TOROIDALIZATION OF MORPHISMS

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Suppose that $f : X \rightarrow Y$ is a morphism of algebraic varieties, over a field $\mathbf{k}$ of characteristic zero. If X and Y are nonsingular, $f : X \rightarrow Y$ is toroidal if there are simple normal crossing divisors D_X on X and D_Y on Y such that $f^*(D_Y) = D_X$, and f is locally given by monomials in appropriate etale local parameters on X. The precise definition of this concept is in [AK] (see also [KKMS]). The problem of toroidalization is to determine, given a dominant morphism $f : X \rightarrow Y$, if there exists a commutative diagram

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \Phi \downarrow & & \downarrow \Psi \\ X & \stackrel{f}{\rightarrow} & Y \end{array} \tag{1}$$

such that Φ and Ψ are products of blow ups of nonsingular subvarieties, X_1 and Y_1 are nonsingular, and there exist simple normal crossing divisors D_{Y_1} on Y_1 and $D_{X_1} = f_1^*(D_{Y_1})$ on X_1 such that f_1 is toroidal (with respect to D_{X_1} and D_{Y_1}). This is stated in Problem 6.2.1. of [AKMW].

Toroidalization, and related concepts, have been considered earlier in different contexts, mostly for morphisms of surfaces. Torodialization is the strongest structure theorem which could be true for general morphisms. The concept of torodialization fails completely in positive characteristic.

In the case when Y is a curve, toroidalization follows from embedded resolution of hypersurface singularities ([H]). When X and Y are surfaces, there are several proofs ([AK], [CP1], Corollary 6.2.3 of [AKMW], [Mat]). All of these proofs make use of special properties of the birational geometry of surfaces.

In [C2], the toroidalization problem is solved in the case when X is a 3-fold and Y is a surface. Torodialization is proven for prepared morphisms from n-folds to surfaces in [CK].

In [C5] toroidalization for birational morphisms of 3-folds is solved.

Toroidalization is solved locally along a (not necessarily discrete) valuation for arbitrary dominant morphisms in [C1] and [C4].

This minicourse will follow the following schedule:

(1) Lecture 1: Toroidalization
   (a) Statement of the problem of toroidalization and examples.
   (b) History of the problem, known results and open problems.
   (c) Proof of toroidalization of morphisms to curves.
   (d) Outline of proof of toroidalization of surfaces.
(2) Lecture 2: Outline of proof of torodialization of dominant morphisms of 3-folds to surfaces.
(3) Lecture 3: Outline of proof of torodialization of birational morphisms of 3-folds.

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