# New Techniques in Lorentz Manifolds

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Recently discovered examples of Lorentz manifolds have renewed interest in the field among group theorists, differential geometers, topologists and dynamicists. The purpose of the November 6 BIRS workshop was to assemble specialists in these fields to discuss these new discoveries.

A *Lorentz manifold* is a manifold with an indefinite metric of index 1. Such structures arise naturally in relativity theory and, more recently, string theory.

Unlike the considerably more familiar *Riemannian manifolds* (with metric tensors of index 0), Lorentzian manifolds are poorly understood. Basic global questions remain unanswered, even for Lorentzian manifolds of constant curvature.

The simplest example is *Minkowski space* $\mathbb{R}^n_1$, a real affine space of dimension $n$, with a non-degenerate inner product of index 1. Although its compact quotients have been classified [17], its noncompact quotients, and more generally manifolds locally isometric or conformal to it are still mysterious. Closely related are the model constant curvature Lorentz manifolds, namely de Sitter space $\mathbb{S}^n_1$ and anti-de Sitter space $\mathbb{AS}^n_1$. Constant curvature *Riemannian* manifolds are also Lorentz manifolds.

Some of the topics discussed during the workshop included:

- Foliations of Lorentz manifolds and globally hyperbolic spacetimes;
- Global hyperbolicity in constant curvature manifolds;
- Conformal Lorentzian dynamics;
- Fundamental domains in anti-de Sitter space;
- Spinors on Lorentz manifolds;
- Topology of the future causal boundary of a spacetime.

We expand here on topics that generated discussion in the "open problems" session, and possible new research directions. The workshop facilitated many discussions which led to several new results.

## 1 Affine spaces; Margulis spacetimes

In 1977 Milnor [23] asked whether a nonabelian free group acts properly by affine transformations of $\mathbb{R}^n$. He suggested taking a discrete free subgroup $\Gamma_0$ of $SO(2,1)$ (for example a Schottky group) and "adding translational components" (that is, an *affine deformation*) to make the group act properly. In 1983 Fried-Goldman [15] reduced the classification of complete affine 3-manifolds to Milnor's

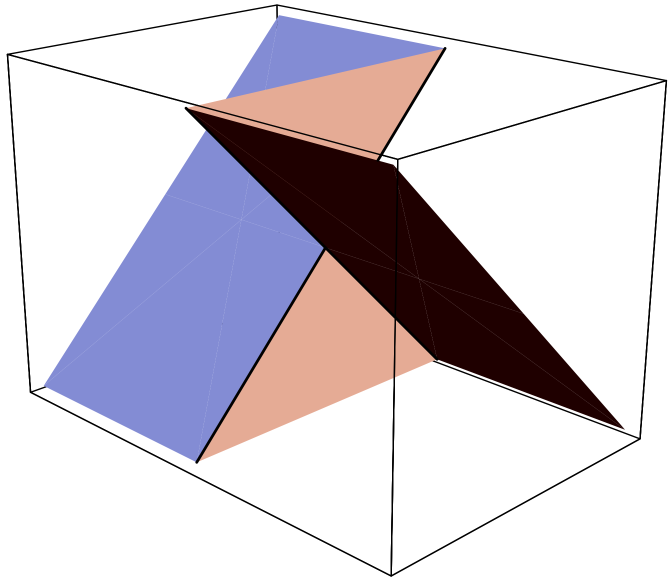

Figure 1: A crooked plane

question. Also in 1983 Margulis [20] constructed proper free actions of nonabelian free groups, answering Milnor's question. Margulis's examples were startling and unexpected.

In his 1990 doctoral thesis, Drumm [11] gave examples by constructing fundamental domains for such actions, using polyhedra called *crooked planes*. A crooked plane is depicted in Figure 1 and the intersections of a tiling of $\mathbb{R}^2_1$ by crooked planes by a horizontal plane are depicted in Figures 2 and 3.

Hence the interest in flat Lorentz 3-dimensional space forms, or *Margulis spacetimes*. Margulis found a criterion for a group $\Gamma$ to *not* act properly. The Margulis invariant of an affine hyperbolic transformation measures signed Lorentzian displacement along an invariant spacelike line. When $\Gamma$ acts properly and contains no parabolics, the quotient spacetime $M = \mathbb{R}^{2,1}/\Gamma$ enjoys the property that every essential loop is freely homotopic to a unique closed geodesic (necessarily spacelike). The absolute value of the Margulis invariant is the *signed Lorentzian length spectrum* of $M$.

Margulis showed that in order for $\Gamma$ to act properly, the sign of the Margulis invariant must be constant over the group. It was conjectured that this is a sufficient condition; it was even hoped that we could find some sort of condition involving only a finite set of elements of $\Gamma$.

In the case where $\Gamma$ is a free group on two generators, this conjecture has already led to surprising findings. If $\Gamma$ is the holonomy of a three-holed sphere, it does act properly if and only if the values of the Margulis invariants for a certain "generating triple" (see below) all carry the same sign [19]. This is equivalent, via a beautiful interpretation of the signed Lorentzian length, to a result by Thurston: all closed geodesics of a hyperbolic three-holed sphere are shortened (resp., lengthened) if the three bounding closed geodesics are shortened (resp., lengthened) [33].

In the case of the punctured torus, the conjecture was answered in the negative by showing that there is no hope to ensure properness of an action by a "same sign" condition on a finite number of elements of the group [6].

In each case, $\Gamma$ is a free rank two subgroup. Thus the moduli space of affine deformations depends on three parameters, namely, the values of the Margulis invariant for a pair of generators and their product – call these a *generating triple*. In fact, since an affine deformation may be considered up to rescaling without loss of generality (this corresponds to rescaling Minkowski space), the real projective plane is the moduli space of affine deformations of $\Gamma$, and by Margulis' result, the proper deformations are bounded by the triangle with homogeneous coordinates (1,0,0), (0,1,0) and (0,0,1).

The contrast between the two cases is evident in Figures 4 and 5. Each line corresponds to a word

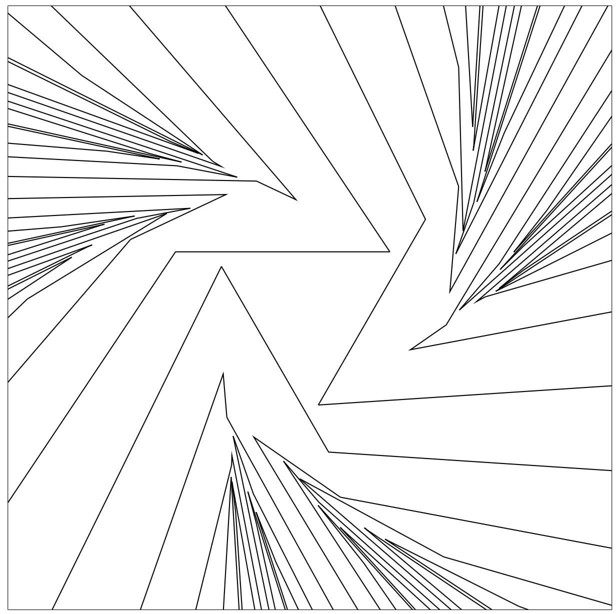

Figure 2: Cross-section of a crooked tiling

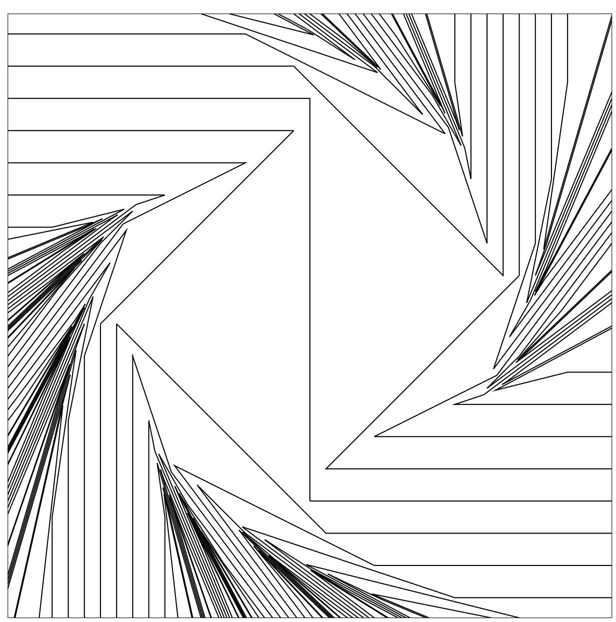

Figure 3: Proper affine deformation of the modular group

in the abstract group associated to $\Gamma$: it is the space of deformations whose Margulis invariant for that word is zero. In the case of the three-holed sphere (Figure 4), the triangle of positive values for the generating triple appears to be contained in the intersection of the positive half-planes. However, in the case of the punctured torus, there are subsets for which every element of the generating triple admits a positive Margulis invariant, but some word in the group does not.

Crooked planes were discussed in Drumm's talk. He described the conjectural relationship between crooked planes and the Margulis invariant. The finite determination of the Margulis invariant was discussed in Charette's lecture [8].

As for the original conjecture, it is now believed to be false and that instead, one must consider an extension of the Margulis invariant, which we outline here.

Set $E$ to be the affine space modeled on $\mathbb{R}^3_1$ and let $\Gamma$ be a free rank two group of isometries of $E$, such that its linear part $\dot{\Gamma}$ is a convex cocompact subgroup of $SO(2,1)$ – thus $\dot{\Gamma}$ is discrete and finitely generated, and $\Sigma = H^2/\dot{\Gamma}$ has no cusps.

Consider the flat Minkowski bundle $\tilde{E} \to U\Sigma$. The affine deformation $\Gamma$ corresponds to a cocycle class in $H^1(\dot{\Gamma}, E)$, which in the de Rham interpretation corresponds to a class $\omega \in H^1(\tilde{E})$. The bundle $\tilde{E} \to U\Sigma$ admits a preferred spacelike section $\nu$, that is an extension of $e_\gamma$, the preferred unit-spacelike eigenvector of $\gamma$ which appears in the definition of the Margulis invariant. The following function is not uniquely defined:

$$\begin{aligned} f : U\Sigma &\longrightarrow \mathbb{R} \\ (x,u) &\longmapsto \langle \omega(X), \nu \rangle, \end{aligned}$$

where $X$ is the generator of the geodesic flow. However, given a probability measure invariant by the geodesic flow, $\lambda$, the following only depends on the cohomology class of $\omega$:

$$\mu(\lambda) = \int_{U\Sigma} f \, d\lambda.$$

Goldman, Labourie and Margulis have shown that $\Gamma$ acts properly on $E$ if and only if the sign of $\mu(\lambda)$ is constant over all $\lambda$ [16]. Here are some open problems remaining around this question. (See also Section 6.)

- Is the Goldman-Labourie-Margulis theorem the sharpest possible? It is believed to be so, that is, that $\Gamma$ may not act properly on $E$, even though the sign of the Margulis invariant is constant over the group.
- Extend the result to the case when $\Sigma$ has cusps, i.e. when $\Gamma$ admits parabolic elements.

Margulis's original definition of the signed Lorentzian length was extended to include parabolic elements by Charette and Drumm [9].

# 2 Surfaces in Lorentz space-forms

This was the subject of Schlenker's talk, as well as Pratoussevitch's talk. While Schlenker discussed the extension of Aleksandrov's theorem to Minkowski space, Pratoussevitch described a surprising construction of fundamental polyhedra for $\mathbb{AS}^3_1$-structures on Seifert 3-manifolds.

A theorem of Aleksandrov states that any metric on the two-sphere $S^2$ with curvature $K > -1$ is induced on a unique convex surface in $H^3$, three-dimensional hyperbolic space. Schlenker and Labourie have worked on the analogous problem in de Sitter space $S^3_1$. In particular, the same result holds in $S^3_1$, except that the curvature is now bounded above by one and the closed geodesics must have length greater than $2\pi$. Let $\Sigma \subset H^3$ be a smooth, strictly convex surface; denote by $I$ the induced metric. We define the third fundamental form on $\Sigma$ to be:

$$III(X,Y) = I(\nabla_X N, \nabla_Y N),$$

where $N$ is the unit normal vector. Then there is a dual statement to Aleksandrov's theorem: any metric $h$ on $S^2$ with curvature less than one and whose closed geodesics have length greater than

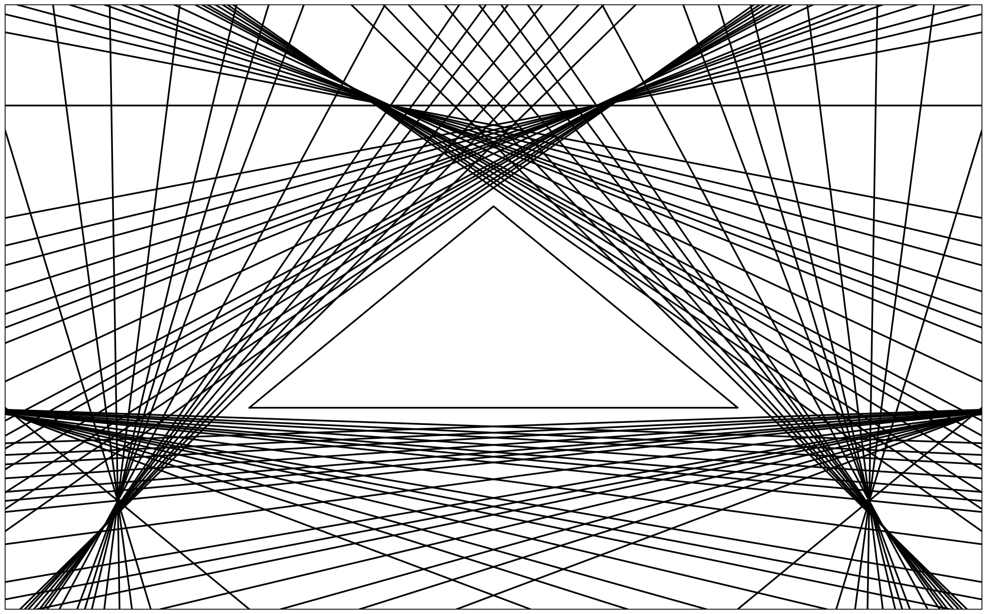

Figure 4: The moduli space of proper affine deformations of a hyperbolic 3-holed sphere

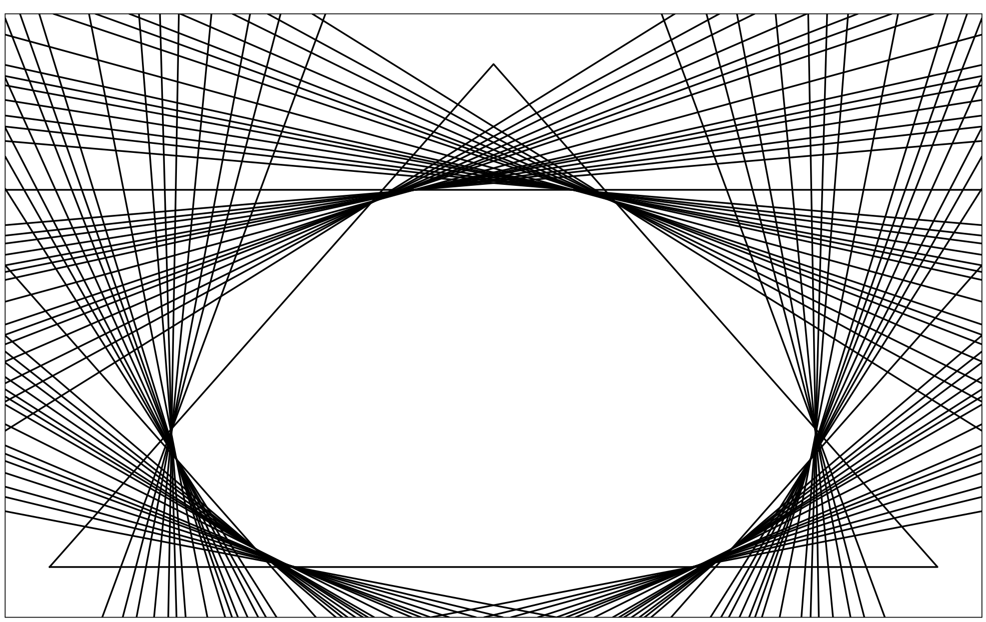

Figure 5: The moduli space of proper affine deformations of a hyperbolic 1-holed torus

$2\pi$ is the third fundamental form of a unique convex surface in $H^3$. This follows from the duality between surfaces in $H^3$ and surfaces in $S^3_1$, which we will outline here.

Let $\Sigma \subset H^3$; for $p \in \Sigma$, the tangent space at $p$ corresponds to a plane in $H^3$, which in turn admits a polar point $p^* \in S^3_1$. The induced metric $I^*$ on the polar surface turns out to be $III$.

Thus any statement concerning convex surfaces in $H^3$ translates to a dual statement in $S^3_1$.

Let us define a *Fuchsian equivariant embedding* of a surface. Given a surface $\Sigma$ of genus $g \geq 2$, an *equivariant embedding* of $\Sigma$ in $H^3$ (resp. $S^3_1$, $H^3_1$) is a pair $(\phi, \rho)$, where:

- $\phi$ is an embedding of $\tilde{\Sigma}$ into $H^3$ (resp. $S^3_1$, $H^3_1$);
- $\rho$ is a monomorphism of $\pi_1(\Sigma)$ into the isometry group of $H^3$ (resp. $S^3_1$, $H^3_1$) such that, for every $x \in \Sigma$ and $\gamma \in \pi_1(\Sigma)$,
$$\phi(\gamma x) = \rho(\gamma)\phi(x).$$

An equivariant embedding $(\phi, \rho)$ is *Fuchsian* if it fixes a totally geodesic plane in $H^3$ (resp. a point in $S^3_1$, $H^3_1$).

In this context, Aleksandrov's theorem is stated as follows: a convex surface $\Sigma$ with curvature $K > -1$ admits a unique Fuchsian equivariant embedding into $H^3$, such that $I = h$. Dually, if $K < 1$ and every closed geodesic has length greater than $2\pi$, $\Sigma$ admits a unique Fuchsian equivariant embedding into $H^3$ such that $III = h$.

In the anti-de Sitter world, analogous statements hold. Namely, a surface $\Sigma$ with metric $h$, whose curvature is bounded *above* by -1, admits a unique equivariant embedding into $H^3_1$ such that $I = h$. Dually, the same result holds with $h = III$ instead of $I$.

Now, Aleksandrov's theorem is a special case of a statement about hyperbolic three-manifolds with convex boundary. Let $h_\pm$ be metrics on a convex surface $\Sigma$ with curvature $K > -1$. Then there exists a unique hyperbolic metric $g$ on $\Sigma \times [-1, 1]$ such that the induced metric on each component of the boundary is given by $h_+$, $h_-$, respectively. Dually, the same statement holds for $K < 1$, as long as the lengths of the closed geodesics are greater than $2\pi$, substituting the third fundamental form for $I$.

In the anti-de Sitter world, all evidence points to the existence of an analogous statement; but this remains conjectural.

# 3 Causality

Perhaps the most salient feature of a Lorentzian structure is its underlying causality structure. Unlike Riemannian manifolds, geodesics (and more generally, smooth curves) come in several flavors, depending on the restriction of the metric tensor to these curves. Steve Harris described the notion of the ideal causal boundary on Lorentz manifolds [18].

In a sequence of talks, Thierry Barbot and François Begun described their joint work [3] with Zeghib, on foliating globally hyperbolic 3-dimensional spacetimes by constant mean curvature surfaces. The principal result is that every maximal such spatially compact Lorentzian manifold admits a *time function,* that is, a function which increases along future-directed timelike curves.

# 4 Lorentzian Foliations and Group Actions

The subject of Riemannian foliations (that is, foliations whose holonomy groupoid preserves a transverse Riemannian metric) was developed in the 1970's and 1980's. Pierre Mounoud presented his recent work [24, 25, 26] on Lorentzian foliations at the workshop.

Frances's lecture dealt with the extension of Obata's theorem to Lorentz manifolds. Obata proved that the only Riemannian manifolds which admit noncompact conformal automorphism groups are Euclidean space $\mathbb{R}^n$ and the Euclidean sphere $\mathbb{S}^{n-1}$. Frances gave surprising examples of compact conformally flat Lorentz manifolds whose automorphism groups are noncompact. Furthermore he discussed which 3-manifolds support such *essential* flat Lorentzian conformal structures. This was the topic of his recent doctoral thesis [12, 13, 14].

In a different direction, Karin Melnick discussed her recent results, concerning which groups can act by Lorentzian isometries on compact manifolds. Building on earlier work of Zeghib [34, 35] and Adams-Stuck [2], she showed that the possible connected isometry groups of a compact connected Lorentz manifold have the form $K \times R^m \times S$, where $K$ is compact and $S$ is locally isomorphic to one of the following:

- $PSL_2(R)$;
- a Heisenberg group $H_n$;
- one of a countable family of solvable extensions isomorphic to $S^1 \ltimes H$, where $H$ is a Heisenberg group.

She went on to describe which manifolds admit an action of the Heisenberg group, particularly one of codimension one–where the dimension of the Heisenberg group is one less than the dimension of the manifold. This work, recently posted to the archives [21], may be part of her forthcoming doctoral thesis.

Closely related to Killing vector fields are *Killing spinor fields* which generalize to *conformal Killing fields.* In her talk, Helga Baum showed how conformal Killing spinor fields lead to new examples of manifolds with essential Lorentzian conformal structures. In particular if the associated vector field is lightlike, then the manifold is one of a few special types (for example, a strictly pseudoconvex boundary of a domain (*a Fefferman space,* or a circle bundle over a Kähler manifold). The proof [4] involves a careful analysis of the zero-set of a conformal Killing spinor field.

# 5 Low-dimensional Topology and other topics

The workshop benefited from several lectures which were not exactly on the topic of the conference, but nonetheless closely related. Suhyoung Choi presented his solution [10] of Marden's Tameness Conjecture for hyperbolic 3-manifolds (proved independently by Agol and Calegari-Gabai).

Dave Morris lectured on which arithmetic groups can act on the line.

Kevin Scannell discussed deformations of hyperbolic 3-manifolds, which through work begun in his thesis [30, 31, 32], closely relate to $\mathbb{R}^3_1$-manifolds.

# 6 Problem session

The items outlined above represent just a sample of the topics discussed at the workshop. On the last day a problem session was held. Here is a list of some of the problems which were suggested:

1. (Labourie) Mess shows that compact oriented orthochronous $2+1$ AdS spacetime with non-empty spacelike boundary $S$ is a product $S \times [0,1]$ and embeds in a domain of dependence. Is it possible to construct a singular AdS manifold with more than two ends, say by branching on a spacelike geodesic in a domain of dependence?

2. (Scannell) Generalize the "no topology change" theorem of Mess noted above to all constant curvature $3+1$ spacetimes. Or (even better) characterize when a constant curvature $3+1$ maximal domain of dependence embeds in a larger constant curvature spacetime.

3. (Schlenker) Let $M$ be a compact AdS cone manifold with $m$ singular curves. Given real numbers $\dot{\alpha_1}, \dots \dot{\alpha_m}$, is there a first order deformation of the AdS structure inducing these derivatives of the cone angles? This is related to the following problem, posed by Mess.

4. (Mess) Let $\rho = (\rho_L, \rho_R)$ be the representation of the fundamental group of a closed surface into $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ corresponding to an AdS domain of dependence.

   (a) Is $\rho$ determined by the two measured laminations on the boundary of the "convex hull"?

(b) Is $\rho$ determined by the hyperbolic structure on the future boundary of the convex hull together with the measured lamination on the past boundary?

(c) Is $\rho$ determined by $\rho_L$ together with the hyperbolic structure on one of the boundary components of the convex hull?

These are analogous to well-known questions about the parameterization of quasi-Fuchsian space by the pair of conformal structures at infinity, and how these relate to the bending laminations and hyperbolic structures on the convex hull boundary.

5. (Barbot) Let $M_0$ be a globally hyperbolic static AdS spacetime with closed spacelike slices and consider $v_0 = vol(M_0)$. Is the volume of a non-static AdS spacetime of the same topological type less than or equal to $v_0$?

6. (Schlenker) Is the volume of the convex core of a $2+1$ AdS domain of dependence strictly concave as the bending lamination varies? This question, and Barbot's question above, can be thought of as refinements of the following question posed by Mess in his preprint:

7. (Mess) For a 2+1 AdS domain of dependence, the volume of the maximal domain of dependence and of the convex core are invariants on $Teich \times Teich$. How do they behave? Are they related, perhaps asymptotically, to invariants of quasi-Fuchsian space, such as the volume of the convex core and the Hausdorff dimension of the limit set?

8. (Harris) A static complete spacetime is conformal to $(\mathbb{L}^1 \times M)/G = U$ with $G \subset Isom(M)$ for a Riemannian manifold $M$. Here $\mu : G \to \mathbb{R}$ is a homomorphism and $G$ acts on $\mathbb{L}^1 \times M$ by $g(t,x) = (t + \mu(g), g \cdot x)$. Does $\hat{\partial}(U)$ depend on $\mu$?

9. (Goldman) Let $M$ be a complete flat $2+1$ spacetime.

   (a) Does $M$ have a fundamental domain bounded by crooked planes?

   (b) Is the interior of $M$ diffeomorphic to a solid handlebody?

   (c) Do there exist natural smooth approximations of crooked planes?

   (d) (Properness conjecture). It is known that if an affine deformation of a Fuchsian group acts properly, then the value of the Margulis invariant is everywhere positive or everywhere negative. Is the converse true?

10. (Goldman) Extend crooked planes to surfaces in AdS space. Are there conformally invariant surfaces that could be used as boundaries of fundamental domains of AdS spacetimes?

11. (Abels)

    (a) Auslander Conjecture: Is every affine crystallographic group (i.e. discrete, cocompact subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ acting properly) virtually solvable?

    (b) Are there properly discontinuous affine groups (not necessarily cocompact) that are neither virtually polycyclic nor virtually free?

12. (Scannell) Characterize closed hyperbolic 3-manifolds which admit affine deformations into $Isom(\mathbb{R}^4)$. Do they always admit quasi-Fuchsian deformations into $Isom(H^4)$?

Many of the talks were influenced by Mess's unpublished preprint [22].

During the problem session it was decided to undertake the project to annotate the preprint (in order to update the results) and eventually publish it.

The organizers solicited papers based on the workshop, possibly including the updated annotated version of Mess's paper. Since one of the organizers of the workshop (Goldman) is editor-in-chief of the journal *Geometriae Dedicata,* that journal seems a particularly appropriate for such a volume.

Kevin Scannell's workshop website `http://borel.slu.edu/lorentz/index.html` facilitates communication between the participants following the workshop. In particular, the summaries of the discussions (and soon the papers arising from the workshop) will be posted there.

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