Topological Methods for Aperiodic Tilings

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In the 1960's and 1970's, mathematicians discovered geometric patterns which displayed a high degree of regularity, and yet were not periodic [7]. The subject also gained enormous importance with the discovery of physical materials (quasicrystals) with pure point x-ray diffraction spectrum, which indicates a highly ordered atomic structure, and yet symmetry patterns in that spectrum which could not be produced by periodic atomic structures [21, 8]. Since that time, the subject has grown substantially. In doing so, it has drawn on a highly diverse collection of mathematical ideas.

One very productive idea is to regard a tiling as producing a dynamical system [14, 17, 16, 15, 20]. First, all translates of the tiling are considered, Then a metric is placed on such tilings. This arises from natural ideas in symbolic dynamics for discrete patterns, but these must be adapted to handle the geometry of Euclidean space. The translation action of the Euclidean space extends to the completion of this metric space. Under fairly mild assumptions, the space obtained, called the hull of the tiling, is compact and so provides a natural setting for using techniques from dynamics. Eventually, it has been realized that this space actually contains a great deal of interesting and computable (from standard topological techniques) information on the original tiling.

The standard assumption through much of the literature is that of 'finite local complexity' or FLC: for a fixed R, the number of different patterns in the tiling of diameter less than R, is finite modulo translation [13]. Moreover, it has been known for a long time that, under FLC the hull is locally the product of a totally disconnected set and $\mathbb{R}^d$, where d is the dimension of the tiling [16, 1, 19]. Moreover, it can been presented as inverse limit of fairly simple cell-complexes. The first natural generalization of FLC is to relax the condition 'modulo translation' to allow more general groups of isometries. This leads to G-FLC, where G is the appropriate group. This has already appeared in the work of many authors (for example, see [4, 18]. However, there are a number of interesting examples where this hypothesis fails, but this can happen in several ways. Several of the participants, Natalie Priebe-Frank, Sadun and Kellendonk, in particular, had been considering such examples, and through the course of the two weeks a unifying view of the metric was achieved. In some cases, the approximation by cell complexes seemed possible. If successful, this could lead to extending computations of cohomology invariants for new classes of tilings. Under study by a fairly large part of the FRG, including Priebe-Frank, Sadun, Kellendonk, Putnam, Hunton, Barge and Diamond, progress was made in understanding them within a global framework. Papers on this subject should be forthcoming shortly. Some other examples of non-FLC tilings were presented by Bellissard, arising from mathematical models of amorphous materials. Here, it seems that new ideas are needed to provide a better understanding of the hull.

There was a general theme for the FRG of trying to understand the nature of the cohomology of the tiling and its physical interpretations. This cohomology is closely linked witht the K-theory of the C^*-algebras associated with the tiling, first constructed by Bellissard and investigated by Forrest, Hunton, Kellendonk, Putnam and others [2, 5, 10, 11, 1, 6]. There were some interesting new interpretations made of how parts of this K-theory could arise from lower dimensional phenomena in the tiling. At a physical level, these could lead to measuring defects in physical materials. Several discussions elaborated links with groupoid cohomology and other interpretations of the cohomology.

A lot of progress was also made in computational methods and results. The use of spectral sequences for these calculations was studied intensely. Recently, tilings were discovered with a non-trivial torsion component in the cohomology. This rather surprising phenomena was investigated and discussed by Gähler, Hunton and Kellendonk. A great deal of progress was made on the calculation of several specific tilings of interest. Most notable was the pinwheel. But there were other examples, where full rotational symmetry was considered. This was the first time sufficient expertise and time had been brought to bear on these computations. Kalugin presented some very novel approaches to the understanding of matching rules from a topological view, leading to new methods for cohomology computation [9]. Recent work of Kellendonk and Putnam on their notion of pattern equivariant cohomology was presented [12]. The group spent some time developing this as an alternate view of cohomology for hulls, and indeed as a view of the hull itself, which seems very useful.

In the special case of one dimensional tilings, Barge and Diamond have a number of quite strong invariants. Moreover, a number of rather precise statements of the hull can be made. These were discussed, especially with an idea to trying to extend this program to higher dimensions.

One of the most popular features of the two weeks were the tutorials. It should be stressed that the common interest was in topological aspects of aperiodic materials, but the participants came from a remarkably wide range of backgrounds: mathematical physics, algebraic topology, operator algebras, dynamical systems, discrete geometry, ... Each day, long tutorials were presented, essentially aimed at novices, of technical tools from these different areas. For example, the use of spectral sequences for these cohomological calculations is crucial, yet only an expert in algebraic topology has this in his tool kit. All participants really gained a lot from some exceptionally revealing presentations.

A large number of other related topics were covered in various presentations: the Aubrey-Mather theory for quasi-crystals, relations with translation surfaces and orbit equivalence for Cantor minimal systems to name a few.

References

[1] J. E. Anderson and I. F. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras*, Ergod. Th. & Dyn. Syst. **18** (1998), 509–537.

[2] J. Bellissard, *K-theory of C^*-algebras in solid state physics*, in: *Statistical Mechanics and Field Theory: Mathematical Aspects*, eds. T. C. Dorlas, N. M. Hugenholtz and M. Winnik, Lecture Notes in Physics, vol. 257, Springer, Berlin (1986), pp. 99–156.

[3] J. Bellissard, *Gap labelling theorems for Schrödinger's Operators*, in: *From Number Theory to Physics*, eds. M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson, Springer, Berlin (1993), pp. 539–630.

[4] R. Benedetti and J.M. Gambaudo, *On the dynamics of G-solenoids; applications to Delone sets*, Ergod. Th. & Dyn. Syst. **23** (2003), 673-692.

[5] A. Connes, *Non-commutative Geometry*, Academic Press, San Diego (1994).

[6] A.H. Forrest, J. Hunton, and J. Kellendonk, *Cohomology groups for projection point patterns*, XIIIth ICMP (London, 2000), 333–339, Int. Press, Boston, MA, (2001),

[7] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, Freeman, New York (1987).

[8] C. Janot, *Quasicrystals – a Primer*, 2nd ed., Clarendon Press, Oxford (1994).

[9] P. Kalugin, *Cohomology of quasiperiodic patterns and matching rules*, J. Phys. A: Math. Gen. 38 No 14 (2005) 3115-3132.

[10] J. Kellendonk, *Non-commutative geometry of tilings and gap labelling*, Rev. Math. Phys. **7** (1995), 1133–1180.

[11] J. Kellendonk, *The local structure of tilings and their integer group of coinvariants*, Commun. Math. Phys. **187** (1997), 115–157.

[12] J. Kellendonk. *Pattern-equivariant functions and cohomology*. J. Phys. A **36**(2003). 1-8.

[13] J. C. Lagarias, *Geometric models for quasicrystals I. Delone sets of finite type*, Discr. Comput. Geom. **21** (1999), 161–191; and: *Geometric models for quasicrystals II. Local rules under isometries*, Discr. Comput. Geom. **21** (1999), 345–372.

[14] S. Mozes, *Tilings, substitution systems and dynamical systems generated by them*, J. d'Analyse Math. **53** (1989), 139–186.

[15] C. Radin and M. Wolff, *Space tilings and local isomorphism*, Geom. Dedicata **42** (1992), 355–360.

[16] E. A. Robinson, Jr., *The dynamical theory of tilings and quasicrystallography*, in: *Ergodic Theory of* $\mathbb{Z}^d$*-Actions*, eds. M. Pollicott and K. Schmidt, London Math. Soc. Lecture Note Series, Cambridge University Press, Cambridge (1996), pp. 451–473.

[17] D. J. Rudolph, *Markov tilings of* $\mathbb{R}^d$ *and representations of* $\mathbb{R}^d$ *actions*, Contemp. Math. **94** (1989), 271–289.

[18] L. Sadun, *Tiling Spaces are Inverse Limits*, Journal of Mathematical Physics **44** (2003) 5410-5414.

[19] L. Sadun and R.F. Williams, *Tiling spaces are Cantor set fibre bundles*, Erd. Th. & Dyn. Sys., **23** (2003), 307-316.

[20] B. Solomyak, *Dynamics of self-similar tilings*, Ergod. Th. & Dyn. Syst. **17** (1997), 695–738.

[21] P. J. Steinhardt and S. Ostlund (eds.), *The Physics of Quasicrystals*, World Scientific, Singapore (1987).