Local Uniformization and Resolution of Singularities

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This research in teams project was devoted to a seemingly impossible classical problem, for which there has recently been encouraging progress: local uniformization (a local form of resolution of singularities) in positive characteristic. In recent years, the participants have made independent progress on this problem, and the main purpose of this meeting was to share the insights and methods that have been developed. One of our participants, Shreeram Abhyankar has proven local uniformization up through dimension 3 in positive characteristic [1]. Although this theorem was proven in the mid 1960s, this is still the strongest general result.

During our discussion we made progress on several central problems which arise in local uniformization and valuation theory, and identified important areas of focus. Below is a short list of some of the focus problems.

1.) Kuhlmann has shown in earlier versions of the papers [9] and [10] that local uniformization is always possible after taking a finite extension, and that no extension is needed in the case of Abhyankar places (places satisfying equality in the Abhyankar inequality). The first result can also be deduced from de Jong's work on resolution after a generically finite extension [7]. However, Kuhlmann's proof uses only valuation theoretic methods, so that it is entirely different from de Jong's approach. We discussed Kuhlmann's proof, and possible removal of the necessity of taking a finite extension in the case of non-Abhyankar places. For this one would need, among other things, to find a transcendence basis of a given valued algebraic function field such that the rational function field generated by it has the same value group as the function field itself. This is closely related with the description of all valuations on rational function fields [11]. We discussed whether it can be done in an "easy" way in characteristic 0; the ideas developed in this discussion shall be worked out in a subsequent research project.

Another ingredient of local uniformization is the elimination of (tame and wild) ramification. Abhyankar's Lemma is an instance of elimination of tame ramification. A short discussion of its proof inspired a generalization to a broader ramification theoretical context [14]. A theorem proved by Epp [8] in 1972 and a theorem proved in Kuhlmann's thesis and applied in [10] are instances of elimination of wild ramification. It would now be desirable to investigate elimination of wild ramification more systematically. This should play a key role in the search of a solution to the problem of avoiding extensions of the function field.

2.) Teissier [15] has recently made outstanding progress on the problem of local uniformization by deformation to the associated graded ring of the valuation. This ring is in general not finitely generated, and requires completion, so a number of very interesting technical problems arise in commutative algebra and valuation theory. We discussed some of these problems. For example, his approach needs an infinite-dimensional form of the Implicit Function Theorem or of Hensel's Lemma. Such a theorem could possibly be proved by general ultrametric techniques, building on work done by Kuhlmann [12]. The results of this paper shall be extended to the infinite-dimensional case.

3.) We discussed and analyzed Zariski's original proof of local uniformization in characteristic zero, and its obstruction to generalization in positive characteristic. We discussed its relationship with the problem of the existence of defect in finite extensions of valued fields, the problem of local monomialization [6] in positive characteristic, and the relationship of this problem with the difficulties arising in 1.) and 2.). We discussed Kuhlmann's work [13] on the classification of Artin-Schreier-extensions with defect and what it could tell us about a crucial example given in [6]. As an improvement of the present form of the paper [13], it was suggested to work out in detail the deformation of Artin-Schreier-extensions with defect into purely inseparable extensions with defect, and to draw the connection with deformations in the sense of algebraic geometry.

4.) Given an analytically unramified local ring R, dominated by a valuation ν, find a prime ideal P in the completion $\hat{R}$ such that ν extends naturally to a valuation of the same rank as ν which dominates $\hat{R}/P$. This problem has arisen independently in the work of Cutkosky [3], [5], [4] and Teissier [15]. P can be defined naturally if ν has rank 1. In fact, P can be chosen to be nonsingular, and have other nice properties reflected in the value group if R has equicharacteristic zero (as shown in [3], [5], [4]). We identified examples showing that if the rank is larger than one then P is not uniquely determined, although it may be possible to impose conditions on P so that it is canonical.

5.) Suppose that R is an analytically unramified local ring, and ν is a valuation which dominates R. In dimension 2 the structure of the semigroup of R and the structure of a generating sequence are well understood. It would be desirable to understand completely the structure of these semigroups, and of a generating sequence in higher dimensions. The connection of these questions with the key polynomials of MacLane, as generalized by Vaquié, and with methods involving pseudo convergent sequences of Kuhlmann [11] were discussed. It appears that key polynomials can be deduced in an easy way from the approach used in [11]; this remains to be worked out in detail.

6.) Knaf has extended Kuhlmann's local uniformization for Abhyankar places to the arithmetic case, working over discrete valuation rings instead of ground fields [9]. In this result certain types of ramification have to be excluded. At BIRS, Knaf and Kuhlmann worked on the paper [10], achieving the same generalization for non-Abhyankar places, and discussed possible approaches to remove the restrictions concerning ramification.

A discussion not directly related to our project produced another little paper [2]. The valuations of interest here are of infinite rank. While such valuations do not appear in the context of local uniformization, they do appear in the context of the model theory of valued fields, which has turned out to be tightly connected to local uniformization through the phenomenon of the defect.

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