FACTORIZATION OF BIRATIONAL MAPS

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The purpose of this mini-series is to discuss the weak and strong versions of the factorization problem. The strong factorization conjecture states that a birational map between complex smooth projective varieties can be factored as a sequence of blowups of smooth centers, followed by inverses of such maps (blowdowns). The weak version of the factorization conjecture asks for a factorization where the blowups and blowdowns can come in any order.

The mini-series can be roughly divided into three parts. In the first and the second part we explain the proofs of the weak factorization conjecture for toric varieties and for general varieties, respectively. In the third part we discuss the strong factorization conjecture.

For toric varieties the factorization problems can be stated combinatorially in terms of star subdivisions of fans. Weak factorization for toric varieties was proved independently by Morelli [7] and Włodarczyk [9] (see also [2]). We explain Morelli's proof that is based on the notion of a cobordism: constructing a sequence of star subdivisions is equivalent to constructing a fan in one dimension higher, such that each maximal cone corresponds to one star subdivision of the original fan.

To go from toric to general varieties one needs to understand the analog of a cobordism for arbitrary varieties. Włodarczyk [10] noted that a cobordism corresponds to a toric variety together with an action of $\mathbb{C}^*$ on it. The intermediate varieties of the factorization are the GIT quotients obtained by varying the linearization of the action. Based on [11, 1], we show how this idea can be extended to arbitrary varieties. Given a birational map $X \dashrightarrow Y$, it is not hard to construct a variety B with a $\mathbb{C}^*$ action on it, so that X and Y are the GIT quotients $B/\mathbb{C}^*$ corresponding to different linearizations. From the work of Dolgachev, Hu [5] and Thaddeus [8] we also know that varying the linearization gives rise to a sequence of locally toric maps between the quotients. Applying the toric factorization to a locally toric map does not work directly: the two varieties have to be toroidal embeddings and the map must be compatible with the embeddings. The essential step in [1] is to convert the locally toric maps into toroidal maps, for which the toric factorization is applicable.

In the final part we discuss the strong factorization conjecture, which is open in almost all cases. For toric varieties we give a simple algorithm that conjecturally achieves strong factorization. However, the termination of this algorithm is not known even in dimension 3. We also explain how this algorithm can be reduced to commutation rules of elementary matrices.

One of the few known results in the strong case is the local strong factorization [3, 6]. The local factorization problem replaces varieties with local rings dominated by a valuation. In the toric case the valuation is given by a vector v and instead of a fan we consider a single cone containing v. We explain, using simple linear algebra, how to prove local strong factorization for toric varieties. Combining this result with Cutkosky's monomialization theorem [4] gives local strong factorization for more general rings.

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