Topology in Dimension 4.5 22w5065

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October 31, 2022 – November 4, 2022

1 Overview of the Field

The use of 4-dimensional perspectives and techniques to study 3-dimensional spaces and knots inside of them is often referred to colloquially as *topology in dimension 3.5*. As the study of 4-manifolds continues to accelerate, the analogous notion of *topology in dimension 4.5* has become increasingly indispensable: the use of 5-dimensional techniques to study 4-dimensional manifolds and knotted surfaces inside of them.

One of the initial, and still most canonical, uses of topology in dimension 4.5 is to study pairs of 4-manifolds that are homeomorphic but not diffeomorphic. Classifying smooth 4-manifolds remains an elusive goal; for instance the generalized Poincaré conjecture remains open smoothly in dimension four. However, if two smooth, closed, simply connected 4-manifolds X and Y are homeomorphic, then there is a smooth, simply connected cobordism W^5 between them that induces homotopy equivalences on either end, i.e an *h-cobordism* [Si]. This provides a valuable bridge between the smooth structures on X and Y, and hence understanding W allows us to compare the two 4-manifolds.

More recent directions of the developing field of topology in dimension 4.5 include: studying knotted surfaces in 4-manifolds up to isotopy and concordance, detecting the difference between topological and smooth isotopy of surfaces, and understanding the mapping class groups of 4-manifolds.

2 Recent Developments and Open Problems

2.1 Corks

In the 1990's, Curtis-Freedman-Hsiang-Stong [CFHS] showed that h-cobordisms are always smoothly products away from contractible sub-cobordisms called *corks*. This localizes the smooth differences between a simply connected exotic pair X, Y, since an h-cobordism W between X and Y is a product away from some cork. Corks have been well-studied in recent years (see e.g. [Ak, MS, AY]) to understand the complexities of h-cobordisms between 4-manifolds and relate exotic structures in general. In particular, Freedman's topological h-cobordism theorem implies the topological 4-dimensional Poincaré conjecture: any simply connected 4-manifold with vanishing self-intersection is homeomorphic to S^4.

Motivating Question 1. Is an h-cobordism between two homotopy 4-spheres always diffeomorphic to $S^4 \times I$?

The above question is one of the most fundamental open problems in smooth 4-dimensional topology, which would imply the smooth 4-dimensional Poincaré conjecture.

Motivating Question 2. Is there a single cork away from which any two homeomorphic, simply-connected smooth 4-manifolds differ?

2.2 Concordance

One of the most prolific topics in 3.5-dimensional topology is the study of an equivalence relation of knots in a 3-manifold called *concordance*. Several recent results have sparked interest in concordance of knotted 2-spheres in 4-manifolds. The notion of *0-concordance* of knotted spheres was introduced by Melvin in 1977, who proved that 0-concordant 2-spheres have diffeomorphic Gluck twists, providing relevance to the smooth 4-dimensional Poincaré Conjecture. The first obstructions to 0-concordance were produced in 2019 by Sunukjian [Su], Dai–Miller [DM], and Joseph [Jo], who showed that the 0-concordance monoid is not finitely generated, and also not a group. Besides these results we know very little about this 4.5-dimensional concept, which could provide a useful lens to study the set of all knotted spheres.

Motivating Question 3. Are there examples of spheres that are n-concordant, for arbitrarily high n? Are there spheres that are 0-cobordant but not 0-concordant?

2.3 Lightbulbs

Gabai's recent '4-dimensional Lightbulb Theorem' [Ga1] has sparked a new interest in multiple classical concordance and isotopy invariants of 2-spheres in 4-manifolds. Gabai used a hands-on geometric approach to characterize the conditions under which homotopy implies isotopy for 2-spheres in certain 4-manifolds. Shortly afterwards, Schneiderman and Teichner [ST2] proved the Lightbulb Theorem in arbitrary 4-manifolds, revitalizing the use of a concordance invariant introduced by Freedman and Quinn [FQ] in the 1990's and later corrected by Stong [St]. Several experts have conjectured an analog to the Lightbulb Theorem for properly embedded disks instead of 2-spheres requiring a powerful refinement of the Freedman-Quinn invariant due to Dax [Da], which obstructs isotopy as opposed to concordance. Examples of concordant disks with vanishing Freedman-Quinn invariant but non-vanishing Dax invariant were recently given by Gabai [Ga2]. Using the Dax invariant along with other homotopy theoretic techniques, Budney and Gabai [BG] were also able to produce pairs of 3-balls in the 4-sphere that are homotopic rel boundary, but not smoothly isotopic.

Motivating Question 4. What invariants obstruct smooth (rather than topological) isotopy and concordance of 2-spheres in 4-manifolds?

2.4 The Disk Embedding Theorem

In another revitalization of classical techniques, several authors have been investigating Freedman and Quinn's seminal work on Freedman's disk embedding theorem [FQ] and other foundational results in the topological category [BKKPR], leading to many generalizations of well-known theorems from that era which fall neatly into the category of 4.5-dimensional topology. For instance, Freedman's proof that every knot in the 3-sphere with Alexander polynomial 1 bounds a topological disk has been recently generalized to the setting of *n-shake sliceness* by Feller–Miller–Nagel–Orson–Powell–Ray [FMNOPR]. In another direction, Conway–Powell [CP] classify locally flat homotopy-ribbon disks in the 4-ball with certain prescribed fundamental groups of their exteriors. Using their result, Hayden recently gave the first examples of exotic ribbon disks in the 4-ball, i.e. disks that are topologically isotopic but smoothly distinct. Guth [Gu] later modified these examples to produce exotic ribbon disks that must be stabilized an arbitrarily high number of times before becoming smoothly isotopic.

Motivating Question 5. What conditions on the exteriors of two disks with the same boundary ensure that the disks are topologically isotopic? What invariants can smoothly distinguish two disks with simple boundary?

2.5 Diffeomorphism Groups

Watanabe uses a strikingly different set of higher dimensional tools to study problems in smooth 4-dimensional topology in his recent proof [Wa1] that the smooth 4-dimensional Smale conjecture is false, and his even more recent work [Wa2] constructing mutually non-isotopic diffeomorphisms of non simply-connected 4-manifolds. Interesting elements of π_k of diffeomorphism groups of 4-manifolds are studied by building 4-manifold bundles over S^{k+1}, which are described using graphs that encode parametrized 4-dimensional surgery instructions. Invariants of these bundles are then computed, intersection-theoretically, in higher dimensional configuration spaces of points in the 4-manifolds. The case $k = 0$, i.e. the study of smooth mapping class groups of 4-manifolds, yields a fascinating interplay between dimensions 4 and 5.

Motivating Question 6. What invariants can distinguish elements of $\pi_n(\text{Diff}(X^4))$? Are they computable?

3 Presentation Highlights

The presentations given during the week-long workshop consisted of:

- **25-minute 'Picture this!' talks:** Instead of a traditional abstract, we let the picture do the talking and serve as the abstract for these talks. Four of the 'Picture this!' talk pictures are included in Figure 1.

Eight 'Picture this!' talks were given in total; six of them by graduate students. The informal and creative format of these talks engaged the conference participants, and gave an opportunity for graduate students to advertise their results in a unique way.

- **50-minute 'Hot-off-the-press' talks:** These were more traditional research talks on recent results of interest, including several talks by early-career researchers.

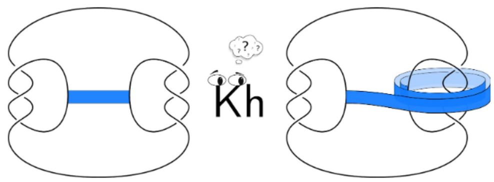
There were 10 'Hot-off-the-press' talks given by a mixture of graduate students, early-career researchers, and established professors who are leaders in the field. For instance: graduate student Kai Nakamura gave a talk on his new techniques for constructing exotic 4-manifolds, titled *Annulus twisting a disk: Standard and exotic*, Slava Krushkal discussed research in-progress with his collaborators at the conference during his talk *Topological pseudoisotopy of 4-manifolds*, and Ryan Budney employed background from the mini-lecture series to explain 'How to show a barbell diffeomorphism is non-trivial'.

- **Three mini-lecture series:** Each mini-lecture series was given by teams of 4-6 conference participants. They presented background, new results, and open questions about selected topics of interest in the field.

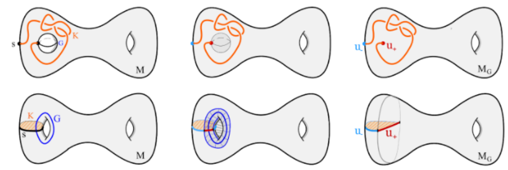
Our mini-lecture series generated many questions and potential avenues of study. It also prompted experts in these topics to go 'back to basics' and welcome a broader 4.5-dimensional topology community into discussions previously privy to only a few. One highlight was Tadayuki Watanabe presenting on his pioneering approaches to the study of diffeomorphism groups of 4-manifolds, one of the topics on which the conference explicitly focused. Section 3 provides more details on the mini-lecture series and a complete list of the questions that were generated.

- **An 'after-hours talk'** given by Rob Kirby on some classical proofs and results in 4-manifold topology.

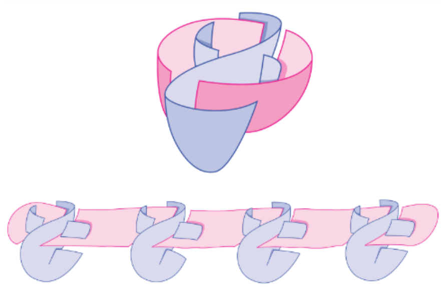
Many also joined the workshop remotely, both to speak and participate, including some graduate students who reached out to us after we had finalized the participant list. Early-career researchers such as Danica Kosanovic and Allison Miller who gave talks, but were not able to attend in-person, also enjoyed the remote option.



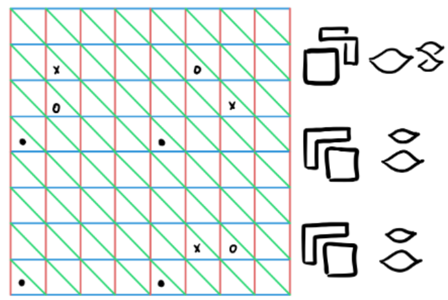
(a) *Speaker*: Isaac Sundberg, *Title*: A non-detection result in Khovanov homology



(b) *Speaker*: Peter Teichner, *Title*: Isotopy classification of half-disks in 4-manifolds



(c) *Speaker*: Nicholas Cazet, *Title*: Broken Sheet Diagrams of Knot Cobordisms



(d) *Speaker*: Sarah Blackwell, *Title*: Triple Grid Diagrams

Figure 1: Selected 'Picture This!' abstracts

4 Scientific Progress Made

During the conference, we asked three groups of 4–6 researchers to present a mini-lecture series on topics related to the theme of this conference. These were, "knotted surfaces," "concordance in dimension 4.5," and "diffeomorphism groups." Each of these groups (and some additional participants) proposed open questions in a range of difficulties. We compiled the following list of suggested open problems, which we made publicly available at https://math.stanford.edu/ maggiehm/TopologyDim4point5.html. We include the list here to illustrate the depth of the conference, although the language is necessarily technical.

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Question 1. [P. Naylor] Is the Gluck twist of the roll-spun 2-knot of the classical knot 7_3 diffeomorphic to the standard 4-sphere?

Naylor and Schwartz [NS] previously showed that the Gluck twist of the roll-spun 2-knot of any classical knot with unknotting number one is diffeomorphic to S^4. The knot 7_3 is the first prime knot with unknotting number two.

Question 2. [J. Boyle [Boy], posed at the conference by P. Naylor] For any classical knot K, is the turned 1-twisted spun torus of K smoothly unknotted?

Question 3. [P. Naylor] What is known about spinning classical links?

Question 4. [P. Naylor] Are there similar constructions to deform spins using non-standard ribbon disks?

Exercise 5. [A. Conway] Find a 2-knot $K \subset S^4$ with trivial Alexander module but nontrivial Rochlin invariant μ.

Question 6. [Unknotting problem, posed at the conference by A. Conway] If $K \subset S^4$ is a 2-knot with $\pi_1(S^4 \setminus K) \cong \mathbb{Z}$, is K smoothly unknotted? What invariants could possibly prove such a 2-knot is smoothly nontrivial?

It follows from work of Freedman [Fr] that a 2-knot in S^4 whose complement has fundamental group $\mathbb{Z}$ is topologically unknotted.

Question 7. [A. Conway] Can you define μ for nullhomologous 2-spheres in other 4-manifolds?

Question 8. [A. Conway] What invariants determine the homotopy type of $S^4 \setminus \nu(K)$?

Problem 9. [A. Ray] Define the Freedman–Quinn invariant $\text{fq}(R, R')$ for more surfaces R, R' (e.g. positive-genus or nonorientable surfaces).

Problem 10. [A. Ray] Extend the lightbulb theorem of Gabai [Ga1] to non-orientable ambient 4-manifolds.

Question 11. [A. Ray] Can the conditions on the dual sphere G in the lightbulb theorem [Ga1] be refined? For example, what if G intersects R and R' in a single point but R is immersed?

Question 12. [well-known; posed at the conference by M. Powell] Is every 2-link slice?

Problem 13. [M. Powell] Classify n-component link maps $\sqcup^n S^2 \to S^4$ up to link homotopy for $n \geq 3$.

Problem 13 is known for $n = 1$, and solved for $n = 2$ by Schneiderman and Teichner [ST1].

Question 14. [R. Schneiderman] Are there more settings in which the Freedman–Quinn invariant can be defined considering unbased homotopies?

Question 15. [R. Schneiderman] Does there exist a self-homotopy J of some $S^2 \subset M^4$ such that the self-intersection invariant J $\mu(J)$ is not in $\mu(\pi_3(M))$?

Question 16. [R. Budney] Do barbell diffeomorphisms generate $\pi_0(\text{Diff}(D^4)$ or $\pi_0(\text{Diff}(S^1 \times D^3)$?

Question 17. [R. Budney] Is θ_2 (as in [BG]) nontrivial?

Budney and Gabai [BG] proved that θ_n is nontrivial for $n \geq 3$.

Question 18. [R. Budney] Can Watanabe's invariants [Wa1] be defined in terms of scanning?

Question 19. [R. Budney] Does knotting of the 2-spheres used in a barbell give any interesting behavior to study?

Question 20. [R. Budney] Are the Hatcher–Wagoner invariants surjective in dimension 4?

Problem 21. [R. Budney] Understand Diff$(S^2 \times S^2)$ or Diff$(\mathbb{CP}^2)$. Can you find generators of π_0?

Question 22. [R. Budney] What is the difference between Diff(spin 4-manifold) and Diff(non-spin 4-manifold)?

Question 23. [well-known; posed at the conference by R. Budney] What is π_0(Diff(D^4))?

Question 24. [S. Krushkal] Barbells generate the subgroup of π_0(Diff$(S^1 \times B^{n-1})$) that is null in pseudoisotopy for $n \geq 6$ [HW]. Does this hold for $n = 5$ too?

Problem 25. [S. Krushkal] Find null-pseudoisotopies for the Budney–Gabai [BG] diffeomorphisms of $S^1 \times D^3$ and compute their Hatcher–Wagoner obstructions.

Question 26. [S. Krushkal] Budney–Gabai [BG] proved π_0(Diff$(S^1 \times B^3, \partial)$ contains an infinite set of linearly independent elements. Are (some of) these elements still nontrivial up to topological isotopy?

Problem 27. [D. Auckly] Compare π_n(Diff(Z, D^4)) to π_n(Homeo(Z, D^4)) up to stabilizing Z^4 with $S^2 \times S^2$ summands.

Question 28. [T. Watanabe] Do the graph classes in π_k(BDiff(D^4)) survive under the map π_k(BDiff$_\partial(D^4)) \to \pi_k$(Bdiff$_\partial(D^4 \#(S^2 \times S^2)))$ from the Weiss fiber sequence?

Question 29. [T. Watanabe] Are the theta-graph (or barbell) classes mapped to nontrivial elements by π_1(BDiff$_\partial(D^3 \times S^1)) \to \pi_1$(BHomeo$_\partial(D^3 \times S^1)$)?

Question 30. [T. Watanabe] Can a configuration space integral invariant be defined on π_1(Bhomeo$_\partial(D^3 \times S^1)$)?

Question 31. [T. Watanabe] What is the image of $p : \pi_1(\mathcal{M}^{\text{psc}}_\partial(X^4)) \to \pi_1$(BDiff$_\partial(X^4)$)? Here, $\mathcal{M}^{\text{psc}}_\partial(X^4)$ refers to the moduli space of positive scalar curvature metrics on X^4.

5 Outcome of the Meeting

Through the mini-lecture series, conference talks, and discussions/collaborations throughout the week at BIRS, our workshop was successful in achieving its stated goals: (i) facilitating in-depth discussions on recent results about knotted surfaces in 4-manifolds which utilize 5-dimensional perspectives, (ii) raising awareness of 4-dimensional isotopy and concordance invariants, as well as their potential applications, and (iii) developing tools to understand smooth mapping class groups of 4-manifolds.

In addition to the creation of the extensive problem list in Section 3, two of the participants who presented in the mini-lecture series created written notes to accompany their portion of the lecture. The first, Anthony Conway, posted his notes 'Invariants of 2-knots' to the arXiv during our workshop [Co]. The latter, Mark Powell, posted his notes 'Concordance and isotopy invariants of surfaces' as an available reference on his website [Po].

The conference at BIRS last week was really enlightening! –Watanabe

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