calculations, that the sequence of gauge transformations and deformations of the contours along which the jump discontinuities arise can be correctly defined by virtue of the analyticity and asymptotic properties of the g-function. Moreover, the presence of sectorial asymptotics in the Riemann-Hilbert problem appears to be consistent with the large N asymptotic properties of the g function, making this approach to the large N asymptotics of the biorthogonal polynomials very likely the correct one for implemention. In addition, these considerations potentially reveal a connection between (i) a physically motivated existence and uniqueness theorem for the equilibrium spectral curve, and (ii) nonlinear steepest descent analysis of the associated Riemann-Hilbert problem.

Much further work will be needed, but the preliminary results, and the general method of approach laid out at this BIRS meeting, seem to give very good promise for further development of an ongoing program that should lead to the resolution of the main unresolved questions on the large N asymptotics of 2-matrix models and biorthogonal polynomials.

References

[1] M. Bertola, B. Eynard and J. Harnad, “Duality, Biorthogonal Polynomials and Multi–Matrix Models”, Commun. Math. Phys. **229**, 73–120 (2002).

[2] M. Bertola, B. Eynard and J. Harnad, “Differential systems for biorthogonal polynomials appearing in 2-matrix models and the associated Riemann-Hilbert problem”, Commun. Math. Phys. **243**, 193–240 (2003).

[3] M. Bertola, B. Eynard and J. Harnad, “Duality of spectral curves arising in two-matrix models” *Theor. Math. Phys. Theor. Math. Phys.* **134**, 27-38 (2003).

[4] M. Bertola , J. Harnad and A. Its “Dual Riemann-Hilbert Approach to Biorthogonal Polynomials”, preprint CRM (2005).

[5] P. Deift, T.Kriecherbauer, K.T.R.McLaughlin, S. Venakides, X. Zhou, “Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory”, *Commun. Pure Appl. Math.* **52**,1335-1425 (1999).

[6] B. Eynard, “Master loop equations, free energy and correlations for the chain of matrices” *J. High En. Phys.* **11**, No. 018 (2003).

[7] B. Eynard, A.Kokotov A and D. Korotkin, “Genus one contribution to free energy in Hermitian two-matrix model”, *Nucl. Phys.* **B 694**, 443-472 (2004).

[8] A. Fokas, A. Its, A. Kitaev, “The isomonodromy approach to matrix models in 2D quantum gravity”, *Commun. Math. Phys.* **147**, 395–430 (1992).

[9] A. B. J. Kuijlaars and K. T-R McLaughlin “A Riemann-Hilbert problem for biorthogonal polynomials”, *J. Comput. Appl. Math.* **178**, 313-320 (2005).