Women in Noncommutative Algebra and Representation Theory 3

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This report summarizes the objectives and scientific progress made at the 3rd workshop for women in noncommutative algebra and representation theory at the Banff International Research Station in Banff, Canada.

1 Objectives

The goals of this workshop were the following.

- To have accessible introductory lectures by participants in the themes of the workshop.
- To have each participant engaged in a stimulating research project and/ or be involved in a expansive research program in noncommutative algebra and/or representation theory.
- To have each participant provide or receive training toward this research activity (before and at the workshop) and to have made significant progress in such directions by the end of the workshop.
- To set-up mechanisms so that the collaborative research groups formed before/ at the workshop can continue research after the workshop, so that their findings will be published eventually.
- To provide networking opportunities and mentoring for its participants at and beyond the workshop

2 Introductory Talks

There were four "What is...?" talks given by participants of the workshop:

- Asilata Bapat: What is... the Bridgeland stability condition?
- Emily Gunawan and Emine Yildirim: What is . . . a connection between cluster algebras and friezes?
- Maryam Khaqan: What is... moonshine?
- Florencia Orosz Hunziker: What is... a vertex algebra?

3 Scientific Progress Made

The group leaders are indicated by (*) below.

3.1 Verlinde-type Formulae for Fusion Coefficients

Group members: Georgia Benkart* (University of Wisconsin, Madison), Sarah Brauner (University of Minnesota), Laura Colmenarejo (North Carolina State University), Francesca Gandini (Kalamazoo College), Ellen Kirkman* (Wake Forest University), and Julia Plavnik (Indiana University).

Let $H = \Bbbk G$ be the group algebra of a finite group over a field of characteristic zero. Then H is a semisimple Hopf algebra. Denote the simple left H-modules by $S_1, \ldots, S_n$. For any fixed left H-module V and each simple module S_j, the tensor product $V \otimes_{\Bbbk} S_j$ is a left H-module so decomposes into a direct sum of simple left H-modules. Hence there are nonnegative integers $n_V(i,j)$ with

$$V \otimes_{\Bbbk} V_j = \sum_{i=1}^{n} n_V(i,j) S_i.$$

The matrix $N_V = (n_V(i,j))$ represents the $\Bbbk$-linear map $N_V : R(H) \to R(H)$ given by left tensoring with V acting on the representation algebra $R(H)$, expressed in terms of the basis of $R(H)$ that consists of isomorphism classes of the simple modules $[S_i]$. The character table of G, thought of as an $n \times n$ matrix S, simultaneously diagonalizes all matrices N_V, so $S^{-1} N_V S = D_V$, for a diagonal matrix D_V, whose entries also come from the character table of G. Solving for $N_V = S D_V S^{-1}$ gives a formula for the fusion coefficients in terms of data from the character table. Such a formula is called a "Verlinde formula", and such formulae occur in various contexts. As one such example, Witherspoon found such a matrix S related to a character table when H is a semi-simple almost cocommutative Hopf algebra (e.g. the Drinfeld Double of a semisimple Hopf algebra). As another example, if $H = \Bbbk G$, when $\Bbbk$ has characteristic p, then H may no longer be semisimple. Using simple composition factors of $\Bbbk G$-modules, instead of simple direct summands, the table of Brauer characters of G was used to study the maps N_V in work of Grinberg, Huang, and Reiner, who noted that the maps N_V can be considered for the modules of any Hopf algebra, since then the tensor product of two H-modules is again an H-module. More generally, onc can consider the fusion relations in a fusion category, and in the setting of a modular fusion category there is a symmetric matrix S, that is a representation of $\mathrm{SL}_2(\mathbb{Z})$, and has other remarkable properties, including the fact that it diagonalizes the fusion relations; in this case the category is braided, and the fusion algebra is semisimple, commutative, symmetric, among other special properties.

Our collaboration group is looking at some examples of Hopf algebras and their related matrices of fusion coefficients, searching for properties that extend the notions described above. In these examples the fusion relations are not always diagonalizable, so their Jordan form is considered, and we are interested in properties of matrices that place the N_V matrices into Jordan form. The N_V matrices' properties usually depend strongly on properties of V. Nevertheless, there is some interest among physicists in producing some sort of Verlinde formula in circumstances beyond the case where the tensor category is modular and the fusion algebra is semisimple and commutative. During our week at BIRS we computed various matrices related to the matrices N_V in several settings. Our work continues in our weekly virtual meetings that were initiated last fall. The group consists of researchers with interests in tensor categories, representation theory, algebraic combinatorics, commutative algebra, and noncommutative algebra, areas that already have come into play in our work.

3.2 Cluster Categories I

Group members: Karin Baur* (University of Leeds), Lea Bittmann (University of Vienna), Emily Gunawan (University of Oklahoma), Gordana Todorov* (Northeastern University), and Emine Yıldırım (Queen's University).

Cluster algebras were introduced by Fomin-Zelevinsky in 2002 in order to give a combinatorial framework for studying algebraic groups, and have since appeared in various fields including representation theory, triangulations of surfaces, Teichmüller theory, Poisson geometry, algebraic combinatorics and frieze patterns. The associated categories of representations, cluster categories, introduced by Buan-Marsh-Reineke-Reiten-Todorov have also had numerous applications throughout mathematics as described in Reiten's ICM 2010 talk on this subject.

During the week of April 4-8, we have looked at questions related to frieze combinatorics and their connections to triangulations of surfaces and to representation theory. Here, a frieze is an array of (possibly infinitely many) rows of integers, starting with a row of 0s and a row of 1s and satisfying the so-called diamond rule: any four entries

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfy $ad - bc = 1$. We consider periodic friezes, i.e. friezes for which all rows have a translational period. An example of a frieze of period 4 is here:

$$\begin{array}{ccccccccccccccccc}
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
& 3 & & 2 & & 2 & & 4 & & 3 & & 2 & & 2 & & 4 & \\
\cdots & & 5 & & 3 & & 7 & & 11 & & 5 & & 3 & & 7 & & \cdots \\
& 18 & & 7 & & 10 & & 19 & & 18 & & 7 & & 10 & & 19 & \\
& & 25 & & 23 & & 27 & & 31 & & 25 & & 23 & & 27 & & \\
& \vdots & & & & \vdots & & & & & & \vdots & & & & \vdots &
\end{array}$$

Friezes have been introduced in the 70s by Coxeter and Conway, [5], [3], [4]. See for example [1] for a survey.

During the WINART3 workshop, our group has studied various types of infinite friezes. Finite friezes are known to arise from cluster algebras of Dynkin types A, D and E. Infinite friezes have been studied in the context of cluster algebras of Dynkin type $\widetilde{A}$ (affine type). A remarkable property of infinite periodic friezes is that their entries grow in a controlled way: if the frieze has period n then for any entry in the nth non-trivial row, the difference to the entry directly above it is a constant, independent of which entry in row n we choose, [2, Theorem 2.2]. This invariant of the frieze is called *growth coefficient* of the frieze. In the example above, the growth coefficient is 20: the first entry shown in row four is 25, the entry directly above it is 5, and so on. Continuing, the difference of an entry in row kn and an entry in row $kn-2$ is also always constant and can be given in terms of the growth coefficient.

Questions and Progress: The main objects we have studied during the week were triangulations of twice punctured disks: these give a geometric model for cluster algebras in type $\widetilde{D}$ (affine type). We were interested in the associated infinite friezes and in their growth coefficients. We found properties of these growth coefficients. In addition, we have considered the affine types $\widetilde{E}$ and determined certain associated friezes. We found that in both affine types, the growth coefficients behave well and are linked to band modules in the corresponding cluster category. In addition, we considered friezes arising from triangulations of a pair of pants, i.e. sphere with three boundary components. These provide new families of triples of infinite friezes which have different behaviour than the above tame types. A third direction we explored was a surgery and gluing construction on the surfaces we worked on. This allows us to go from triangulations of annuli to triangulations of pairs of pants and back, i.e. between triangulations of a sphere with two boundary components and a sphere with three boundary components.

3.3 Cluster Categories II

Group members: Ilke Çanakçı*(Vrije University Amsterdam), Francesca Fedele (Università degli Studi di Verona - Università degli Studi di Padova), Ana Garcia Elsener* (University of Glasgow - Universidad Nacional de Mar del Plata), Khrystyna Serhiyenko (University of Kentucky).

Cluster algebras are a class of commutative rings that were introduced by Fomin and Zelevinsky [7] in 2002. Their original motivation was coming from studying canonical bases in Lie Theory. Today, cluster algebras are connected to various fields of mathematics, including geometry, combinatorics, and representation theory of associative algebras. The study of cluster variables, the distinctive set of generators for a cluster algebra, can be simplified by working with their geometric model or their representation theoretic interpretation.

We focus on cluster algebras of Dynkin type A_n. Their geometric model consists of a triangulation of a disk with $n + 3$ marked points, where the initial cluster variables are in bijection with the arcs of the triangulation and the remaining cluster variables correspond to the remaining arcs. The cluster variables can be computed combinatorially using a snake graph formula: each internal diagonal of the triangulated disk is associated to a snake graph and the corresponding cluster variable can be written down using its perfect matchings (also known as dimer covers).

Alternatively, the cluster variables can be computed homologically using representation theoretic means. Denoting by kA_n the path algebra of A_n, the cluster variables are in bijection with the indecomposable kA_n-modules. Moreover, the Caldero-Chapoton map [6] applied to each indecomposable in mod(kA_n) gives its corresponding cluster variable.

Our project. Starting from the above classic theory, our project aims to give a deeper understanding of the super cluster algebras of Dynkin type A_n, as studied combinatorially by Musiker, Ovenhouse and Zhang [8] in 2021. A super algebra is a $\mathbb{Z}_2$-graded algebra and it is generated by a set of even variables $\underline{x}$, which commute with each other, and a set of odd variables $\underline{\theta}$, which anticommute with each other and commute with the even ones. The geometric model of these algebras consists of an oriented triangulation (without internal triangles) of a disk with $n + 3$ marked points, where the initial even variables are in bijection with the arcs of the triangulation and the remaining super cluster variables are in bijection with the remaining arcs. Moreover, the initial odd variables are associated to each triangle of the triangulation. As in the classic case, the super cluster variables can be computed combinatorially using a snake graph formula: each (oriented) internal diagonal in the disk is associated to a snake graph and the corresponding super cluster variable can be written down using its double dimer covers (obtained by superimposing two perfect matchings). Since each tile in a snake graph is obtained by gluing together two triangles, it has two associated odd variables which also play a role in the super snake graph formula.

We aim to give a representation theoretic interpretation of super cluster algebras of type A_n. Our proposal is to study the algebra $\widetilde{kA_n} := kA_n \otimes_k k[\epsilon]$, obtained by tensoring the path algebra of A_n with the dual numbers $k[\epsilon]$. We claim that the super cluster variables are in bijection with the indecomposable induced modules over $\widetilde{kA_n}$, that is the modules of the form $M \otimes_k k[\epsilon]$, where M is an indecomposable kA_n-module. In order to show this correspondence, we define a super Caldero-Chapoton map from the induced modules to the set of super cluster variables.

- We have worked out a bijection between the lattice of double dimer covers of a snake graph and the submodule lattice of the corresponding induced module. We are in the process of writing down a formal proof.
- We defined a super Caldero-Chapoton map and we understand the role of the odd variables combinatorially. We are exploring if these variables can be described in a representation-theoretic way.
- During the WINART3 workshop week we have also fully worked out the A_2 case, and we believe we understand the general case. We exchanged emails with technical questions with the super cluster algebra paper authors. We explored bibliography that will be used in our project and agreed on a writing plan that will take place in the next months. We will have zoom meetings periodically.

3.4 On the Monster Lie algebra

Group members: Darlayne Addabbo (University of Arizona), Lisa Carbone* (Rutgers University), Elizabeth Jurisich* (College of Charleston), Maryam Khaqan (Stockholm University), and Sonia Vera (Universidad Nacional de Cordoba).

This work concerns open questions about the structure of the Monster Lie algebra. Borcherds constructed the Monster Lie algebra $\mathfrak{m}$ to prove part of the Conway–Norton *Monstrous Moonshine Conjecture*. Let $\mathbb{M}$ denote the Monster finite simple group. A fundamental component of Borcherds' construction was Frenkel, Lepowsky and Meurman's Moonshine Module $V^\natural$, a graded $\mathbb{M}$-module with $\mathrm{Aut}(V^\natural) = \mathbb{M}$. $V^\natural$ is an example of a vertex operator algebra. The Monster Lie algebra $\mathfrak{m}$ is a quotient of the 'physical space' of the vertex algebra $V = V^\natural \otimes V_{1,1}$, where $V_{1,1}$ is a vertex algebra for the even unimodular 2-dim Lorentzian lattice $II_{1,1}$. We describe this construction in more detail below. The Monster Lie algebra $\mathfrak{m}$ also has a realization as the Borcherds (generalized Kac–Moody) algebra $\mathfrak{m} = \mathfrak{g}(A)/\mathfrak{z}$ where $\mathfrak{g}(A)$ is the Lie algebra with infinite generalized Cartan matrix A and $\mathfrak{z}$ is the center of $\mathfrak{g}(A)$:

$$A = \begin{array}{c} \\ c(-1) \\ \\ c(1) \\ \\ \\ c(2) \\ \\ \\ \end{array} \left(\begin{array}{c|ccc|ccc|c}
\overset{c(-1)}{2} & 0 & \overset{c(1)}{\cdots} & 0 & -1 & \overset{c(2)}{\cdots} & -1 & \cdots \\ \hline
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \hline
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \hline
\vdots & & \vdots & & & \vdots & &
\end{array}\right)$$

The numbers $c(j)$ are coefficients of q in the modular function $J(q) = j(q) - 744 =$

$$\sum_{i \geq -1} c(j) q^j = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

so $c(-1) = 1$, $c(0) = 0$, $c(1) = 196884, \dots$. If we set $I_0 = \{-1, 1, 2, 3, \dots\}$ and let $(i,k) \in I_0 \times \mathbb{Z}_{>0}$ for $1 \leq k \leq c(i)$. Then $\mathfrak{g}(A)$ has generators

$$e_{ik},\ f_{ik},\ h_{ik}$$

for $(i,k) \in I = \{(i,k) | i,k \in \mathbb{Z}, 1 \leq k \leq c(i)\}$ and simple roots α_{ik} for $(i,k) \in I_0 \times \mathbb{Z}_{>0}$, $1 \leq k \leq c(i)$. For $(i,k) = (-1,1)$ we write the generators as e_{-1}, f_{-1}, h_{-1}.

The Lie algebra $\mathfrak{g}(A)$ has defining relations:
(**R1**) $[h_{jk}, h_{i\ell}] = 0$,
(**R2**) $[h_{jk}, e_{i\ell}] = -(j+i)e_{i\ell}$,
(**R3**) $[h_{jk}, f_{i\ell}] = (j+i)f_{i\ell}$,
(**R4**) $[e_{jk}, f_{i\ell}] = \delta_{ji}\delta_{k\ell}h_{jk}$,
(**R5**) $(\mathrm{ad}\, e_{-1})^i e_{ik} = 0$ and $(\mathrm{ad}\, f_{-1})^i f_{ik} = 0$

for $(j,k), (i,\ell) \in I$. The h_{ik} are linearly dependent, so $\mathfrak{m} = \mathfrak{g}(A)/\mathfrak{z}$ has a two dimensional Cartan subalgebra $\mathfrak{h}$ with basis elements denoted h_1, and h_2.

The Monster Lie algebra has the usual triangular decomposition $\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^\pm$ are direct sums of the positive (respectively negative) root spaces of $\mathfrak{m}$. We define the extended index set

$$E = \{(\ell, j, k) \mid (j,k) \in I^{\mathrm{im}},\ 0 \leq \ell < j\} = \{(\ell, j, k) \mid j \in \mathbb{N},\ 1 \leq k \leq c(j),\ 0 \leq \ell < j\}.$$

and set

$$e_{\ell,jk} := \frac{(\mathrm{ad}\, e_{-1})^\ell e_{jk}}{\ell!} \quad \text{and} \quad f_{\ell,jk} := \frac{(\mathrm{ad}\, f_{-1})^\ell f_{jk}}{\ell!},$$

for $(\ell, j, k) \in E$. The following non-trivial result gives an additional non-standard decomposition of $\mathfrak{m}$.

Theorem 3.1. *([9], [10]) Let* $\mathfrak{gl}_2(-1)$ *be the subalgebra of* $\mathfrak{m}$ *with basis* $\{e_{-1}, f_{-1}, h_1, h_2\}$. *Then*

$$\mathfrak{m} = \mathfrak{u}^- \oplus \mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$$

where $\mathfrak{gl}_2(-1) := \langle e_{-1}, f_{-1}, h_1, h_2 \rangle \cong \mathfrak{gl}_2$, $\mathfrak{u}^+$ *is a subalgebra freely generated by* $\{e_{\ell,jk} \mid (\ell,j,k) \in E\}$ *and* $\mathfrak{u}^-$ *is a subalgebra freely generated by* $\{f_{\ell,jk} \mid (\ell,j,k) \in E\}$.

To construct $\mathfrak{m}$ as a quotient of the 'physical space' of a vertex algebra V, we define $V = V^\natural \otimes V_{1,1}$, where $V_{1,1}$ is a vertex algebra for the even unimodular 2-dim Lorentzian lattice $II_{1,1}$. We have $\mathfrak{m} = P_1/R$ where

$$P_1 = \{\psi \in V^\natural \otimes V_{1,1} \mid L(n)\psi = \delta_{n0}\psi,\ n \geq 0\}$$

is the space of weight one primary vectors of the vertex algebra $V^\natural \otimes V_{1,1}$ and

$$R := \{v \in V \mid (u, v) = 0 \text{ for } u \in V\}$$

is the radical of the symmetric bilinear form $(\cdot, \cdot)$. It is an open question to construct specific elements of $V^\natural \otimes V_{1,1}$ that determine the generators of certain distinguished subalgebras of $\mathfrak{m}$ which are known only in terms of sets of generating root vectors.

Goals. The following are the main goals of this work. In the next subsection, we describe our current progress towards these goals.

- Find vertex operators that correspond to generators of the 'imaginary' $\mathfrak{gl}_2$ subalgebras $\mathfrak{gl}_2(\mathrm{im}_{(ik)})$ in $V^\natural \otimes V_{1,1}$ corresponding to imaginary simple root vectors $e_{ik},\ f_{ik},\ h_{ik}$ and imaginary simple roots α_{ik} for $(i,k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $1 \leq k \leq c(i)$.
- Find the vertex operators that correspond to the imaginary root vectors $\{e_{\ell,jk}\}$ and $\{f_{\ell,jk}\}$ which generate the free subalgebras $\mathfrak{u}^\pm$ of $\mathfrak{m}$ respectively.
- Clarify our understanding of how the action of $\mathbb{M}$ carries through $V^\natural \otimes V_{1,1}$ to $\mathfrak{m}$ as a quotient.
- Understand the role of the vectors in $V^\natural \otimes V_{1,1}$ that give rise to the co-dimension 1 free Lie algebra in $\mathfrak{n}^+$ and the structure they generate in $V^\natural \otimes V_L$.

Progress. During the months leading up to the WINART program, our WINART group members participated in a reading group lead by Prof. Carbone, which helped us to learn necessary background material for the project. Before our arrival at BIRS, we gave the vertex operators for the $\mathfrak{gl}_2$ subalgebra in $V^\natural \otimes V_L$ corresponding to the unique real simple root vector of $\mathfrak{m}$. This was previously also given in [11]. A main goal for our group during the WINART workshop was to begin identifying vertex operators for $\mathfrak{gl}_2$ subalgebras in $V^\natural \otimes V_L$ corresponding to imaginary simple root vectors $e_{ik},\ f_{ik},\ h_{ik}$ for $(i,k) \in I$ and simple roots α_{ik} for $(i,k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $1 \leq k \leq c(i)$. During the workshop, we made conjectures as to how to construct such vertex operators, and completed several preliminary calculations necessary for proving our conjectures. Upon returning from the WINART workshop, we proved the existence of certain vectors in $V^\natural$ that we need for our proposed construction and completed more computations in this direction.

3.5 Combinatorial models in representation theory: additive friezes

Group members: Asilata Bapat (Australian National University), Véronique Bazier-Matte (University of Connecticut), Eleonore Faber* (University of Leeds), Bethany Marsh* (University of Leeds), Kunda Kambaso (RWTH Universität Aachen), and Yadira Valdivieso (UDLAP University of the Americas Puebla).

Recently, many combinatorial models have arisen in the representation theory of algebras. Examples include the categorification of Coxeter–Conway friezes using cluster categories (first pointed out by Caldero–Chapoton [14]), the description of module categories via Dyck paths (Moreno Cañadas-Bravo Riós [21]) and the description of categories associated to gentle algebras via surface triangulations and ribbon graphs (see Baur–Coelho Simoes [12], Lekili–Polishchuk [19], Opper–Plamondon–Schroll [22]).

During the workshop we looked at a variant of Coxeter–Conway's frieze patterns, so-called *additive friezes*, and studied their representation-theoretic properties.

The notion of a (multiplicative) frieze pattern was introduced by Coxeter in [16] and further studied by Conway and Coxeter in the 1970s in [17, 18], where it was shown that integral friezes of finite rank correspond to triangulations of n-gons. After the introduction of cluster algebras in the 2000s, it was shown that the triangulations define a categorification of cluster algebras of type A [15] (see also [13]). Since then, friezes have been studied from various points of view in representation theory; in particular Morier-Genoud's survey [20] gives a good overview of recent developments.

While multiplicative friezes are well explained in the context of cluster categories, their additive counterparts (see [23] for definitions) have been studied much less from a representation theoretic point of view. In particular, we were interested in additive friezes of non-negative integers (AIFs), and how to enumerate them, fixing the rank n of the frieze.

Results: For small n, we could determine the number of AIFs using a computer program. Our model for additive friezes of rank n is the Auslander–Reiten quiver of $D^b(\mathrm{mod} kQ)$, where Q is a quiver of type A_n. We showed how to interpret additive friezes as elements of the dual of the Grothendieck group of $D^b(\mathrm{mod} kQ)$ and determined a suitable basis of the dual. Using cluster category methods, we were able to interpret additive friezes as points in an associahedron in some $\mathbb{R}^m$. The main task was to the find integer points that correspond to AIFs: we claim that these are integer points in a certain "symmetrization" of the associahedron. For small n we were able to verify this claim by direct computation and we are working on a proof for arbitrary n.

3.6 Hochschild (co)homology I

Group members: Hongdi Huang (Rice University), Monique Müller (Universidade Federal de São João del-Rei), María Julia Redondo* (Universidad Nacional del Sur), Fiorela Rossi Bertone* (Universidad Nacional del Sur), Pamela Suárez (Universidad Nacional de Mar del Plata).

The aim of this group is to consider a particular family of algebras, the gentle algebras, and study their deformations in terms of Maurer-Cartan elements.

The gentle algebras are a particular case of monomial algebras, and they can be described as path algebras kQ/I, with some particular conditions on the quiver Q and the monomial relations I.

It is well-known, see [24], that the deformations of an algebra are parametrized by the Maurer-Cartan elements, that is, elements f of degree one in $C^*(A)[1]$, the shifted Hochschild complex which has structure of DGLA, satisfying the equation

$$df + \frac{1}{2}[f, f] = 0.$$

The Hochschild complex $C^*(A)$ is obtained by applying the functor $\mathrm{Hom}_{A-A}(-, A)$ to the Hochschild resolution $C_*(A)$. When A is an algebra over a field k, the Hochschild resolution is a projective resolution of A in the category of A-bimodules. Usually, when dealing with computation of Hochschild cohomology, it is convenient to replace the Hochschild complex $C^*(A)$ by any other complex obtained from another projective resolution of A.

In the particular case of monomial algebras, Bardzell's complex $B^*(A)$ has shown to be an efficient replacement of Hochschild complex when dealing with computations. However, with this replacement we loose the DGLA structure needed to study deformations. One needs to consider L_∞-structures, which is a generalization of DGLA, in order to recover the connection with deformations, which is now given in terms of the generalized Maurer-Cartan equation.

Since Bardzell's complex is a retract of Hochschild complex, one can describe a L_∞-structure on $B^*(A)$ that induces a quasi-isomorphism of L_∞-algebras.

In [26] we have described explicitly this L_∞-structure on $B^*(A)$ for any monomial algebra A, using some comparison morphisms between $C^*(A)$ and $B^*(A)$ that have been introduced in [25].

Finally, in order to study deformations of gentle algebras, we have to compute the L_∞-structure of their Barzdell's complex, and describe their Maurer-Cartan elements. The problem with the generalized Maurer-Cartan equation is that it is a series and it could be divergent. So, the project for this group contemplates, in the case of gentle algebras, giving conditions under which the generalized Maurer-Cartan equation is convergent and, when possible, giving a description of the Maurer-Cartan elements.

We have started working a few weeks ago through Zoom meetings. We are already familiar with the problem and with the calculations we have to do using the comparison morphisms described in [25] and the L_∞-structure given in [26].

3.7 Hochschild (co)homology II

Group members: Dalia Artenstein (Universidad de la República), Janina Letz (University of Bielefeld), Amrei Oswald (University of Iowa), Sibylle Schroll* (University of Cologne), and Andrea Solotar* (Universidad de Buenos Aires).

Due to pandemic reasons and also due to personal reasons of some of the members of the group, the work of our group has been completely online.

Andrea Solotar explained in the first meeting the aim and scopes of our work: Our aim is to study a family of algebras called *string algebras* using their Hochschild cohomology and possibly the associated geometric models. String algebras are monomial special biserial algebras and as such they are an important testing ground for conjectures and ideas. Many other classes of well-studied algebras are part of this class of algebras, the most well-known ones being the so-called gentle algebras which relate with many other areas of mathematics. The Hochschild cohomology of an associative algebra endowed with the cup product and the Gerstenhaber bracket has a very rich structure. Andrea explained some results concerning gentle algebras that are part of unpublished work by Schroll and Solotar in collaboration with Cristian Chaparro Acosta and Mariano Suárez-Alvarez and which are important to the project on string algebras.

At a later stage, Andrea Solotar explained in detail the definition of string algebras and showed how to compute the Hochschild cohomology of a particular family of string algebras: namely, the above mentioned family of gentle algebras. For this, it is important to have a precise knowledge of how to use Bardzell's resolution for monomial algebras.

Some of the members of the team were not familiar with Hochschild cohomology, so this part of the project is taking longer than initially expected, but it is nevertheless important to spend time on it since it is a fundamental tool for our work.

At the end of the WINART3 week, Dalia Artenstein explained in her short talk the framework of our project and some preliminary examples that we have been discussing. Some interesting suggestions resulted from her talk.

Once all the members of our team acquire enough experience with the required methods of computation, we will be able to study a wide subfamily of non necessarily quadratic string algebras form the homological and the representation theoretic points of view.

3.8 Generalized Quantum Symmetry via Hopf algebroids

Group members: Bojana Femić (Mathematical Institute of the Serbian Academy of Sciences and Arts), Florencia Orosz Hunziker (University of Denver), Chelsea Walton* (Rice University), and Elizabeth Wicks* (Microsoft Corporation).

We seek to understand more about the representation categories of Hopf algebroids. Notions of Hopf algebroids naturally arise in various fields such as Poisson geometry, category theory, and algebraic geometry/topology says something about stable homotopy theory. From our perspective as non-commutative algebraists, bialgebroids and Hopf algebroids arise naturally in the study of symmetries.

What do we mean by this? A symmetry is a property-preserving transformation from an object A to itself. We will take A to be a $\Bbbk$-algebra here, where $\Bbbk$ is the ground field.

Classical symmetry: When A is a polynomial ring, its symmetries are fairly well-understood. For example, we have that the group $GL_2(\mathbb{C})$ acts on $\mathbb{C}[x, y]$ naturally by automorphisms, and the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ acts on $\mathbb{C}[x, y]$ naturally by derivations.

However, if we want to alter A (for example by a deformation that makes A noncommutative), we have to alter our acting object as well. This was one of the motivations to develop quantum symmetry.

Quantum symmetry: Let us suppose that we deform our original algebra as follows:

$$\mathbb{C}[x, y] \rightsquigarrow \mathbb{C}_q[x, y] := \frac{\mathbb{C}\langle x, y\rangle}{(yx - qxy)}, \quad q \in \mathbb{C}^\times.$$

In this case, groups or Lie algebras do not suffice to capture the symmetries of this algebra, since we cannot deform them in the same manner. However, there is a more general structure called a Hopf algebra that captures these symmetries. A Hopf algebra is defined to be a tuple $(H, m, u, \Delta, \epsilon, S : H \to H)$ where (H, m, u) is an algebra, (H, Δ, ϵ) is a coalgebra, and S is the antipode, such that these structures satisfy certain compatibility conditions. The Hopf algebras $\mathcal{O}_q(GL_2), U_q(\mathfrak{gl}_2)$ are Hopf algebras that act on $\mathbb{C}_q[x, y]$.

To be precise, we can say that a Hopf algebra H acts on an algebra A if and only if A is an algebra in the monoidal category $H\text{-mod}$. The category $H\text{-mod}$ is known to be monoidal with tensor product $\otimes_{\Bbbk}$.

Weak quantum symmetry: Now let us suppose that we want to alter A further. One natural operation is to take direct sums. We could imagine that if H is a Hopf algebra acting on A, then $H \oplus H$ acts on $A \oplus A$. For example,

$$\mathbb{C}[x, y] \rightsquigarrow \mathbb{C}[x, y] \oplus \mathbb{C}[x, y] \cong (\mathbb{C} \oplus \mathbb{C})[x, y].$$

However, $H \oplus H$ is no longer a Hopf algebra, so this does not fit into the quantum symmetry framework. We must alter our notion of acting object in order to allow for direct sums.

A natural candidate is the weak Hopf algebra. A weak Hopf algebra is defined to be a tuple $(H, m, u, \Delta, \epsilon, S : H \to H)$ where (H, m, u) is an algebra, (H, Δ, ϵ) is a coalgebra, and S is the antipode, such that these structures satisfy weaker compatibility conditions than a Hopf algebra. It is known that the direct sum of two weak Hopf algebras is again a weak Hopf algebra, and Hopf algebras are special cases of weak Hopf algebras.

In the previous example we generalized the base of the algebra: the base of $\mathbb{C}[x, y]$ is $\mathbb{C}$ while the base of $\mathbb{C}[x, y]^{\oplus 2}$ is $\mathbb{C} \oplus \mathbb{C}$. In both cases the base is commutative and also Frobenius separable. If we want to consider symmetries of algebras with any noncommutative base, we need to expand our notion of symmetry even further.

Towards Generalized Quantum Symmetry: We want to extend our algebra A as follows:

$$\mathbb{C}[x, y] \rightsquigarrow B[x, y],$$

where B is any $\Bbbk$-algebra. We think that the natural object to act on such an algebra is a Hopf algebroid with base B. We omit the definition of Hopf algebroid here, but remark that a weak Hopf algebra is a special case of a Hopf algebroid where the base is Frobenius separable.

Project: We want to continue the theme of understanding symmetries by understanding module categories over Hopf-type objects. In particular, we want to study categorical properties of $H\text{-mod}$ for H a Hopf algebroid. For Hopf algebras and weak Hopf algebras, it is well-known that such categories are monoidal, and work has been done to describe conditions under which the categories inherit desirable properties such as rigidity, braidedness, semisimplicity, and even modularity. We would like to extend similar conclusions to the Hopf algebroid case.

References

`For Section 1.2:`

[1] K. Baur. Frieze patterns of integers. *The Mathematical Intelligencer*, June 2021.

[2] Karin Baur, Klemens Fellner, Mark J. Parsons, and Manuela Tschabold. Growth behaviour of periodic tame friezes. *Revista Matemática Iberoamericana*, 35 (2):575–606, 2019.

[3] John H. Conway and Harold S. M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(400):87–94, 1973.

[4] John H. Conway and Harold S.M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(401):175–183, 1973.

[5] Harold S. M. Coxeter. Frieze patterns. *Acta Arith.*, 18:297–310, 1971.

`For Section 1.3:`

[6] Caldero, P., & Chapoton, F. (2006). Cluster algebras as Hall algebras of quiver representations. Commentarii Mathematici Helvetici, 81(3), 595-616.

[7] Fomin, S., & Zelevinsky, A. (2002). Cluster algebras I: foundations. Journal of the American Mathematical Society, 15(2), 497-529.

[8] Musiker, G., Ovenhouse, N., & Zhang, S. W. (2021). An Expansion Formula for Decorated Super-Teichmüller Spaces. Symmetry, Integrability and Geometry: Methods and Applications, 17(0), 80-34.

For Section 1.4:

[9] Jurisich, E., *Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra*, J. Pure Appl. Algebra 126 (1998), no. 1-3, 233–266.

[10] Jurisich, E., *An exposition of generalized Kac-Moody algebras. Lie algebras and their representations*, (Seoul, 1995), 121–159, Contemp. Math., 194, Amer. Math. Soc., Providence, RI, (1996).

[11] Jurisich, E., Lepowsky, J. and Wilson, R. L. *Realizations of the monster Lie algebra*, Selecta Math. (N.S.) 1 (1995), no. 1, 129–161.

For Section 1.5:

[12] Karin Baur and Raquel Coelho Simões. A geometric model for the module category of a gentle algebra. Int. Math. Res. Not. IMRN 2021, no. 15, 11357–11392.

[13] Aslak Bakke Buan, Bethany Marsh, Markus Reineke, Idun Reiten and Gordana Todorov. Tilting theory and cluster combinatorics. Adv. Math. 204, no. 2, 572–618, 2006.

[14] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3):595–616, 2006.

[15] P. Caldero, F. Chapoton and R. Schiffler. Quivers with relations arising from clusters (A_n case). Trans. Amer. Math. Soc. 358, no. 3, 1347–1364, 2006.

[16] H. S. M. Coxeter, Frieze patterns. *Acta Arith.* 18, 297–310, 1971.

[17] John H. Conway and Harold S. M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(400):87–94, 1973.

[18] John H. Conway and Harold S.M. Coxeter. Triangulated polygons and frieze patterns. *Math. Gaz.*, 57(401):175–183, 1973.

[19] Yankı Lekili and Alexander Polishchuk. Derived equivalences of gentle algebras via Fukaya categories. Math. Ann. 376, no. 1-2, 187–225, 2020.

[20] Sophie Morier-Genoud. Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics. *Bull. Lond. Math. Soc.*, 47(6):895–938, 2015.

[21] Agustín Moreno Cañadas and Gabriel Bravo Ríos. Dyck paths categories and its relationships with cluster algebras. Preprint arXiv:2102.02974 [math.RT], 2021.

[22] Sebastian Opper, Pierre-Guy Plamondon and Sibylle Schroll. A geometric model for the derived category of gentle algebras. Preprint arXiv:1801.09659 [math.RT], 2018.

[23] G. C. Shephard. Additive frieze patterns and multiplication tables. *Math. Gaz.*, 60(413):178–184, 1976.

For Section 1.6:

[24] M. Gerstenhaber, 'On the deformation of rings and algebras', *Ann. of Math.* (2) 79 (1964) 59–103.

[25] M. J. Redondo L. Román, 'Comparison morphisms between two projective resolutions of monomial algebras', *Rev. Un. Mat. Argentina* 59, no. 1 (2018) 1–31.

[26] M. J. Redondo, F. Rossi Bertone. *L_∞-structure on Bardzell's complex for monomial algebras*. J. Pure Appl. Algebra 226 (2021), 106935.