Report: Localization Behavior in Reaction-Diffusion Equations: BIRS Meeting: August 10th-16th

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The purpose of this workshop was to discuss advances in the mathematical study of localized structures in reaction-diffusion systems arising in applications. In recent years there have been many important developments in the mathematical treatment of localization phenomena in reaction-diffusion problems, especially those connected with the equilibrium or static theory associated with certain traditional classes of elliptic PDE's. However, many open problems remain, most notably those in the realm of nonlinear dynamics, bifurcation behavior, and the numerical computation of localized structures in reaction-diffusion systems. In addition, there is a strong need for researchers in this area to be exposed to new classes of PDE's involving localization phenomena that arise from a more sophisticated modeling of biological and physical problems.

To illustrate the importance and relevance of this research area, over the past four years there have been many conferences devoted to the analysis of localization phenomena in reaction-diffusion systems. A sample of these include a one-week conference in Crete, Greece in June 1999 on nonlinear dynamics for PDE's related to materials science (organized by N. Alikakos of U. Tennessee), a one-week conference on reaction-diffusion systems at CUHK in Dec. 1999 (organized by J. Wei of CUHK and M. Mimura of U. Hiroshima), a two-week conference in pattern formation at the Lorentz Institute in Leiden, Holland in March 2001 (organized by D. Hillhorst of U. Paris-Sud and H. Matano of U. Tokyo), the PIMS conference on point-condensation phenomena in Vancouver, Canada in July 2001 (organized by C. Gui of U. Conneticut and N. Ghoussoub of UBC), and a six-month programme (Jan. 2001-Jun. 2001) in reaction-diffusion equations at the Newton Institute in Cambridge, England (organized by N. Dancer of U. Sydney and H. Brezis of Paris 6).

In this workshop we focussed on three main sub-areas of localization behavior in reaction-diffusion systems: (1) localized patterns in chemotaxis; (2) localized patterns in activator inhibitor systems including the Gierer-Meinhardt and Gray-Scott models; (3) localized patterns in emerging areas, including di-block co-polymers, chemical reactions on surfaces, combustion etc.

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In this workshop we solicited four survey talks: *Horstmann* on chemotaxis; *Nishiura* on the Gray-Scott model; *Ren* on the di-block co-polymer problem, and *Wei* on the Gierer-Meinhardt model. In addition, each of these people also gave a more specialized lecture. There were 13 other lectures given by other participants. We now summarize results and open problems that were brought forth to each of the three themes of the workshop.

1 Chemotaxis and Oriented Movement

One major focus of the meeting was to investigate pattern formation in reaction-advection-diffusion systems that occur in population dynamics, and in chemotaxis.

An essential characteristic of living organisms is the ability to sense signals in the environment and adapt their movement accordingly. This allows for the location of food, the avoidance of predators, or the search for mates. When the response involves the detection of a chemical, it is termed *chemotaxis*, *chemokinesis*, or generally *chemosensitive movement*. The term *chemotaxis* is used broadly in the mathematical literature to describe general chemosensitive movement responses, and it is in this context that we use the term here. Models for chemotaxis have been successfully applied to bacteria, slime molds, skin pigmentation patterns, leukocytes and many other examples.

The following participants gave presentations about population dynamics *Chris Cosner (Miami)*, *Thomas Hillen (Edmonton)*, *Dirk Horstmann (Köln, Germany)*, *Hans Othmer (Minneapolis)*, and *Angela Stevens (Leipzig, Germany)*.

1.1 Diffusion Based Models for Chemotaxis

Patlak (1953) and Keller and Segel (1970) were the first to derive a mathematical model for chemotaxis. The Keller-Segel model in it's general form consists of four coupled reaction diffusion equations. It can be reduced to "only two" essential parameters, the population density $u(t,x)$ and the concentration of a chemical signal $v(t,x)$. The Keller-Segel model reads

$$\begin{aligned} u_t &= \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v) \\ v_t &= k_c\Delta v - k_3(v)v + uf(v). \end{aligned} \tag{1.1}$$

This system has been studied on unbounded and on bounded domains with various boundary conditions (Dirichlet, Neumann, mixed). In his survey talk, *Dirk Horstmann* gave an excellent review of the existing literature and the analytical results on the Keller-Segel model (1.1). Most results deal with finite time blow-up solutions. In the case of constant k_1, k_2, k_3, and f it is known that the qualitative behavior strongly depends on the space dimension. In 1-D the system has globally existing solutions. The 2-D case in ambiguous and thresholds have been found. If the total initial mass exceeds its threshold, then the solution blows up in finite time. When the initial mass is below this threshold, then solutions exist globally in time.

The blow-up solutions of (1.1) show the existence of a very strong instability and a large aggregational force. In certain situations, however, it is desirable to obtain stable aggregation patterns, which do not blow up in finite time. *Thomas Hillen* gave a continuation of *Horstmann's* review in the sense that he discussed various mechanisms which prevent blow up. The mechanisms were (i) saturation effects, (ii) volume filling effects, (iii) quorum sensing effects, and (iv) finite sampling radius. Ad (i): Saturation effects in $k_2(u, v)$ occur very naturally if cell surface receptor kinetics are taken into account. Chemotaxis models with saturation effects have been studied analytically and have been used in many applications. Ad (ii): The volume filling effect was introduced by Hillen and Painter (2002). It is assumed that particles have a finite volume and that cells can not move into regions which are already filled by other cells. In a simple form this leads to a term $k_2(u, v) = \chi u(1 - u)$. It was shown analytically that this form of k_2 leads to globally existing solutions in all space dimensions. Ad (iii): Quorum sensing occurs if the cells release an extra chemical which is repulsive to other cells. The resulting equation has two competing drift terms, chemotactic attraction and quorum sensing repulsion. It is an open mathematical challenge to find general conditions such that solutions blow up, or exist globally. Ad (iv): Also the inclusion of a finite sampling radius leads to global existence, at least in 2-D, as was shown by Hillen and Schmeiser.

1.2 Transport Models for Chemotaxis

The models mentioned so far are based on the assumption that the particles carry out an uncorrelated random walk, which is modeled by diffusion. However, some species movement can be characterized better by a transport equation. The bacterium *Eschirichia coli* moves via rotation of flagella and, when rotating anticlockwise, these flagella bundle together resulting in a period of smooth swimming - a "run". Clockwise rotation, however, results in the flagella spraying outwards resulting in a random reorientation, or a "tumble". Normal swimming is characterized by periods of smooth runs punctuated by tumbling. In the presence of a chemoattractant, *E. coli* bias their behavior by tumbling less frequently in an increasing attractant gradient, resulting in the general movement toward high concentrations. Detection of the attractant is made by the binding of attractant molecules to cell surface receptors, which subsequently initiates a cell internal pathway which transduces the signal to the movement machinery.

Both of these aspects, the microscopic level of signal transduction, and the macroscopic level of population movement have been modeled separately. *Hans Othmer*, in his talk, attempts to merge these levels of modeling to find one model for the whole process of bacterial chemotaxis which also includes the internal dynamics. The analysis is based on the velocity jump process for describing the motion of individuals, wherein each individual carries an internal state that evolves according to a system of ordinary differential equations forced by a time- and/or space-dependent external signal. He derives a macroscopic system of hyperbolic differential equations from this velocity jump process using moment closure techniques. He also reduces this macroscopic system to a single second order hyperbolic equation which, in a suitable limit, reduces to a classical chemotaxis equation in which the chemotactic sensitivity is now a known function of parameters of the internal dynamics.

Angela Stevens also studies transport equations for chemotaxis. She discusses in detail how the variety of evaluations of the chemical stimulus influences the motion of the respective species on the macroscopic level. It is known that in a formal limit the transport equations lead to chemotaxis models of Keller-Segel type. Chalub, Markowich, Perthame and Schmeiser (2003) rigorously proved that in three dimensions, these kind of kinetic models lead to the classical Keller-Segel model as its drift-diffusion limit, when the equation for the chemical signal is of elliptic type and local and non-local effects are taken into account.

In the talk of *Stevens* the rigorous derivation of Keller-Segel type systems and their variants is given in case of parabolic or elliptic signal equation (also in 2 dimensions). Under suitable structure conditions existence of global solutions for the kinetic model can be shown.

1.3 Forward-Backward Diffusion

Chris Cosner presented reaction diffusion population models which lead to aggregations. This includes models with taxis-terms (chemotaxis or prey-taxis). Classical models for population dynamics with spatial dispersal typically assume that dispersal occurs by passive diffusion, perhaps with advection. Replacing passive diffusion with nonlinear diffusion of the type proposed to model aggregation can lead to changes in the possible dynamics supported by the models. In logistic models it can create an Allee effect, where small populations go extinct but populations above some threshold persist. Stronger versions of nonlinear diffusion can lead to ill-posed problems. Cosner described some nonlinear diffusion models in population dynamics, presented analytical results, and discussed some topics for further research on such phenomena. The analytic results are based on bifurcation theory and classical methods in partial differential equations.

In a second talk, *Dirk Horstmann* studied a very specific reduced Keller-Segel model, where forward-backward diffusion arises. The equation takes the form

$$p_t = \Delta(T(p)p), \qquad \text{with} \qquad T(p) = \frac{1}{K + p^2}\,.$$

This is an ill-posed forward-backward problem. However this equation can be approximated by a systems of a PDE and an ODE that takes time delays into account. Horstmann presented joint results with H.G. Othmer and K.J. Painter where he compared for one specific model the asymptotic behavior of the solution of the non-local in time or time delay problem with the asymptotic behavior of the solution of the corresponding single equation problem, which is a forward-backward problem in this specific case. Horstmann discussed the variational structure of the problem above and he drew connections to the Perrona-Malik formalism in image processing.

1.4 Conclusion

It appears that the pattern forming mechanisms are related to those for the Gray-Scott model or for Cahn-Hillard type problems. Whereas in many physical contexts there is usually an energy functional which guides the dynamics, in many biological application there is no such energy functional. Nevertheless, it is an open challenge for the future to describe the analogy between Cahn-Hillard patterns and chemotaxis patterns. Moreover, some chemotaxis patterns appear to be metastable. In the context of Gierer-Meinhardt systems a theory of non local eigenvalue problems and metastability has been developed recently. We expect that this theory can be adapted to the biological applications as well.

2 An Update on the Study of the Gierer-Meinhardt and Gray-Scott Systems

Through the use of a linearized analysis, Alan Turing in 1952 showed how stable spatially complex patterns can develop from small perturbations of spatially homogeneous initial data for a coupled system of reaction-diffusion equations. He then proposed that this type of localization could be responsible for the process of morphogenesis. Since that time, there have been many reaction-diffusion models proposed for pattern formation, including the following well-known activator-inhibitor system of Gierer and Meinhardt 1972

$$\text{(GM)} \qquad \frac{\partial a}{\partial t} = \epsilon^2 \Delta a - a + \frac{a^p}{h^q}, \qquad \tau \frac{\partial h}{\partial t} = D\Delta h - h + \frac{a^r}{h^s}, \qquad \frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0 \ \text{ on } \partial\Omega,$$

where

$$\epsilon << 1\,, \quad \frac{\epsilon^2}{D} << 1 \quad \tau > 0\,, \quad \frac{qr}{(p-1)(s+1)} > 1 \quad p > 1\,, \quad q > 0\,, \quad r > 1\,, \quad s \geq 0\,. \tag{2.1}$$

Extensions of the Gierer-Meinhardt model have been used to model localization phenomena in developmental biology and pattern formation on sea shells. Previous studies have been concentrated on using various types of weakly nonlinear theories for the onset of instabilities of spatially homogeneous steady-state solutions. However, such a theory no longer works in the study of highly inhomogeneous solutions, notably, spike-type solutions.

A related system, introduced by Gray and Scott (1988), models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a reservoir of one of the two chemicals in the reaction. In nondimensional variables the resulting system, known as the Gray-Scott model, can be written as

$$\begin{aligned} v_t &= \epsilon^2 \Delta v - v + Auv^2\,, \qquad x \in \Omega\,, \quad t > 0\,, \\ \tau u_t &= D\Delta u + (1-u) - uv^2 \qquad x \in \Omega\,, \quad t > 0\,, \\ \frac{\partial v}{\partial \nu} &= \frac{\partial u}{\partial \nu} = 0\,, \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

Here $A > 0$, D, $\tau > 1$, and $\epsilon \ll 1$, are constants. For various ranges of these parameters, the Gray-Scott model is known to posses a rich solution structure including the existence of stable standing pulses, the propagation of traveling waves, pulse-replication behavior, and spatio-temporal chaos (cf. Pearson (1993), Reynolds et al. (1997), Nishiura and Ueyama (1999,2001), Doelman, Gardner, and Kaper (1998)).

Following on the many studies on spikey phenomena for singularly perturbed scalar quasilinear elliptic PDE's originating in the papers of W. M. Ni and collaborators, there has been considerable focus on the mathematical study of spike-type solutions for activator-inhibitor systems. This meeting has been an excellent platform for mathematicians from different areas including, dynamical systems, nonlinear PDEs, variational methods, and numerical analysis, to share their insights and results on pattern formation in the GM and Gray-Scott models, and related systems. Many open questions on mathematical problems related to spikey patterns were presented in the survey talk of *Wei*.

There were six lectures related to spikey patterns in the GM and Gray-Scott models, and related systems. *Dancer's* talk concerns steady-states of shadow systems, *Kolokolnikov's* lecture dealt pulse-splitting behavior in the Gray-Scott model, *Kaper's* talk was focused on semi-strong interactions in the Gray-Scott model, *Nishiura* introduced the notion of scattors in dissipative systems and he gave a survey talk of pulse-type behavior in the Gray-Scott model, *Ward's* discussed Hopf bifurcations of spike solutions and the dynamics of spikes, while *Wei* gave a survey lecture highlighting open problems and accomplishments to date. He also gave a lecture on spotty patterns in the two-dimensional Gray-Scott model and an analysis of the instability of ring solutions in the Gray-Scott model and their evolution to spotty patterns.

For the GM model, we begin with the Shadow System obtained by letting $D \to \infty$;

$$\frac{\partial a}{\partial t} = \epsilon^2 \Delta a - a + \frac{a^p}{\xi^q}, \qquad \tau \frac{\partial \xi}{\partial t} = -\xi + \xi^{-s} \frac{1}{|\Omega|} \int_\Omega a^r, \qquad \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega .$$

By the scaling $a = \xi^{\frac{q}{p-1}} u$, the corresponding steady-state problem becomes

$$(I) \ \ \epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \qquad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Since the fundamental works of Ni-Takagi (91-93), many authors have studied problem (I) in search of multiple interior or boundary spike solutions. They include:Alikakos, Bates, Cao, Dancer, del Pino, Felmer, Fusco, Ghoussoub, Grossi, Gui, Kowlaczyk, Y.Y. Li, A. Pistoia, C.-S. Lin, Ni, Takagi, Noussair, XB Pan, Shi, Ward, Wei, Winter, Yan... Gui-Wei's result asserts that for any fixed integers K, l, there is a solution to (I) with K boundary spikes and l interior spikes.

Dancer reported his recent study on (I), where he proved that if the Morse index is finite and dimension N is $N = 2, 3$, then all solutions must have multiple spikes. His result also extends to the study of a bistable-type nonlinearity. There are still many questions remained towards understanding (I):

Question 1: uniqueness and Morse index of multiple spike solutions.

Question 2: characterize and construct solutions that concentrate on a higher-dimensional subset of Ω.

Question 3: do concentrated solutions exist for the supercritical case $p > \frac{N+2}{N-2}$?
Question 4: characterize the global bifurcation branches of solutions as a function of ϵ.

The stability of multiple spikes solutions for the shadow system has been studied by Ni-Takagi-Yanagida in the one-dimensional case and by Wei (1999) in the 2D case. Wei's result only applies to the following exponent case

$$(*) \text{ either } r = 2, \; 1 < p \leq 1 + \frac{4}{N} \quad \text{or} \quad r = p+1, 1 < p < \frac{N+2}{N-2}.$$

Question 5: develop a general method for studying the spectrum of the following nonlocal eigenvalue problem (NLEP) in the right half plane $\mathrm{Re}(\lambda) \geq 0$:

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \lambda\phi \,.$$

Here w is the ground-state solution.

The most interesting case is $r = p$ which arises in the study of Selkov's model, the Keller-Segel model in chemotaxis, an Epidemic SIS model, and in the study of hot-spot behavior in microwave heating models.

Next we discuss the near-shadow system case $D \sim e^{\frac{d}{\epsilon}}$. Kolokolnikov and Ward (preprint:2003) have done some bifurcation analysis of one-spike solutions for dumbbell shape domains. Still the following question remains to be resolved:
Question 6: under what condition on D, can one construct multiple boundary or multiple interior spike solutions to the GM model without any additional condition on Ω? Where are the locations of spike equilibria in an arbitrarily shaped two-dimensional domain as a function of D.

For the strong-coupling case $D = O(1)$, there have been many results in one and two space dimensions. Takagi (1986) constructed multiple symmetric spike solutions in 1D, M. Ward and Wei (2001) constructed multiple asymmetric spikes in 1D generated by exactly two types. Wei and Winter 99-2002 constructed multiple symmetric and asymmetric spots in 2D. Doelman-Kaper-Ploeg, Chen-Del Pino-Kowalczyk (2000) constructed multiple pulse ground states in 1D. Del Pino-Kowalczyk-Wei (2000) constructed multi-bump ground state solutions on a regular $K-$polygon or concentric polygons or honey-comb. Wei and Winter (2002) constructed multiple symmetric and asymmetric clusters in 1D. Ni-Wei (2003) constructed ring-like solutions in higher dimensions.
Question 7: why are there only two types of patterns? Can one construct a reaction-diffusion systems with more than two types of spikes in a given spike pattern? The phenomena of only two types of spikes is believed to be related to properties of certain Green's functions.
Question 8: Suppose $N \geq 3$, are there ground state solutions with a single bump? This problem

can be reduced to the study of the following simple-looking elliptic system

$$\Delta A - A + \frac{A^2}{H} = 0\,, \qquad \text{in } R^N$$
$$\Delta H + A^2 = 0 \qquad \text{in } R^N,$$
$$A = A(|y|), \quad H = H(|y|), \quad A, H \to 0 \quad \text{as} \quad |y| \to +\infty\,.$$

There have been many recent works on the stability of spike patterns in the strong coupling case. In 1D, Iron-Ward-Wei (2001) (matched asymptotics), Wei-Winter (2002) (rigorous) showed that there exists a sequence of decreasing numbers:$D_1 > D_2 > D_3 > ... > D_K > ...$ such that for $D < D_K$, $K-$symmetric spikes are stable when $\tau = 0$ and for $D > D_K$, K-symmetric spikes are unstable for any $\tau \geq 0$. Similar results are also established in 2D by Wei-Winter (2002). In that case,

$$D_K \sim \frac{|\Omega| \log \frac{\sqrt{|\Omega|}}{\epsilon}}{2\pi K} \tag{2.2}$$

Question 9: What about stability of multiple asymmetric K-spikes in 1D? Numerically, they are believed to be all unstable.

Wei mentioned an interesting comparison between D_K in 2D with Ozawa's asymptotic expansion of small eigenvalues for domains with small holes.

Question 10 : What is the $O(1)$ term in the asymptotic expansion of $D_K(\epsilon)$?

For the case $\tau > 0$, the stability of spike patterns for the 1-D GM model has been studied in Ward-Wei (2003) using a combination of rigorous and asymptotic analysis. This was the subject of *Ward's* lecture. The stability diagram in the $\tau - D$ parameter plane has been now largely understood. For a fixed $D < D_K$, it was shown that there is a Hopf bifurcation as τ increases past some critical value τ_h. This oscillatory instability has the effect of synchronizing the amplitude and phase of the amplitude oscillations of the spikes.

The nonlocal eigenvalue problem that is central to the analysis in Ward and Wei (2003) has the form

$$\Delta\phi - \phi + pw^{p-1}\phi - \chi(\tau\lambda)(p-1)\frac{\int_{R^1} w^{r-1}\phi}{\int_{R^1} w^r} w^p = \lambda_0\phi, \qquad \phi \in H^2(R^N)\,.$$

There are many open questions regarding eigenvalue problems of this form in one space dimension

In *Ward's* talk, it is shown that for multiple spikes in 1D, one has

$$\chi = \chi(z; j) \equiv qr\left(s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)}\left[\tanh(\theta_\lambda/k) + \frac{(1-\cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)}\right]\right)^{-1},$$

where

$$z \equiv \tau\lambda\,, \qquad \theta_\lambda \equiv \theta_0\sqrt{1+z}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{2.3}$$

In his talk, *M. Ward* showed that if $r = 2$, $1 < p < 1 + \frac{4}{N}$, and $chi(z) = \frac{a}{b+cz}$, NLEP has a unique Hopf bifurcation. He also presented results on the Hopf bifurcation and oscillatory instability of multiple spikes in 1D. Some new instability are discovered: competition instabilities and synchronous oscillations. Dancer (2001) showed the existence of two positive unstable eigenvalues for τ large. However, a key open problem concerns proving a strict transversal crossing condition for χ of the form given above and for general values of r.

Question 11: uniqueness of Hopf bifurcation for other exponents and general $\chi(z)$. Can there exist several values of τ where there is a Hopf bifurcation? Is this bifurcation subcritical or supercritical?

In *Ward's* lecture it was shown that in the low-feed rate regime of the 1D Gray-Scott model where $A = O(\epsilon^{1/2})$ there is a spectral equivalence principle between the Gray-Scott and the Gierer-Meinhardt model in the sense that one can identify appropriate exponents (p, q, r, s) in the GM model to obtain the exact spectral problem associated with the Gray-Scott model in the low feed-rate regime. This implies that oscillatory and competition instabilities also occur for the Gray-Scott model.

The lecture of *Kolokolnikov* concerned the analysis of pulse-splitting behavior of the 1D Gray-Scott model in the regime where $A = O(1)$, and where the finite domain places an important role. He introduced the following core problem that is connected to pulse splitting behavior:

$$\begin{gathered} V'' - V + V^2 U = 0\,, \qquad 0 < y < \infty\,, \\ U'' = UV^2\,, \qquad 0 < y < \infty\,, \\ V'(0) = U'(0) = 0\,; \qquad V \to 0\,, \qquad U \sim By\,, \qquad \text{as} \quad y \to \infty\,, \end{gathered}$$

where $U > 0$, $V > 0$, and

$$B \equiv \frac{A}{\coth(\theta_0/k)}\,, \qquad \theta_0 = D^{-1/2}\,. \tag{2.4}$$

As shown numerically in *Kolokolnikov's* lecture, there are two positive solutions to this core problem when $0 < B < 1.347$. He then showed formally that there are no k-spike equilibria in the pulse-splitting regime $A = O(1)$ when $A > A_{pk}$, where

$$A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right)\,.$$

In terms of this core problem, when τ is sufficiently small, *Kolokolnikov* predicted that a one-spike solution centered at the midpoint of a 1D domain will undergo 2^{m-1} spitting events, and that the final equilibrium state will have 2^m spikes where, for some smallest value of m, A lies in the interval

$$A_{p2^{m-1}} < A < A_{p2^m}\,.$$

He also analyzed the thresholds for the existence of traveling waves.

The core problem for V and U, without the effect (2.4) of the finite domain, was one focus of the lecture by *Kaper*. He gave specific results on the core problem and a bounding region that contains all solutions to the core problem.

Question 12: Give rigorous results on the global solution branches of the core problem. Estimate the fold point value 1.347 rigorously and describe the shape of the solutions on these branches.

The problem of the dynamics of multiple spikes is largely open: there are only scarce results (both formal and rigorous) on dynamics of spikes. Chen-Kowalzyk considered the dynamics of interior spikes shadow system case. D. Iron and M. Ward derived the dynamics of boundary spikes for the shadow systems and multiple spikes in 1D.

Kaper's lecture gave some results on the semi-strong dynamics of two spikes for the Gray-Scott model and for a generalized GM model. He derived a formula for the speed of separation of two pulses, and showed that blow-up solutions are possible for certain variants of the GM model.

Question 13: give a rigorous treatment of dynamics of multiple spike solutions spikes in 1D and 2D. Determine the stability of quasi-equilibrium solutions consisting of k spikes.

Nishiura gave an excellent survey of instabilities of particle-like solutions in reaction-diffusion systems. He showed self-replication and self-destruction phenomena in several systems, and showed different collision properties between traveling spots. In his other lecture, he showed that the notion of scattors is a very useful concept for understanding the input-output relation for the collision of two pulses. When pulses collide they can either annihilate, repel like fixed particles, repel and produce complicated oscillatory phenomena in their trailing edges, or produce a bound pair. *Nishiura* showed that special types of unstable steady or time-periodic solutions called scattors act as saddle points in phase space and are critical for determining the fate of colliding pulses. This concept was illustrated using the Complex Ginzburg Landau equation, the Gray-Scott model, and a three-component reaction diffusion model arising in gas-discharge phenomena.

3 Other Applications of Localization

There were several other talks of localization behavior in other systems.

Ren discussed interface phenomena in non-local variational problems associated with di-block co-polymers. He gave a survey talk on the physics of the co-polymer problem and a sketch of the derivation of the nonlocal energy functional that determines the formation of interfaces. A basic summary of the set-up is that there are type A and type B monomers in a di-block co-polymer system which often form A-rich and B-rich microdomains. On a larger scale these phase domains give rise to a morphological pattern. The widely observed lamellar and wriggled lamellar patterns are modeled using the Ohta-Kawasaki model in which the free energy density field depends nonlocally on the monomer composition field. In his other lecture, *Ren* showed that in one dimension the Gamma-convergence technique can be used to reduce to this problem to a finite dimensional minimization problem. For each K there exists a 1D local minimizer with K+1 microdomains and K domain walls. Among these 1D local minimizers there is the 1D global minimizer that has optimal spacing between the domain walls. These 1D local minimizers are extended trivially to two dimensions to give solutions of lamellar patterns to the Euler-Lagrange equation. The stability of these solutions was studied and their spectra are found. A 1D local minimizer is stable in 2D only if it has sufficiently many domain walls. The 1D global minimizer is near the borderline of 2D stability. This interesting phenomenon was found to be related to the existence of wriggled lamellar solutions as seen from the bifurcation theory. The stability properties of the wriggled lamellar solutions was determined in the theory after some careful calculations.

The lecture of *Choksi* also concerned the dib-block co-polymer problem, but had a more global focus. He presented two physical problems in which pattern formation can be modeled via the minimization of a nonlocal free energy. In each case, he explained the origin and derivation of the free energy. He then used gamma convergence techniques to determine the scales and patterns for minimizers in several space dimensions.

The lecture of *Pearson* focused on reaction-diffusion and traveling wave patterns that are concentrated on a thin sheet, representing the cell boundary. In the *Xenopus laevis* oocyte, calcium ion channels are clustered in a thin shell near the outer cell wall. Motivated by this morphology he studied the effect of "sheet excitability" in an idealized reaction–diffusion system with a 2-dimensional sheet of sources embedded in 3-dimensional space. He found that waves undergo propagation failure with *increasing* diffusion coefficient and a scaling regime in which the wave speed is independent of the diffusion coefficient.

The lecture of *Matkowsky* focussed on the dynamics of hot-spot solutions in a gasless combustion model, representing a solid sample where combustion occurs only on the surface of a cylinder of radius R. Different hot-spot behavior was observed in their numerical computations as R was increased. For a fixed value of the Zeldovich number, if R is sufficiently small, slowly propagating planar pulsating flames are the only modes observed. As R is increased transitions to more complex modes of combustion occur, including (i) traveling waves (TWs), i.e., spin modes in which one or several symmetrically spaced hot spots (localized temperature maxima) rotate around the cylinder as the flame propagates along the cylindrical axis, thus following a helical path, (ii) counterpropagating (CP) modes, in which spots propagate in opposite angular directions around the cylinder, executing various types of dynamics, (iii) alternating spin CP modes (ASCP), where rotation of a spot around the cylinder is interrupted by periodic events in which a new spot is spontaneously created ahead of the rotating spot. The new spot splits into counterpropagating daughter spots, one of which collides with the original spot leading to their eventual mutual annihilation, while the other continues to spin. Other more complicated features were observed as R is increased. Since the hot-spots are localized it would be interesting to try to explain some of these different behaviors analytically using recent techniques developed for the Gierer-Meinhardt and Gray-Scott models.

Finally the lecture by *Kuske* concerned modulation theory of patterns. Modulation equations describe the behavior of complex systems over long scales. However, their validity is often limited to near-criticality. A new multi-scale approach, combining energy arguments and balance of nonlinearities, yields modulation equations for localized buckling of a strut away from the critical load, where standard asymptotics and normal forms fail. Immediate connections to heterogeneous patterns in other applications are shown. Her approach was illustrated via simple one-dimensional models, motivated by numerics and experiments.

List of Speakers:

- Prof. Rustum Choksi (Simon Fraser U: choksi@cs.sfu.ca)
- Prof. Chris Cosner (U. Miami: gcc@math.miami.edu)
- Prof. E. N. Dancer (U. Sydney: normd@maths.usyd.edu.au)
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